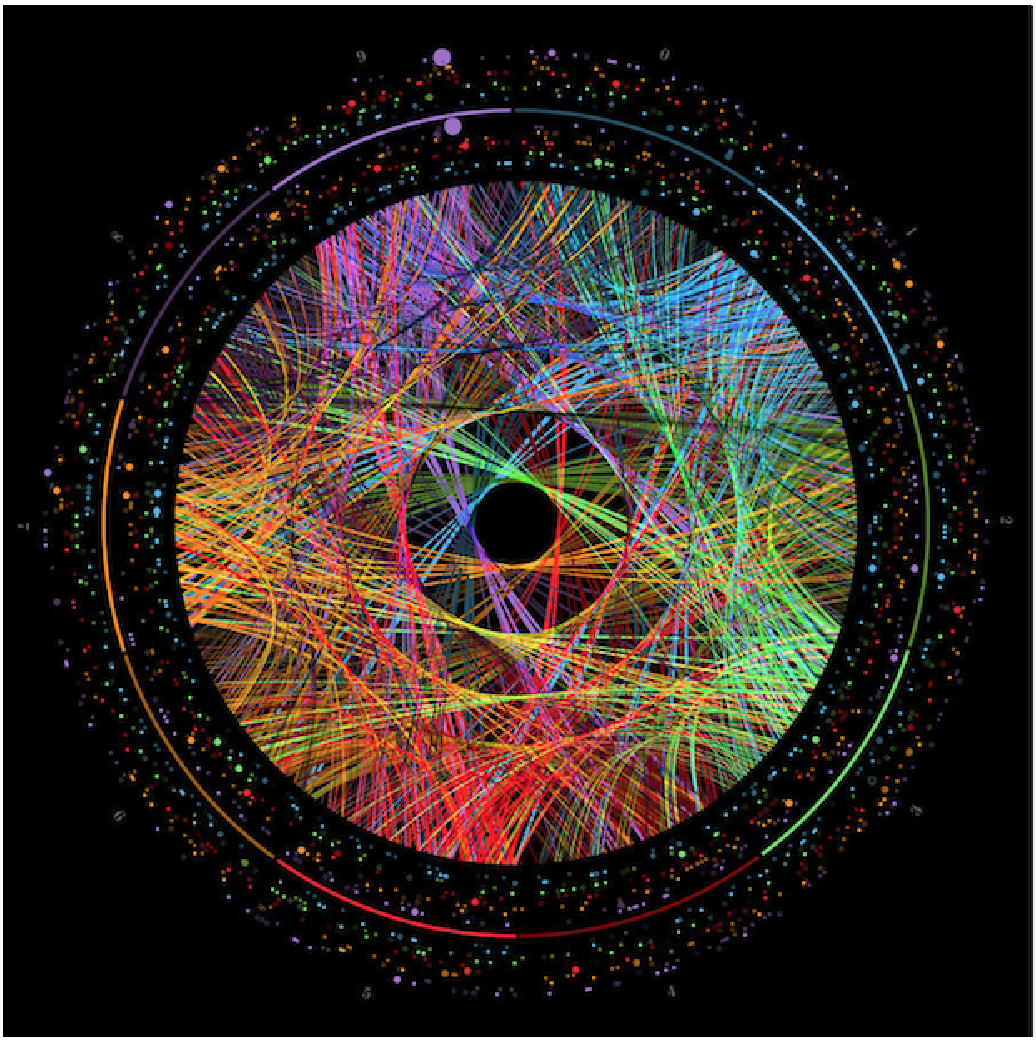
Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

MATRIX SYSTEMS OF EQUATIONS



“Pi Transition Paths” by Martin Krzywinski

M A T R I X
S Y S T E M S   O F   E Q U A T I O N S

Lesson 1

Further A-Level Pure Mathematics : Core 1

Matrix Systems of Equations

1.1 Matrix as Organiser

A matrix can be viewed as an organiser; a tool that allows information to be processed in a systematic, compact and efficient manner. When solving, for example, a system of five simultaneous equations with five unknowns, using matrices keeps the working both manageable and presentable. Computerising the solution techniques developed is then straight forward as most computer languages have matrix arithmetic built in.

1.2 Two Equations, Two Unknowns

As a simple example consider the following two simultaneous equations with two unknown variables x and y,

$$2x + 5y = 4$$

$$6x + 4y = 1$$

These are rewritten as the following matrix calculation,

$$\begin{pmatrix} 2 & 5 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

The theory for this rewrite in matrix form was covered in Lesson 2 of *Matrix Transformations* where how to transform a point $(x,\ y)$ using a matrix was studied.

Here is a reminder of that key previous result;

Point Transformation by a Matrix (Two Dimensions)

The point $(x,\ y)$ is written $\begin{pmatrix} x \\ y \end{pmatrix}$ and placed right of the transforming matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

The point (x, y) has been transformed to the point $(ax + by,\ cx + dy)$

From GCSE it is known that the algebra of solving two simultaneous equations has a geometric interpretation, namely, of finding the point where two straight lines intersect. So matrices, previously used to manipulate points, are also of use in solving simultaneous equations because solving them is all about finding a point; the point (x, y) of intersection of the two straight lines.

