9.4.1 Solution to 9.2 Example

$\mathbf{C}^2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ is in the form of a rotation matrix, $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

with $\cos\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \theta = \frac{\pi}{6}, \frac{11\pi}{6}$

and $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore$ $\mathbf{C}^2$ represents a rotation of $\frac{\pi}{6}$ and so $\mathbf{C}$ must represent a rotation of $\frac{\pi}{12}$

$$\mathbf{C}^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{pmatrix}$$

$$a^2 - b^2 = \frac{\sqrt{3}}{2} \quad \text{and} \quad 2ab = \frac{1}{2}$$

$$b = \frac{1}{4a}$$

$$a^2 - \left(\frac{1}{4a}\right)^2 = \frac{\sqrt{3}}{2}$$

$$a^2 - \frac{1}{16a^2} = \frac{\sqrt{3}}{2}$$

$$16a^4 - 1 = 8\sqrt{3}\,a^2$$

$$16a^4 - 8\sqrt{3}\,a^2 - 1 = 0$$

$$a^2 = \frac{8\sqrt{3} \pm \sqrt{64 \times 3 - 4 \times 16 \times (-1)}}{32}$$

$$= \frac{8\sqrt{3} \pm 16}{32}$$

$$= \frac{\sqrt{3} + 2}{4} \quad \text{as } a > 0$$

$$a = \frac{\sqrt{\sqrt{3} + 2}}{2}$$

$$\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{\sqrt{3} + 2}}{2} \quad \text{which is in the requested form}$$

[7 marks]

9.4.2 Solutions to 9.3 Exercise

Answer 1

Compare, $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & \frac{\sqrt{3}}{2} \\ 0 & b & -\frac{1}{2} \end{pmatrix}$ with $\mathbf{R}_{x,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

$$a = -\frac{1}{2} \text{ and } b = -\frac{\sqrt{3}}{2}$$

[2 marks]

Answer 2

Anna should be making a comparison between the matrix in the question,

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$ and $\mathbf{R}_{x,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

Anna should have deduced that $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2}$

The first gives $\theta = 30°, 150°$ and the second gives $\theta = 150°, 210°$

So the value of θ that satisfies both equations is $\theta = 150°$

[2 marks]

Answer 3

The matrix given is recognized as being a rotation about the *z*-axis.

So, compare $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\mathbf{R}_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

from which is deduced that, $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = -\frac{1}{2}$

The first gives $\theta = 60°, 120°$ and the second gives $\theta = 120°, 240°$

So the value of θ that satisfies both equations is $\theta = 120°$

The matrix is a rotation about the *z*-axis of $120°$

[3 marks]

Answer 4

(a) **R** must be of the form $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The inverse of **R** must be a rotation about the z-axis through an angle $(-\theta)$

$$\therefore \mathbf{R}^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(As $\cos\theta$ is an even function, and $\sin\theta$ is odd)

$$\mathbf{AR} = \mathbf{B}$$

$$\mathbf{AR}\,\mathbf{R}^{-1} = \mathbf{B}\,\mathbf{R}^{-1}$$

$$\mathbf{A} = \mathbf{B}\,\mathbf{R}^{-1}$$

$$= \begin{pmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\cos^2\theta - \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ which shows **A** is independent of θ

[3 marks]

(b) **A** is a reflection in the plane $x = 0$

[1 mark]

Answer 5

(a) The matrix given is recognized as being a rotation about the y-axis.

So, compare $\begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$ with $\mathbf{R}_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

from which is deduced that, $\sin\theta = \frac{1}{2}$ and $\cos\theta = \frac{\sqrt{3}}{2}$

The first gives $\theta = 30°, 150°$ and the second gives $\theta = 30°, 330°$

So the value of θ that satisfies both equations is $\theta = 30°$

The matrix is a rotation about the y-axis of 30°

[3 marks]

(b) $\begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} + \frac{1}{2} \\ -2 \\ \frac{1}{2} + \frac{\sqrt{3}}{2} \end{pmatrix}$

The image of the point $(-1, -2, 1)$ is $\left(\frac{1-\sqrt{3}}{2}, -2, \frac{1+\sqrt{3}}{2} \right)$

[2 marks]

Answer 6

(a) $\mathbf{P} = \mathbf{R}_{z,120} = \begin{pmatrix} \cos 120 & -\sin 120 & 0 \\ \sin 120 & \cos 120 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[1 mark]

(b) $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} + \frac{\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} + \frac{1}{2} \\ 0 \end{pmatrix}$

The image of the point $(3, -1, 0)$ is $\left(\frac{-3+\sqrt{3}}{2}, \frac{3\sqrt{3}+1}{2}, 0 \right)$

[1 mark]

(c) $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} k \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} -\frac{k}{2} \\ \frac{k\sqrt{3}}{2} \\ k \end{pmatrix}$

The image of the point $(k, 0, k)$ is $\left(-\frac{k}{2}, \frac{\sqrt{3}k}{2}, k \right)$

[1 mark]

Answer 7

$\mathbf{C}^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is in the form of a rotation matrix, $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

with $\cos\theta = \frac{1}{\sqrt{2}} \quad \Rightarrow \; \theta = \frac{\pi}{4}, \frac{7\pi}{4}$

and $\sin\theta = \frac{1}{2} \; \Rightarrow \; \theta = \frac{\pi}{4}, \; \frac{3\pi}{4}$

