4.3 Answers

4.3.1 Solution to 4.1 Example (Teaching Video)

$$
\begin{aligned}
2x + 4y - z &= 12 \\
x - y + 4z &= 6 \\
4x + 5y - z &= 17
\end{aligned}
$$

In matrix form these become,

$$
\begin{pmatrix} 2 & 4 & -1 \\ 1 & -1 & 4 \\ 4 & 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ 17 \end{pmatrix}
$$

Using a calculator to get the inverse matrix, and pre multiplying both sides by it gives,

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{21}\begin{pmatrix} -19 & -1 & 15 \\ 17 & 2 & -9 \\ 9 & 6 & -6 \end{pmatrix}\begin{pmatrix} 12 \\ 6 \\ 17 \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}
$$

$\therefore$ The unique solution is $x = 1, \quad y = 3, \quad z = 2$

[4 marks]

4.3.2 Solutions to 4.2 Exercise

Answer 1

(a) $\mathbf{M} = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -2 & 1 \\ 3 & 5 & -2 \end{pmatrix} \Rightarrow \mathbf{M}^{-1} = \dfrac{1}{69}\begin{pmatrix} 1 & 13 & 5 \\ -11 & -5 & 14 \\ -26 & 7 & 8 \end{pmatrix}$

[2 marks]

(b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dfrac{1}{69}\begin{pmatrix} 1 & 13 & 5 \\ -11 & -5 & 14 \\ -26 & 7 & 8 \end{pmatrix}\begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix}$

$$
= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}
$$

$\therefore$ The unique solution is $x = 2, \quad y = 1, \quad z = 3$

[2 marks]

Answer 2

(i)

$$\begin{pmatrix} 1 & -1 & 1 \\ 4 & 0 & 1 \\ 2 & a & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2a \\ a \end{pmatrix}$$

[1 mark]

(ii)

$$\begin{vmatrix} 1 & -1 & 1 \\ 4 & 0 & 1 \\ 2 & a & 2 \end{vmatrix} = 1 \times \begin{vmatrix} 0 & 1 \\ a & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 4 & 0 \\ 2 & a \end{vmatrix}$$

$$= -a + 6 + 4a$$

$$= 3(a + 2)$$

[2 marks]

(iii)

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ a & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ 2 & a \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ a & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & a \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -a & 6 & 4a \\ -2-a & 0 & a+2 \\ -1 & -3 & 4 \end{pmatrix}$$

[4 marks]

(iv)

$$\mathbf{C} = \begin{pmatrix} -a & -6 & 4a \\ 2+a & 0 & -a-2 \\ -1 & 3 & 4 \end{pmatrix}$$

[1 mark]

(v)

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -a & 2+a & -1 \\ -6 & 0 & 3 \\ 4a & -a-2 & 4 \end{pmatrix}$$

[1 mark]

(vi)

$$\mathbf{S}^{-1} = \frac{1}{3(a+2)}\begin{pmatrix} -a & 2+a & -1 \\ -6 & 0 & 3 \\ 4a & -a-2 & 4 \end{pmatrix}$$

[1 mark]

(vii)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3(a+2)} \begin{pmatrix} -a & 2+a & -1 \\ -6 & 0 & 3 \\ 4a & -a-2 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 2a \\ a \end{pmatrix}$$

$$= \frac{1}{3(a+2)} \begin{pmatrix} -4a + 4a + 2a^2 - a \\ -24 + 0 + 3a \\ 16a - 2a^2 - 4a + 4a \end{pmatrix}$$

$$= \frac{1}{3(a+2)} \begin{pmatrix} a(2a-1) \\ 3(a-8) \\ 2a(8-a) \end{pmatrix}$$

[2 marks]

(viii) With $a = 3$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3(3+2)} \begin{pmatrix} 3(2 \times 3 - 1) \\ 3(3-8) \\ 2 \times 3(8-3) \end{pmatrix}$$

$$= \frac{1}{15} \begin{pmatrix} 15 \\ -15 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\therefore \; x = 1, \quad y = -1, \quad z = 2$$

which shows that x, y and z are integers. as required. □

[2 marks]

Answer 3

The given system of equations can be written in the following matrix form,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

Using a calculator to obtain the inverse matrix, this becomes,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 3 & 3 & 0 \\ 5 & 2 & -3 \\ 1 & -5 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\therefore \; x = 1, \quad y = 2, \quad z = 0$$

[5 marks]

Answer 4

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 3 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 7 & -3 & 0 & 1 \\ -8 & 4 & 1 & -2 \\ 2 & -1 & 1 & 0 \\ 2 & -1 & -1 & 1 \end{pmatrix}$$

[4 marks]

Answer 5

With such a simple matrix the inverse is easiest to find by setting up a general matrix as the inverse and using the fact that a matrix multiplied by its inverse is the identity matrix. Then study the multiplications...

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ kd & ke & kf \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a = 1, \quad b = 0, \quad c = 0$$

$$d = 0, \quad e = \frac{1}{k}, \quad f = 0$$

$$g = 0, \quad h = 0, \quad i = 1$$

Thus, $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{k} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{k}\begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}$$

[3 marks]