To advance the example, another previous result is needed, that of writing down the inverse matrix of a given matrix,

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The Inverse of a Matrix

In general, the inverse of a non-singular matrix $\mathbf{M}$ is the matrix $\mathbf{M}^{-1}$ such that

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

In particular, if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \dfrac{1}{det\,\mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

This result in turn required knowing how to find the determinant of a matrix,

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Definition of the Determinant

Given the generalised 2×2 square matrix, $\mathbf{M}$, where,

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant $\mathbf{M}$ is given by,

$$|\mathbf{M}| = ad - bc$$

This can also be written $det\,\mathbf{M}$ or $\Delta\mathbf{M}$

- If $|\mathbf{M}| = 0$ then $\mathbf{M}$ is a singular matrix
- If $|\mathbf{M}| \neq 0$ then $\mathbf{M}$ is a non-singular matrix

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At last, armed with this background knowledge, the example can be progressed !
Teaching Video : http://www.NumberWonder.co.uk/v9095/1.mp4

Watch the video and then write out a full solution here:

☞

[5 marks]

1.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

Let **P** be the matrix $\begin{pmatrix} 7 & 5 \\ 3 & 2 \end{pmatrix}$ and let **Q** be the matrix $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$

Calculate the matrix **PQ**

[3 marks]

Question 2

In the teaching video, it was claimed that when the Left Hand Side of

$$\begin{pmatrix} 2 & 5 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

was front multiplied by $\mathbf{M}^{-1}$ the result was just $\begin{pmatrix} x \\ y \end{pmatrix}$

Show in more detail exactly how this comes about.

[4 marks]

Question 3

Let $\mathbf{A}$ be the matrix $\begin{pmatrix} 10 & 4 \\ 7 & 3 \end{pmatrix}$

Determine $\mathbf{A}^{-1}$

[4 marks]

Question 4

Let $\mathbf{M}$ be the matrix $\begin{pmatrix} 0.5 & 0.25 \\ 2 & 1.5 \end{pmatrix}$

(i) Calculate the value of $det(\mathbf{M})$

[2 marks]

(ii) Determine $\mathbf{M}^{-1}$

Simplify your answer

[2 marks]

Question 5

Solve the following simultaneous equations using matrix methods,

$$\begin{aligned} 3x + 4y &= 18 \\ 4x - y &= 5 \end{aligned}$$

[5 marks]

Question 6

Solve the following simultaneous equations using matrix methods,

$$5x - 2y = 16$$

$$7x + 6y = -4$$

[5 marks]

Question 7

The matrix **A** $= \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$ and **AB** $= \begin{pmatrix} 4 & 1 & 9 \\ 1 & 9 & 4 \end{pmatrix}$

Find the matrix **B**

[5 marks]

1.4 Answers

1.4.1 Solution to 1.2 Example (Teaching Video)

$$2x + 5y = 4$$

$$6x + 4y = 1$$

The representation in matrix form is then,

$$\begin{pmatrix} 2 & 5 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Let $\mathbf{M} = \begin{pmatrix} 2 & 5 \\ 6 & 4 \end{pmatrix}$ be the matrix of the coefficients, in which case,

$$\begin{aligned} \mathbf{M}^{-1} &= \frac{1}{2 \times 4 - 5 \times 6}\begin{pmatrix} 4 & -5 \\ -6 & 2 \end{pmatrix} \\ &= \frac{1}{22}\begin{pmatrix} -4 & 5 \\ 6 & -2 \end{pmatrix} \\ \therefore \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{22}\begin{pmatrix} -4 & 5 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= \frac{1}{22}\begin{pmatrix} -4 \times 4 + 5 \times 1 \\ 6 \times 4 + (-2) \times 1 \end{pmatrix} \\ &= \frac{1}{22}\begin{pmatrix} -11 \\ 22 \end{pmatrix} \\ &= \begin{pmatrix} -0.5 \\ 1 \end{pmatrix} \end{aligned}$$

That is, $x = -0.5$, $y = 1$

[5 marks]

1.4.2 Solutions to 1.3 Exercise

Answer 1

$$\mathbf{PQ} = \begin{pmatrix} 7 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \times 6 + 5 \times 1 \\ 3 \times 6 + 2 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 47 \\ 20 \end{pmatrix}$$

[3 marks]

Answer 2

$$\mathbf{M}^{-1}\begin{pmatrix} 2 & 5 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{22}\begin{pmatrix} -4 & 5 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{22}\begin{pmatrix} (-4) \times 2 + 5 \times 6 & (-4) \times 5 + 5 \times 4 \\ 6 \times 2 + (-2) \times 6 & 6 \times 5 + (-2) \times 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{22}\begin{pmatrix} 22 & 0 \\ 0 & 22 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \end{pmatrix}$$

[4 marks]

Answer 3

$$\mathbf{A} = \begin{pmatrix} 10 & 4 \\ 7 & 3 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{10 \times 3 - 4 \times 7}\begin{pmatrix} 3 & -4 \\ -7 & 10 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3 & -4 \\ -7 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 1.5 & -2 \\ -3.5 & 5 \end{pmatrix}$$

