7.6 Answers

7.6.1 Solution to 7.4 Example

(i) In matrix form the system of equations will be...

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -3 & 2 \\ -1 & -4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 \\ 2 \\ 9 \end{pmatrix}$$

Expanding along the upper row to obtain the determinant...

$$1\,((-3)(7) - (-4)(2)) - 2\,(-1)(7) - (-1)(2)) + 3\,((-1)(-4) - (-1)(-3))$$

$$= (-21 + 8) - 2\,(-7 + 2) + 3\,(4 - 3)$$

$$= -13 + 10 + 3$$

$$= 0 \text{ as anticipated}$$

This tells us that the three planes do not meet is a single point.

[4 marks]

(ii) The normals are $\boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\boldsymbol{m} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$ and $\boldsymbol{l} = \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix}$

None of these are parallel because there is no constants k_{um}, k_{ul} or k_{ml}

for which $\boldsymbol{u} = k_{um}\,\boldsymbol{m}$, $\boldsymbol{u} = k_{ul}\,\boldsymbol{l}$ or $\boldsymbol{m} = k_{ml}\,\boldsymbol{l}$

As none of the normals are in the same direction (parallel), none of the planes are parallel.

[4 marks]

(iii) Adding together the upper and middle equations...

$$\begin{aligned} x + 2y + 3z &= 13 \\ -x - 3y + 2z &= 2 \\ \hline -y + 5z &= 15 \end{aligned}$$

Adding together upper and lower equations...

$$\begin{aligned} x + 2y + 3z &= 13 \\ -x - 4y + 7z &= 9 \\ \hline -2y + 10z &= 22 \\ -y + 5z &= 11 \end{aligned}$$

The inconsistency shows that the three planes form a prism.

[4 marks]

7.6.2 Solutions to 7.5 Exercise

Answer 1

$$\begin{pmatrix} 1 & 4 & -5 \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{58} \begin{pmatrix} 7 & -3 & 19 \\ -1 & 17 & -11 \\ -11 & 13 & -5 \end{pmatrix} \begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

The point of intersection is $(2, -1, -2)$

[4 marks]

Answer 2

As Kevin has worked out that the determinant is zero we know the three planes do not intersect in a unique point (Thanks, Kevin).

So, we now look at the normals which are,

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad \boldsymbol{m} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} \qquad \boldsymbol{l} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

As $\boldsymbol{u} = \frac{1}{3}\boldsymbol{m}$ there are at least two parallel planes.

As there are no constants k_{ul} or k_{ml} for which $\boldsymbol{u} = k_{ul}\,\boldsymbol{l}$ or $\boldsymbol{m} = k_{ml}\,\boldsymbol{l}$

the remaining plane is not also parallel to the other two.

So the configuration is “exactly two parallel planes”.

[4 marks]

Answer 3

(a) In matrix form the system of equations will be...

$$\begin{pmatrix} 1 & \lambda & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$$

Expanding along the upper row to obtain the determinant...

$$\begin{aligned} &1\,((1)(2) - (-2)(5)) - \lambda((2)(2) - (1)(5)) + 3((2)(-2) - (1)(1)) \\ &= (2 + 10) - \lambda(4 - 5) + 3(-4 - 1) \\ &= 12 + \lambda - 15 \\ &= \lambda - 3 \end{aligned}$$

The determinant will be zero if $\lambda = 3$.

So the planes will meet in a unique point except when $\lambda = 3$.

