6.2 Answers

Answer 1

(i) Expanding along the top row,

$$\begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & 4 \\ -4 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ -4 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ -4 & 2 \end{vmatrix}$$

$$= 2 \times (-8) - 1 \times 17 - 1 \times 2$$

$$= -35$$

[2 marks]

(ii)

$$\begin{vmatrix} 3 & 1 & 2 \\ k & 4 & 5 \\ 0 & 2 & 3 \end{vmatrix} = 2$$

$$3\begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} k & 5 \\ 0 & 3 \end{vmatrix} + 2\begin{vmatrix} k & 4 \\ 0 & 2 \end{vmatrix} = 2$$

$$3 \times 2 - 1 \times 3k + 2 \times 2k = 2$$

$$6 - 3k + 4k = 2$$

$$k = -4$$

[3 marks]

Answer 2

(i) Using the result, if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \dfrac{1}{\det \mathbf{M}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$|\mathbf{A}| = \begin{vmatrix} a & 2 \\ 3 & 2a \end{vmatrix}$$

$$= 2a^2 - 6$$

$$= 2(a^2 - 3)$$

$$\mathbf{A}^{-1} = \frac{1}{2(a^2 - 3)}\begin{pmatrix} 2a & -2 \\ -3 & a \end{pmatrix}$$

[3 marks]

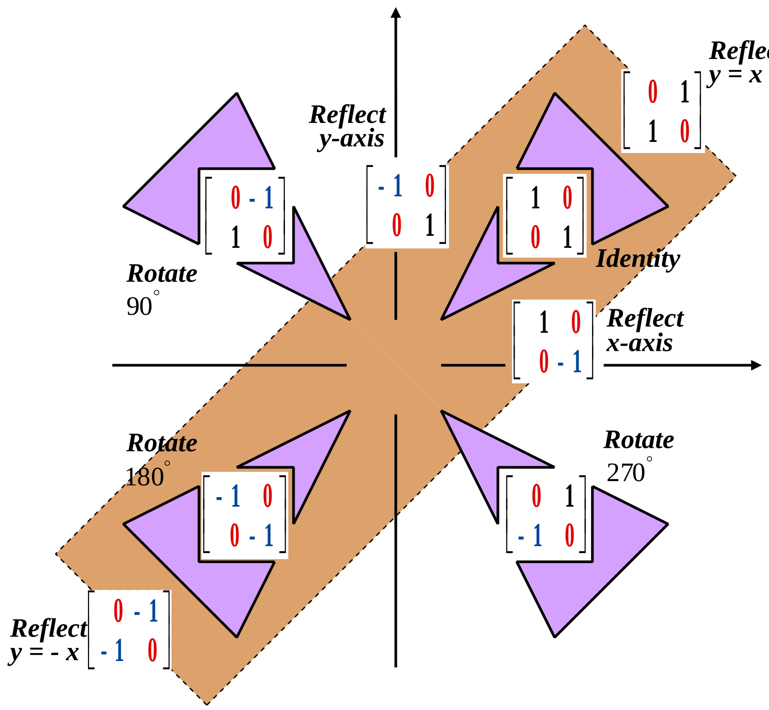
(ii) The inverse matrix does nor exist if the determinant is zero.

That is, when $a = \pm\sqrt{3}$

[1 mark]

Answer 3

The obvious answer is to give a 2×2 reflection matrix or the $180°$ rotation matrix.



Any answer, **A**, should satisfy $\mathbf{A}^2 = \mathbf{I}$ which can be used to check answers given.

[1 mark]

Answer 4

(i) Expanding along the top row,

$$\begin{vmatrix} 2 & 0 & 3 \\ k & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 2 \times 3 + 3 \times (k - 1)$$

$$= 3(k + 1) \qquad \square$$

[3 marks]

(ii) $$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} k & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ k & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ k & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4k - 1 & k - 1 \\ -3 & 5 & 2 \\ -3 & 2 - 3k & 2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 3 & 1 - 4k & k - 1 \\ 3 & 5 & -2 \\ -3 & 3k - 2 & 2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 3 & 3 & -3 \\ 1 - 4k & 5 & 3k - 2 \\ k - 1 & -2 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{3(k + 1)}\begin{pmatrix} 3 & 3 & -3 \\ 1 - 4k & 5 & 3k - 2 \\ k - 1 & -2 & 2 \end{pmatrix}$$

[4 marks]

Answer 5

Let m represent the number of adult male mole-rats.
Let f represent the number of adult female mole-rats.
Let y represent the number of youngster mole-rats.
Thus the system of equations becomes,

$$\begin{aligned} m + f + y &= 1000 \\ 0.02m + 0.03f - 0.04y &= -20 \\ -m + f + 0y &= 100 \end{aligned}$$

The middle equation can be multiplied through by 100 to make the numbers a little easier to handle,

$$\begin{aligned} m + f + y &= 1000 \\ 2m + 3f - 4y &= -2000 \\ -m + f + 0y &= 100 \end{aligned}$$

As a matrix equation this will be,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -4 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} m \\ f \\ y \end{pmatrix} = \begin{pmatrix} 1000 \\ -2000 \\ 100 \end{pmatrix}$$

Using a calculator to get the inverse of the leftmost matrix results in,

$$\begin{aligned} \begin{pmatrix} m \\ f \\ y \end{pmatrix} &= \frac{1}{13}\begin{pmatrix} 4 & 1 & -7 \\ 4 & 1 & 6 \\ 5 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1000 \\ -2000 \\ 100 \end{pmatrix} \\ &= \begin{pmatrix} 100 \\ 200 \\ 700 \end{pmatrix} \end{aligned}$$

The initial number of each type of mole-rat is thus,

adult male : 100

adult female : 200

youngsters : 700

[6 marks]

Answer 6

(a) Finding the determinant by expanding along the top row,

$$\begin{vmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{vmatrix} = 2\begin{vmatrix} k & 4 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & k \\ 3 & 2 \end{vmatrix}$$

$$= 2\,(-k - 8) + 1 \times (-15) + 1 \times (6 - 3k)$$

$$= -2k - 16 - 15 + 6 - 3k$$

$$= -5k - 25$$

$$= (-5)(k + 5)$$

$\therefore$ **M** has an inverse for all real values of k, except $k = -5$

(b) The given equations can be written in the matrix form,

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & -6 & 4 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix}$$

The leftmost matrix is the part (a) matrix with $k = -6$

Using a calculator to get the inverse of the leftmost matrix,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 & 1 & 2 \\ 15 & -5 & -5 \\ 24 & -7 & -9 \end{pmatrix}\begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -2p + 1 \\ 15p - 5 \\ 24p - 7 \end{pmatrix}$$

The point of intersection is thus,

$$\left(\frac{1 - 2p}{5},\ 3p - 1,\ \frac{24p - 7}{5} \right)$$

[5 marks]

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