[4 marks]

Answer 4

(i)

$$\mathbf{M} = \begin{pmatrix} 0.5 & 0.25 \\ 2 & 1.5 \end{pmatrix}$$

$$\begin{aligned} det\ (\mathbf{M}) &= 0.5 \times 1.5 - 0.25 \times 2 \\ &= 0.75 - 0.5 \\ &= 0.25 \end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned} \mathbf{M}^{-1} &= \frac{1}{0.25} \begin{pmatrix} 1.5 & -0.25 \\ -2 & 0.5 \end{pmatrix} \\ &= 4 \begin{pmatrix} 1.5 & -0.25 \\ -2 & 0.5 \end{pmatrix} \\ &= \begin{pmatrix} 6 & -1 \\ -8 & 2 \end{pmatrix} \end{aligned}$$

[2 marks]

Answer 5

$$\begin{aligned} 3x + 4y &= 18 \\ 4x - \ \ y &= \ \ 5 \end{aligned}$$

The representation in matrix form is then,

$$\begin{pmatrix} 3 & 4 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 5 \end{pmatrix}$$

Let $\mathbf{M} = \begin{pmatrix} 3 & 4 \\ 4 & -1 \end{pmatrix}$ be the matrix of the coefficients, in which case,

$$\begin{aligned} \mathbf{M}^{-1} &= \frac{1}{3 \times (-1) - 4 \times 4} \begin{pmatrix} 3 & 4 \\ 4 & -1 \end{pmatrix} \\ &= -\frac{1}{19} \begin{pmatrix} -1 & -4 \\ -4 & 3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \therefore \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{19} \begin{pmatrix} 1 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 18 \\ 5 \end{pmatrix} \\ &= \frac{1}{19} \begin{pmatrix} 1 \times 18 + 4 \times 5 \\ 4 \times 18 - 3 \times 5 \end{pmatrix} \\ &= \frac{1}{19} \begin{pmatrix} 38 \\ 57 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{aligned}$$

That is, $x = 2, \ y = 3$

[5 marks]

Answer 6

$$5x - 2y = 16$$

$$7x + 6y = -4$$

The representation in matrix form is then,

$$\begin{pmatrix} 5 & -2 \\ 7 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 \\ -4 \end{pmatrix}$$

Let $\mathbf{M} = \begin{pmatrix} 5 & -2 \\ 7 & 6 \end{pmatrix}$ the matrix of the coefficients, in which case,

$$\mathbf{M}^{-1} = \frac{1}{5 \times 6 - (-2) \times 7}\begin{pmatrix} 6 & 2 \\ -7 & 5 \end{pmatrix}$$

$$= \frac{1}{44}\begin{pmatrix} 6 & 2 \\ -7 & 5 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{44}\begin{pmatrix} 6 & 2 \\ -7 & 5 \end{pmatrix}\begin{pmatrix} 16 \\ -4 \end{pmatrix}$$

$$= \frac{1}{44}\begin{pmatrix} 6 \times 16 + 2 \times (-4) \\ (-7) \times 16 + 5 \times (-4) \end{pmatrix}$$

$$= \frac{1}{44}\begin{pmatrix} 88 \\ -132 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

That is, $x = 2, \ y = -3$

[5 marks]

Answer 7

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{1 \times 1 - (-3) \times 2}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$$

$$\mathbf{B} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 & 9 \\ 1 & 9 & 4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 & 28 & 21 \\ -7 & 7 & -14 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 & 3 \\ -1 & 1 & -2 \end{pmatrix}$$

[5 marks]

2.1 The 3×3 Matrix

To solve n equations with n unknowns, the working is with an $n \times n$ matrix. In this course, the focus is upon 3 equations with 3 unknowns, and on working with a 3×3 general matrix.

The general 3×3 matrix can be expressed in two ways.

Firstly, and the method preferred by the A-Level course,

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Secondly, and more easily extendable to larger matrices,

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{pmatrix} \end{array}$$

However it is expressed, for the 3×3 matrix, two questions immediately arise.

- How is its determinant worked out ?
- How is its inverse, assuming one exists, found ?

The first will be addressed in this lesson, the second in the next.

2.2 The Determinant of a 3×3 Matrix

Before looking at the method of finding the determinant of a 3×3 matrix it is useful to be able to find what is termed the minor of such a matrix.

The Minor of a 3×3 Matrix

The minor of an element in a 3×3 matrix is the determinant of the 2×2 matrix that remains after the row and column containing that element are crossed out.

Example #1

Find the minor of the element 6 in the matrix $\mathbf{A} = \begin{pmatrix} 5 & -3 & 6 \\ 2 & 7 & -4 \\ 8 & 4 & 9 \end{pmatrix}$

Teaching Video : http://www.NumberWonder.co.uk/v9095/2a.mp4

Watch the video and then write out a full solution here:

☞

[3 marks]

Spot Check

Show that, for $\mathbf{A} = \begin{pmatrix} 5 & -3 & 6 \\ 2 & 7 & -4 \\ 8 & 4 & 9 \end{pmatrix}$, $\quad M_{21} = -51$

[3 marks]

The determinant of a 3×3 matrix is found by "expanding along a row or column". As beginners, always expand along the top row. Using other rows or columns will be explored in the exercise. All, of course, arrive at the same answer !