[3 marks]

(b) With $\lambda = -2$ we have,

$$\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -12 & 2 & 13 \\ -1 & 1 & -1 \\ 5 & 0 & -5 \end{pmatrix}\begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

Thus, the point of intersection is $(1,\ 2, -3)$

[3 marks]

Answer 4

$$\begin{pmatrix} 1 & 4 & -5 \\ 2 & 3 & 1 \\ 3 & 7 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 5 \end{pmatrix}$$

Determinant :

$$1((3)(-4) - (7)(1)) - 4((2)(-4) - (3)(1)) - 5((2)(7) - (3)(3))$$

$$= (-12 - 7) - 4(-8 - 3) - 5(14 - 9)$$

$$= -19 + 44 - 25$$

$$= 0 \qquad \text{So planes do not meet in a single point}$$

The normals are $\boldsymbol{u} = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}$, $\boldsymbol{m} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{l} = \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$

None of these are parallel because there is no constants k_{um}, k_{ul} or k_{ml} for which $\boldsymbol{u} = k_{um}\,\boldsymbol{m}$, $\boldsymbol{u} = k_{ul}\,\boldsymbol{l}$ or $\boldsymbol{m} = k_{ml}\,\boldsymbol{l}$

As none of the normals are in the same direction (parallel), none of the planes are parallel.

Let the equations be upper, *U*, middle, *M*, and lower, *L*, as follows,

$$\begin{aligned} x + 4y - 5z &= 8 \qquad U \\ 2x + 3y + z &= -1 \qquad M \\ 3x + 7y - 4z &= 5 \qquad L \end{aligned}$$

Working out $U + 5M$

$$\begin{aligned} x + 4y - 5z &= 8 \\ 10x + 15y + 5z &= -5 \\ \hline 11x + 19y \qquad &= 3 \end{aligned}$$

Working out $L + 4M$

$$\begin{aligned} 3x + 7y - 4z &= 5 \\ 8x + 12y + 4z &= -4 \\ \hline 11x + 19y \qquad &= 1 \end{aligned}$$

As these two equations for $11x + 19y$ are inconsistent, the three planes form a prism.

[4 marks]

Answer 5

In matrix form the planes can be expressed as the system of equations given by,

$$\begin{pmatrix} -4 & p & 7 \\ 1 & -2 & 5 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ q \\ 2 \end{pmatrix}$$

To form a sheaf, the determinant must be zero.

$$\begin{aligned} -4((-2)(1) - (3)(5)) - p((1)(1) - (2)(5)) + 7((1)(3) - (2)(-2)) &= 0 \\ -4(-17) - p(-9) + 7 \times (7) &= 0 \\ 68 + 9p + 49 &= 0 \\ 9p &= -117 \\ p &= -13 \end{aligned}$$

Substituting in the value for p gives the equations as,

$$\begin{aligned} -4x - 13y + 7z &= 4 \qquad U \\ x - 2y + 5z &= q \qquad M \\ 2x + 3y + z &= 2 \qquad L \end{aligned}$$

where the equations have be labelled upper, U, middle, M, and lower, L

Working out $7L - U$

$$\begin{aligned} 14x + 21y + 7z &= 14 \\ -4x - 13y + 7z &= 4 \\ \hline 18x + 34y &= 10 \\ 9x + 17y &= 5 \end{aligned}$$

Working out $5L - M$

$$\begin{aligned} 10x + 15y + 5z &= 10 \\ x - 2y + 5z &= q \\ \hline 9x + 17y &= 10 - q \end{aligned}$$

For a sheaf these two equations for $9x + 17y$ must be consistent.

Thus, $q = 5$

[6 marks]

Answer 6

(a) With $k = -1$ the system of equations is,

$$\begin{pmatrix} -1 & 1 & -2 \\ 2 & 3 & -6 \\ 3 & -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}$$

Determinant:

$$(-1)((3)(5) - (-2)(-6)) - 1((2)(5) - (3)(-6)) - 2((2)(-2) - (3)(3))$$

$$= (-1)(15 - 12) - 1(10 + 18) - 2(-4 - 9)$$

$$= -3 - 28 + 26$$

$= -5$ so the three planes will intersect in a unique point.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 1 & 0 \\ 28 & -1 & 10 \\ 13 & -1 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

Thus, there is a point of intersection at $(-1,\ 3,\ 2)$

[3 marks]

(b) With $k = \frac{2}{3}$ the system of equations is,

$$\begin{pmatrix} \frac{2}{3} & 1 & -2 \\ 2 & 3 & -6 \\ 3 & -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}$$