$\therefore$ $\mathbf{C}^2$ represents a rotation of $\frac{\pi}{4}$ and so $\mathbf{C}$ must represent a rotation of $\frac{\pi}{8}$

$$\mathbf{C}^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{pmatrix}$$

$$a^2 - b^2 = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad 2ab = \frac{1}{\sqrt{2}}$$

$$b = \frac{1}{2\sqrt{2}\,a}$$

$$a^2 - \left(\frac{1}{2\sqrt{2}\,a}\right)^2 = \frac{1}{\sqrt{2}}$$

$$a^2 - \frac{1}{8\,a^2} = \frac{\sqrt{2}}{2}$$

$$8\,a^4 - 1 = 4\sqrt{2}\,a^2$$

$$8\,a^4 - 4\sqrt{2}\,a^2 - 1 = 0$$

$$a^2 = \frac{4\sqrt{2} \pm \sqrt{16 \times 2 - 4 \times 8 \times (-1)}}{32}$$

$$= \frac{4\sqrt{2} \pm 8}{32}$$

$$= \frac{\sqrt{2} + 2}{8} \qquad \text{as } a > 0$$

$$a = \frac{\sqrt{\sqrt{2} + 2}}{2}$$

$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{\sqrt{2} + 2}}{2} \text{ which is in the requested form}$$

[7 marks]

Answer 8

(a) The matrix given is recognized as being a rotation about the y-axis.

So, compare $\begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}$ with $\mathbf{R}_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

from which is deduced that, $\sin\theta = -\dfrac{1}{2}$ and $\cos\theta = -\dfrac{\sqrt{3}}{2}$

The first gives $\theta = 210°, 330°$ and the second gives $\theta = 150°, 210°$

So the value of θ that satisfies both equations is $\theta = 210°$

The matrix is a rotation about the y-axis of 210°

[3 marks]

(b) $$\begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} k \\ -k \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}\,k}{2} \\ -k \\ \frac{k}{2} \end{pmatrix}$$

The image of the point $(k, -k,\ 0)$ is $\left(\dfrac{-\sqrt{3}\,k}{2}, -k, \dfrac{k}{2}\right)$

[3 marks]

Answer 9

(a) $$|\mathbf{M}| = \begin{vmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{vmatrix} = 2\sqrt{3} \times 2\sqrt{3} - 2 \times (-2) = 16$$

The determinant is the area scale factor so the length scale factor is 4

[2 marks]

(b) Thus we have that,

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{R}_\theta = \begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix}$$

The opposite of an enlargement by 4 is an enlargement by $\dfrac{1}{4}$

$$\mathbf{R}_\theta = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \text{ compared with } \mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

from which is deduced that, $\sin\theta = \dfrac{1}{2}$ and $\cos\theta = \dfrac{\sqrt{3}}{2}$

So the value of θ that satisfies both equations is $\theta = 30°$

[3 marks]

(c) Let the coordinates of P be (x, y) in which case...

$$\begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 330 & -\sin 330 \\ \sin 330 & \cos 330 \end{pmatrix}\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{\sqrt{3}}{8} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} \sqrt{3}\,a + b \\ -a + \sqrt{3}\,b \end{pmatrix}$$

So P has the coordinates $\left(\dfrac{\sqrt{3}\,a + b}{8}, \ \dfrac{-a + \sqrt{3}\,b}{8} \right)$

[4 marks]

Answer 10

(a) $\mathbf{R}_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

$$\text{So } \mathbf{R}_{y,315} = \begin{pmatrix} \cos 315 & 0 & \sin 315 \\ 0 & 1 & 0 \\ -\sin 315 & 0 & \cos 315 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

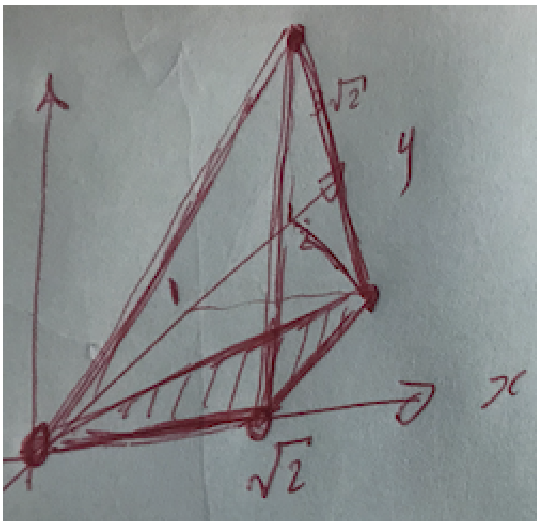
[2 marks]

(b) $\begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & -1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3\sqrt{2} & 0 \end{pmatrix}$

That is,
$$(1,\ 0,\ -1) \to (\sqrt{2},\ 0,\ 0)$$
$$(1,\ 1,\ -1) \to (\sqrt{2},\ 1,\ 0)$$
$$(3,\ 2,\ 3) \to (0,\ 2,\ 3\sqrt{2})$$
$$(0,\ 0,\ 0) \to (0,\ 0,\ 0)$$

[3 marks]

(c)



$$V_{TET} = \frac{1}{3} \times \textit{base area} \times \textit{height}$$
$$= \frac{1}{3} \times \frac{1}{2} \times \sqrt{2} \times 1 \times 3\sqrt{2}$$
$$= 1$$

[2 marks]