The Determinant of a 3×3 Matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Notice that this could be written using the minors of the matrix as,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\,M_{11} - b\,M_{12} + c\,M_{13}$$

which is easier to remember.

Example #2

Find the value of $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix}$

Teaching Video : http://www.NumberWonder.co.uk/v9095/2b.mp4

Watch the video and
then write out a full
solution here:

☞

[4 marks]

2.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 25

Question 1

The matrix **H** is $\begin{pmatrix} -3 & 1 & 9 \\ 2 & 5 & 3 \\ -1 & 6 & 8 \end{pmatrix}$

Without using a calculator, show that **H** has a determinant of 68

[4 marks]

Question 2

Further A-Level Examination Question from June 2017, FP3, Q6 (a) (Edexcel)

The matrix **M** is given by,

$$\mathbf{M} = \begin{pmatrix} 1 & k & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}, \quad k \in \mathbb{R}, \quad k \neq \frac{1}{2}$$

Show that $det\, \mathrm{M} = 1 - 2k$

[2 marks]

Question 3

If a matrix is described as being singular, what does this tell you about its determinant ?

[1 mark]

Question 4

Further A-Level Examination Question from June 2016, FP3, Q1 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & -3 \\ k & 1 & 3 \\ 2 & -1 & k \end{pmatrix}, \text{ where } k \text{ is a constant}$$

Given that the matrix **A** is singular, find the possible values of k

[4 marks]

Question 5

The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{pmatrix}$, where k is a constant

Given that the determinant of **A** is 8, find the possible values of k

[5 marks]

Question 6

Every element a_{ij} in a matrix has associated with it a cofactor.

The cofactor c_{ij} is minor M_{ij} which may or may not have its sign changed.

If $i + j$ is even the cofactor is the unchanged minor, otherwise its sign is changed. In practice, this means that the cofactors in a 3×3 matrix are the minors adjusted in accordance with the following pattern,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

When expanding the matrix by the top row, this is where the signs in front of *a*, *b* and *c* came from; they were the top row of of this pattern matrix, +, –, +

That is,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +a\,M_{11} - b\,M_{12} + c\,M_{13}$$

Let G be the matrix, $G = \begin{pmatrix} -7 & 3 & 5 \\ 0 & 2 & 0 \\ 5 & 6 & 4 \end{pmatrix}$

Show that the determinant obtained by expansion along the top row is the same as that obtained by expansion along the middle row.

[5 marks]

Question 7

Show that, for all real values of x, the matrix $\begin{pmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{pmatrix}$ is non singular.

[4 marks]

2.4 Answers

2.4.1 Solution to Example #1 (Teaching Video)

The minor of the element 6 is given by,

$$
\begin{aligned}
M_{13} &= \begin{vmatrix} 2 & 7 \\ 8 & 4 \end{vmatrix} \\
&= 2 \times 4 - 8 \times 7 \\
&= 8 - 8 \times 7 \\
&= 8(1 - 7) \\
&= 8 \times (-6) \\
&= -48
\end{aligned}
$$

[3 marks]

Solution to Spot Check

$$
\mathbf{A} = \begin{pmatrix} 5 & -3 & 6 \\ 2 & 7 & -4 \\ 8 & 4 & 9 \end{pmatrix}
$$

$$
\begin{aligned}
M_{21} &= \begin{vmatrix} -3 & 6 \\ 4 & 9 \end{vmatrix} \\
&= (-3) \times 9 - 4 \times 6 \\
&= -27 - 24 \\
&= -51 \qquad \square
\end{aligned}
$$

[3 marks]

Solution to Example #2 (Teaching Video)

$$
\begin{aligned}
\begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix} &= 1 \times \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} - 2 \times \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} + 4 \begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix} \\
&= 1(2 \times 3 - 4 \times 1) \\
&\quad -2(3 \times 3 - (-1) \times 1) \\
&\quad +4(3 \times 4 - (-1) \times 2) \\
&= 2 - 20 + 56 \\
&= 38
\end{aligned}
$$

[4 marks]

2.4.2 Solutions to 2.3 Exercise

Answer 1

$$\begin{vmatrix} -3 & 1 & 9 \\ 2 & 5 & 3 \\ -1 & 6 & 8 \end{vmatrix} = (-3)\begin{vmatrix} 5 & 3 \\ 6 & 8 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ -1 & 8 \end{vmatrix} + 9\begin{vmatrix} 2 & 5 \\ -1 & 6 \end{vmatrix}$$

$$\begin{aligned} &= -3(5 \times 8 - 6 \times 3) \\ &\quad -1(2 \times 8 - (-1) \times 3) \\ &\quad +9(2 \times 6 - (-1) \times 5) \\ &= -66 - 19 + 153 \\ &= 68 \qquad \square \end{aligned}$$

[4 marks]

Answer 2

$$\mathbf{M} = \begin{pmatrix} 1 & k & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}, \ k \in \mathbb{R}, \ k \neq \frac{1}{2}$$

$$|\mathbf{M}| = 1\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - k\begin{vmatrix} 2 & 1 \\ -4 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & -2 \\ -4 & 1 \end{vmatrix}$$

$$= 1\,(\,(-2) \times (-1) - 1 \times 1\,) - k\,(\,2 \times (-1) - (-4) \times 1\,) + 0$$

$$= 1 - 2k \qquad \square$$

[2 marks]

Answer 3

It tells you that the determinant is zero.