Determinant:

$$\left(\frac{2}{3}\right)((3)(5) - (-2)(-6)) - 1((2)(5) - (3)(-6)) - 2((2)(-2) - (3)(3))$$

$$= \left(\frac{2}{3}\right)(15 - 12) - 1(10 + 18) - 2(-4 - 9)$$

$$= 2 - 28 + 26$$

$= 0$ So planes do not meet in a single point

So, we now look at the normals which are,

$$\boldsymbol{u} = \begin{pmatrix} \frac{2}{3} \\ 1 \\ -2 \end{pmatrix} \qquad \boldsymbol{m} = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} \qquad \boldsymbol{l} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

As $\boldsymbol{u} = \frac{1}{3}\boldsymbol{m}$ there are at least two parallel planes.

As there are no constants k_{ul} or k_{ml} for which $\boldsymbol{u} = k_{ul}\,\boldsymbol{l}$ or $\boldsymbol{m} = k_{ml}\,\boldsymbol{l}$

the remaining plane is not also parallel to the other two.

So the configuration is “exactly two parallel planes” with the third separately intersecting each in a line.

[4 marks]

Answer 7

In matrix form the planes can be expressed as the system of equations given by,

$$\begin{pmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & b & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ c \end{pmatrix}$$

(a) **(i)** To **not** intersect at a unique point, the determinant must be zero.

$$(-1)((3)(1) - b(1)) - a((2)(1) - (1)(1)) = 0$$

$$-3 + b - a = 0$$

$$b = a + 3$$

[4 marks]

(ii) With $b = a + 3$ (because a sheaf has determinant zero) the equations are,

$$\begin{aligned} -x + \quad\quad ay \quad\quad &= 2 \quad U \\ 2x + \quad\quad 3y + z &= -3 \quad M \\ x + (a+3)y + z &= c \quad L \end{aligned}$$

are labelled upper, U, middle, M, and lower, L

Working out $L - M$

$$\begin{aligned} x + (a+3)y + z &= c \\ 2x + \quad\quad 3y + z &= -3 \\ \hline -x + \quad\quad ay \quad\quad &= c + 3 \end{aligned}$$

For a sheaf this result must be consistent with the upper equation.

Thus, $c = -1$

[3 marks]

(b) For $b = a$ and $c = 1$ the system of equations becomes,

$$\begin{pmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & a & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Let $\mathbf{A} = \begin{pmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & a & 1 \end{pmatrix}$

$$\begin{aligned} |\mathbf{A}| &= \begin{vmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & a & 1 \end{vmatrix} \\ &= (-1) \times \begin{vmatrix} 3 & 1 \\ a & 1 \end{vmatrix} - a \times \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ &= (-1)(3 - a) - a \\ &= -3 \end{aligned}$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 3 & 1 \\ a & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & a \end{vmatrix} \\ \begin{vmatrix} a & 0 \\ a & 1 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -1 & a \\ 1 & a \end{vmatrix} \\ \begin{vmatrix} a & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & a \\ 2 & 3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3-a & 1 & 2a-3 \\ a & -1 & -2a \\ a & -1 & -3-2a \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 3-a & -1 & 2a-3 \\ -a & -1 & 2a \\ a & 1 & -3-2a \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 3-a & -a & a \\ -1 & -1 & 1 \\ 2a-3 & 2a & -3-2a \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{(-3)} \begin{pmatrix} 3-a & -a & a \\ -1 & -1 & 1 \\ 2a-3 & 2a & -3-2a \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{(-3)} \begin{pmatrix} 3-a & -a & a \\ -1 & -1 & 1 \\ 2a-3 & 2a & -3-2a \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$= \frac{1}{(-3)} \begin{pmatrix} 6-2a+3a+a \\ -2+3+1 \\ 4a-6-6a-3-2a \end{pmatrix}$$

$$= \frac{1}{(-3)} \begin{pmatrix} 2a+6 \\ 2 \\ -4a-9 \end{pmatrix}$$

The point of intersection is $\left(\dfrac{2a+6}{(-3)}, \dfrac{2}{(-3)}, \dfrac{-4a-9}{(-3)} \right)$

$$= \left(-\frac{2a+6}{3}, -\frac{2}{3}, \frac{4a-9}{3} \right)$$

[6 marks]