[1 mark]

Answer 4

As the matrix is singular, $det\,\mathbf{A} = 0$

$$\begin{vmatrix} -2 & 1 & -3 \\ k & 1 & 3 \\ 2 & -1 & k \end{vmatrix} = 0$$

$$(-2)\begin{vmatrix} 1 & 3 \\ -1 & k \end{vmatrix} - 1\begin{vmatrix} k & 3 \\ 2 & k \end{vmatrix} - 3\begin{vmatrix} k & 1 \\ 2 & -1 \end{vmatrix} = 0$$

$$(-2)(k + 3) - \left(k^2 - 6\right) - 3(-k - 2) = 0$$

$$-2k - 6 - k^2 + 6 + 3k + 6 = 0$$

$$-k^2 + k + 6 = 0$$

$$k^2 - k - 6 = 0$$

$$(k - 3)(k + 2) = 0$$

Either $k = 3$ or $k = -2$

[4 marks]

Answer 5

$$\det A = 8$$

$$\begin{vmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{vmatrix} = 8$$

$$2\begin{vmatrix} 2 & 4 \\ 1 & k+3 \end{vmatrix} + 1\begin{vmatrix} k & 4 \\ -2 & k+3 \end{vmatrix} + 3\begin{vmatrix} k & 2 \\ -2 & 1 \end{vmatrix} = 8$$

$$\begin{aligned} 2(2(k+3) - 1\times 4) & \\ + 1(k(k+3) - (-2)\times 4) & \\ + 3(k\times 1 - (-2)\times 2) &= 8 \\ 4k + 12 - 8 + k^2 + 3k + 8 + 3k + 12 &= 8 \\ k^2 + 10k + 16 &= 0 \\ (k+8)(k+2) &= 0 \end{aligned}$$

Either $k = -8$ or $k = -2$

[5 marks]

Answer 6

Top row expansion,

$$\begin{vmatrix} -7 & 3 & 5 \\ 0 & 2 & 0 \\ 5 & 6 & 4 \end{vmatrix} = -7\begin{vmatrix} 2 & 0 \\ 6 & 4 \end{vmatrix} - 3\begin{vmatrix} 0 & 0 \\ 5 & 4 \end{vmatrix} + 5\begin{vmatrix} 0 & 2 \\ 5 & 6 \end{vmatrix}$$

$$\begin{aligned} &= -7 \times 8 - 3\times 0 + 5\times(-10) \\ &= -56 - 0 - 50 \\ &= -106 \end{aligned}$$

Middle row expansion,

$$\begin{vmatrix} -7 & 3 & 5 \\ 0 & 2 & 0 \\ 5 & 6 & 4 \end{vmatrix} = -0\begin{vmatrix} 3 & 5 \\ 6 & 4 \end{vmatrix} + 2\begin{vmatrix} -7 & 5 \\ 5 & 4 \end{vmatrix} - 0\begin{vmatrix} -7 & 3 \\ 5 & 6 \end{vmatrix}$$

$$\begin{aligned} &= 0 + 2(\,(-7)\times 4 - 5\times 5\,) - 0 \\ &= 2\,(-28 - 25\,) \\ &= 2\times(-53)) \\ &= -106 \quad \text{As before} \qquad \square \end{aligned}$$

[5 marks]

Answer 7

$$\begin{vmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{vmatrix} = 2\begin{vmatrix} x & -2 \\ 3 & x \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ -1 & x \end{vmatrix} + 4\begin{vmatrix} 3 & x \\ -1 & 3 \end{vmatrix}$$

$$= 2\left(x^2 + 6\right) + 2\left(3x - 2\right) + 4\left(9 + x\right)$$

$$= 2x^2 + 12 + 6x - 4 + 36 + 4x$$

$$= 2x^2 + 10x + 44$$

$$= 2\left[x^2 + 5x\right] + 44$$

$$= 2\left[\left(x + \frac{5}{2}\right)^2 - \frac{25}{4}\right] + 44$$

$$= 2\left(x + \frac{5}{2}\right)^2 - \frac{25}{2} + \frac{88}{2}$$

$$= 2\left(x + \frac{5}{2}\right)^2 + \frac{63}{2}$$

This expression is always positive.

$\therefore$ There is no real value for x that will cause the matrix to be singular.

That is, the matrix is always non singular. ☐

[4 marks]

3.1 Inverting a 3×3 Matrix

Finding the inverse of a 3×3 matrix makes use of a “Cookbook Recipe”.
Before listing the five steps in the recipe, there is one matrix manipulation that has not been previously mentioned:

The Transpose of a 3×3 Matrix

Given, for example, the matrix $\mathbf{G} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ the transpose of matrix **G**

is denoted $\mathbf{G}^{\mathrm{T}}$ and is formed by an interchange of rows and columns.

Thus, $$\mathbf{G}^{\mathrm{T}} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

Starting with a matrix, **A**, the recipe cooks up the matrix, $\mathbf{A}^{-1}$

Inverse Matrix “Cookbook Recipe”

Step 1 : Find the determinant of of **A**, *det* **A**

Step 2 : Form, **M**, the matrix of minors of **A** by replacing each of the nine elements of the matrix **A** with that element's minor.

Step 3 : Form, **C**, the matrix of cofactors by reversing the sign of some elements of the matrix of minors according to the pattern matrix,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$+$ indicates no change whereas $-$ indicate change

Step 4 : Write down, $\mathbf{C}^{\mathrm{T}}$, the transpose of the matrix of cofactors.

Step 5 : $\mathbf{A}^{-1} = \dfrac{1}{det\,\mathbf{A}}\,\mathbf{C}^{\mathrm{T}}$

3.2 Example

Find the inverse of the matrix, $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

Teaching Video : http://www.NumberWonder.co.uk/v9095/3.mp4

Watch the video and
then write out a full
solution here:

☞

[6 marks]

3.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 25

Question 1

Calculate the product of $\mathbf{A}^{-1}$, in the above example, with $\mathbf{A}$. That is, $\mathbf{A}^{-1} \times \mathbf{A}$
Explain why the answer is not a surprise.

[3 marks]

Question 2

By use of the "Cookbook Recipe", find the inverse of $\mathbf{W} = \begin{pmatrix} -4 & 5 & 2 \\ -5 & 6 & 2 \\ 8 & -9 & -3 \end{pmatrix}$

In your solution, label each of the five steps.

[6 marks]

Question 3

By use of the "Cookbook Recipe", find the inverse of $\mathbf{R} = \begin{pmatrix} 3 & 2 & -2 \\ -2 & k & 0 \\ -1 & -3 & 3 \end{pmatrix}$

In this matrix k is a constant, $k \neq 0$.

Your answer will, of course, be in terms of k

[6 marks]

Question 4

In examinations, if a matrix contains only numbers and no unknown constants, you may use your calculator to obtain the inverse matrix.
Use your calculator to find the inverse of the following matrix,

$$\mathbf{S} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

[2 marks]

Question 5

(i) Prove that if $\mathbf{A} = \mathbf{A}^{-1}$ then $\mathbf{A}^2 = \mathrm{I}$

[2 marks]

(ii) The matrix $\mathbf{A} = \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix}$

Given that $\mathbf{A} = \mathbf{A}^{-1}$, find the values of the constants a, b and c

[6 marks]

3.4 Answers

3.4.1 Solution to 3.3 Example (Teaching Video)

Step 1 : Find the determinant of of **A**, *det* **A**

$$\begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 2 \times 3 - 1 \times 4 + 1 \times (-1)$$

$$= 1$$

Step 2 : Form, **M**, the matrix of minors of **A** by replacing each of the nine elements of the matrix **A** with that element's minor.

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 3 & 4 & -1 \\ 1 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

Step 3 : Form, **C**, the matrix of cofactors by reversing the sign of some elements of the matrix of minors according to the pattern matrix,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$+$ indicates no change whereas $-$ indicate change

$$\mathbf{C} = \begin{pmatrix} 3 & -4 & -1 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

Step 4 : Write down, $\mathbf{C}^{\mathrm{T}}$, the transpose of the matrix of cofactors.

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Step 5 : $\mathbf{A}^{-1} = \dfrac{1}{det\ \mathbf{A}}\mathbf{C}^{\mathrm{T}}$

$$\mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

[6 marks]

3.4.2 Solutions to 3.3 Exercise

Answer 1

$$
\begin{aligned}
\mathbf{A}^{-1}\mathbf{A} &= \begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 6-3-2 & 3-2-1 & 3-1-2 \\ -8+6+2 & -4+4+1 & -4+2+2 \\ -2+0+2 & -1+0+1 & -1+0+2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \mathbf{I}
\end{aligned}
$$

This is not a surprise as for any matrix $\mathbf{X}$, which has an inverse $\mathbf{X}^{-1}$,

$$\mathbf{X}\,\mathbf{X}^{-1} = \mathbf{X}^{-1}\,\mathbf{X} = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix

[3 marks]

Answer 2

Step 1 : Find the determinant of of **W**, *det* **W**

$$\begin{vmatrix} -4 & 5 & 2 \\ -5 & 6 & 2 \\ 8 & -9 & -3 \end{vmatrix} = -4\begin{vmatrix} 6 & 2 \\ -9 & -3 \end{vmatrix} - 5\begin{vmatrix} -5 & 2 \\ 8 & -3 \end{vmatrix} + 2\begin{vmatrix} -5 & 6 \\ 8 & -9 \end{vmatrix}$$

$$= (-4) \times 0 - 5 \times (-1) + 2 \times (-3)$$

$$= -1$$

Step 2 : Form, **M**, the matrix of minors of **A** by replacing each of the nine elements of the matrix **A** with that element's minor.