Answer 8

(a) $$\begin{vmatrix} 1 & 5 & 3 \\ 4 & -2 & p \\ 8 & 5 & -11 \end{vmatrix} = 1 \times \begin{vmatrix} -2 & p \\ 5 & -11 \end{vmatrix} - 5 \times \begin{vmatrix} 4 & p \\ 8 & -11 \end{vmatrix} + 3 \times \begin{vmatrix} 4 & -2 \\ 8 & 5 \end{vmatrix}$$

$$= 22 - 5p + 5 \times 44 + 40p + 60 + 48$$

$$= 350 + 35p$$

$$= 35(10 + p) \quad \text{restriction on } p \text{ is that } p \neq -10$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -2 & p \\ 5 & -11 \end{vmatrix} & \begin{vmatrix} 4 & p \\ 8 & -11 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ 8 & 5 \end{vmatrix} \\ \begin{vmatrix} 5 & 3 \\ 5 & -11 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 8 & -11 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 8 & 5 \end{vmatrix} \\ \begin{vmatrix} 5 & 3 \\ -2 & p \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 4 & p \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 4 & -2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 22 - 5p & -44 - 8p & 36 \\ -70 & -35 & -35 \\ 5p + 6 & p - 12 & -22 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 22 - 5p & 44 + 8p & 36 \\ 70 & -35 & 35 \\ 5p + 6 & -p + 12 & -22 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 22 - 5p & 70 & 5p + 6 \\ 44 + 8p & -35 & -p + 12 \\ 36 & 35 & -22 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{35(10 + p)} \begin{pmatrix} 22 - 5p & 70 & 5p + 6 \\ 44 + 8p & -35 & -p + 12 \\ 36 & 35 & -22 \end{pmatrix}$$

[6 marks]

(b) **(i)**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{35(10+p)} \begin{pmatrix} 22-5p & 70 & 5p+6 \\ 44+8p & -35 & -p+12 \\ 36 & 35 & -22 \end{pmatrix} \begin{pmatrix} 5 \\ 24 \\ -30 \end{pmatrix}$$

$$= \frac{1}{35(10+p)} \begin{pmatrix} 110-25p+1680-150p-180 \\ 220+40p-840+30p-360 \\ 180+840+660 \end{pmatrix}$$

$$= \frac{1}{35(10+p)} \begin{pmatrix} 1610-175p \\ -980+70p \\ 1680 \end{pmatrix}$$

$$= \frac{1}{10+p} \begin{pmatrix} 46-5p \\ -28+2p \\ 48 \end{pmatrix}$$

The point of intersection is $\left(\dfrac{46-5p}{10+p}, \dfrac{-28+2p}{10+p}, \dfrac{48}{10+p} \right)$

[4 marks]

(ii) With $p = 2$ the equations become,

$$\begin{aligned} x + 5y + 3z &= 5 \\ 4x - 2y + 2z &= 24 \\ 8x + 5y - 11z &= -30 \end{aligned}$$

Showing 1st and 2nd are mutually perpendicular;

$$\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} = 1 \times 4 + 5 \times -2 + 3 \times 2 = 0$$

Showing 1st and 3rd are mutually perpendicular;

$$\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 8 \\ 5 \\ -11 \end{pmatrix} = 1 \times 8 + 5 \times 5 + 3 \times (-11) = 0$$

Showing 2nd and 3rd are mutually perpendicular;

$$\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 8 \\ 5 \\ -11 \end{pmatrix} = 4 \times 8 + (-2) \times 5 + 2 \times (-11) = 0$$

Thus, via the scalar products of all possible pairs of the planes' normal vectors it is shown that the planes are mutually perpendicular.

[4 marks]