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 6 & 2 \\ -9 & -3 \end{vmatrix} & \begin{vmatrix} -5 & 2 \\ 8 & -3 \end{vmatrix} & \begin{vmatrix} -5 & 6 \\ 8 & -9 \end{vmatrix} \\ \begin{vmatrix} 5 & 2 \\ -9 & -3 \end{vmatrix} & \begin{vmatrix} -4 & 2 \\ 8 & -3 \end{vmatrix} & \begin{vmatrix} -4 & 5 \\ 8 & -9 \end{vmatrix} \\ \begin{vmatrix} 5 & 2 \\ 6 & 2 \end{vmatrix} & \begin{vmatrix} -4 & 2 \\ -5 & 2 \end{vmatrix} & \begin{vmatrix} -4 & 5 \\ -5 & 6 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -1 & -3 \\ 3 & -4 & -4 \\ -2 & 2 & 1 \end{pmatrix}$$

Step 3 : Form, **C**, the matrix of cofactors by reversing the sign of some elements of the matrix of minors according to the pattern matrix,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

+ indicates no change whereas – indicate change

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -3 \\ -3 & -4 & 4 \\ -2 & -2 & 1 \end{pmatrix}$$

Step 4 : Write down, $\mathbf{C}^T$, the transpose of the matrix of cofactors.

$$\mathbf{C}^T = \begin{pmatrix} 0 & -3 & -2 \\ 1 & -4 & -2 \\ -3 & 4 & 1 \end{pmatrix}$$

Step 5 : $\mathbf{W}^{-1} = \dfrac{1}{det\ \mathbf{A}} \mathbf{C}^T$

$$\mathbf{W}^{-1} = \frac{1}{(-1)} \begin{pmatrix} 0 & -3 & -2 \\ 1 & -4 & -2 \\ -3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 2 \\ -1 & 4 & 2 \\ 3 & -4 & -1 \end{pmatrix}$$

[6 marks]

Answer 3

Step 1 : Find the determinant of of **R**, *det* **R**

$$\begin{vmatrix} 3 & 2 & -2 \\ -2 & k & 0 \\ -1 & -3 & 3 \end{vmatrix} = 3\begin{vmatrix} k & 0 \\ -3 & 3 \end{vmatrix} - 2\begin{vmatrix} -2 & 0 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} -2 & k \\ -1 & -3 \end{vmatrix}$$

$$= 3 \times 3k - 2 \times (-6) - 2 \times (6 + k)$$

$$= 9k + 12 - 12 - 2k$$

$$= 7k$$

Step 2 : Form, **M**, the matrix of minors of **A** by replacing each of the nine elements of the matrix **A** with that element's minor.

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} k & 0 \\ -3 & 3 \end{vmatrix} & \begin{vmatrix} -2 & 0 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} -2 & k \\ -1 & -3 \end{vmatrix} \\ \begin{vmatrix} 2 & -2 \\ -3 & 3 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -1 & -3 \end{vmatrix} \\ \begin{vmatrix} 2 & -2 \\ k & 0 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -2 & k \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 3k & -6 & 6 + k \\ 0 & 7 & -7 \\ 2k & -4 & 3k + 4 \end{pmatrix}$$

Step 3 : Form, **C**, the matrix of cofactors by reversing the sign of some elements of the matrix of minors according to the pattern matrix,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$+$ indicates no change whereas $-$ indicate change

$$\mathbf{C} = \begin{pmatrix} 3k & 6 & 6 + k \\ 0 & 7 & 7 \\ 2k & 4 & 3k + 4 \end{pmatrix}$$

Step 4 : Write down, $\mathbf{C}^{\mathrm{T}}$, the transpose of the matrix of cofactors.

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 3k & 0 & 2k \\ 6 & 7 & 4 \\ 6 + k & 7 & 3k + 4 \end{pmatrix}$$

Step 5 : $\mathbf{W}^{-1} = \dfrac{1}{det\ \mathbf{A}} \mathbf{C}^{\mathrm{T}}$

$$\mathbf{W}^{-1} = \frac{1}{7k}\begin{pmatrix} 3k & 0 & 2k \\ 6 & 7 & 4 \\ 6 + k & 7 & 3k + 4 \end{pmatrix}$$

[6 marks]

Answer 4

$$\mathbf{S} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix} \Rightarrow \mathbf{S}^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ 8 & -7 & -4 \end{pmatrix}$$

[2 marks]

Answer 5

(i)

$$\mathbf{A} = \mathbf{A}^{-1}$$

$$\mathbf{A}\,\mathbf{A} = \mathbf{A}\,\mathbf{A}^{-1} \qquad \text{Pre-multiply both sides by A}$$

$$\mathrm{A}^2 = \mathbf{I} \qquad \square$$

[2 marks]

(ii) From part (i) if $\mathbf{A} = \mathbf{A}^{-1}$ then $\mathbf{A}^2 = \mathrm{I}$

$$\therefore \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix} \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 25 + ab + 8 & 5a - 7a - 8 & 20 + 8a + 4c \\ 5b - 7b + 16 & ab + 49 - 16 & 4b - 56 + 8c \\ 10 - 2b + 2c & 2a + 14 - 2c & 8 - 16 + c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_{2\,1} = 0$$

$$5a - 7a - 8 = 0$$

$$-2a = 8 \Rightarrow a = -4$$

$$a_{1\,2} = 0$$

$$5b - 7b + 16 = 0$$

$$-2b = -16 \Rightarrow b = 8$$

$$a_{2\,3} = 0$$

$$4b - 56 + 8c = 0$$

$$32 - 56 + 8c = 0$$

$$8c = 24 \Rightarrow c = 3$$

[6 marks]

4.1 Three Equations, Three Unknowns

The previous two lessons have developed the mathematics necessary to solve a set of three simultaneous equations in three unknowns using matrix methods.

The strategy employed is exactly the same as that used in Lesson 1 when questions of two equations in two unknowns where tackled.

Example

Use your calculator help find the unique solution to the system of equations,

$$\begin{aligned} 2x + 4y - z &= 12 \\ x - y + 4z &= 6 \\ 4x + 5y - z &= 17 \end{aligned}$$

Teaching Instructions :

How to use a CASIO fx-991EX to help solve this is presented on the next page.

[4 marks]

Calculator Assisted Solution using the CASIO CLASSWIZ fx-991EX

First write the system of equations as a matrix equation,

$$\begin{pmatrix} 2 & 4 & -1 \\ 1 & -1 & 4 \\ 4 & 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ 17 \end{pmatrix}$$

In what follows $\mathbf{A} = \begin{pmatrix} 2 & 4 & -1 \\ 1 & -1 & 4 \\ 4 & 5 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 12 \\ 6 \\ 17 \end{pmatrix}$

Use the calculator to get the inverse of matrix **A** as follows,

- Turn the calculator **ON** and **MENU 4** to get into matrix mode .
- Press **1** to define matrix **A**
- Press **3** and **3** again to specify 3 rows and 3 columns for matrix **A**
- Enter the nine elements of the matrix **A** pressing **=** after each entry
- Press **AC** to tell the calculator the matrix **A** is now defined
- Press **OPTN 3** to initiate a calculation involving matrix **A**
- Press the button $\boldsymbol{x}^{-1}$ followed by **=**
- Scroll through the elements of the inverse matrix and write down,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{21}\begin{pmatrix} -19 & -1 & 15 \\ 17 & 2 & -9 \\ 9 & 6 & -6 \end{pmatrix}\begin{pmatrix} 12 \\ 6 \\ 17 \end{pmatrix}$$

The above line of working is worth half marks.

Now the calculator will be used to perform the matrix multiplication $\mathbf{A}^{-1}\mathbf{B}$ and so yield the values of *x*, *y* and *z*

- Press **MENU 4** to again enter the Define Matrix screen
- Press **2** to define matrix **B**
- Press **3** and **1** to specify 3 rows and 1 column for matrix **B**
- Enter the three elements of the matrix **B** pressing **=** after each entry
- Press **AC** to tell the calculator the matrix **B** is now defined
- Press **OPTN 3** to initiate a calculation involving matrix **A**
- Press the button $\boldsymbol{x}^{-1}$ followed by $\times$
- Press **OPTN 4** to enter matrix **B** into the evolving calculation
- Now press **=** and write down,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

$\therefore$ The unique solution is $x = 1, \quad y = 3, \quad z = 2$

[4 marks]

4.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Further A-Level Examination Question, May 2018, Core 1, Q1 (a), (b) (Edexcel)

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -2 & 1 \\ 3 & 5 & -2 \end{pmatrix}$$

(a) Find $\mathbf{M}^{-1}$ giving each element in exact form.

[2 marks]

(b) Solve the simultaneous equations,

$$\begin{aligned} 2x + y - 3z &= -4 \\ 4x - 2y + z &= 9 \\ 3x + 5y - 2z &= 5 \end{aligned}$$

[2 marks]

Question 2

In the following system of equations, a is an unknown constant, $a \neq -2$,

$$\begin{aligned} x - y + z &= 4 \\ 4x + z &= 2a \\ 2x + ay + 2z &= a \end{aligned}$$

(i) Construct a suitable matrix equation with a view to preparing to solve this system of equations by matrix methods.

[1 mark]

(ii) Find, in terms of a, an expression for the determinant of the matrix,

$$\mathbf{S} = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 0 & 1 \\ 2 & a & 2 \end{pmatrix}$$

[2 marks]

(iii) From **S**, form the matrix of minors, **M**, in terms of a

[4 marks]

(iv) From **M**, form the matrix of cofactors, **C**, in terms of a

[1 mark]

(v) Write down the transpose, $\mathbf{C}^{\mathrm{T}}$, of the matrix of cofactors, in terms of a

[1 mark]

(vi) Write down in terms of a the inverse matrix $\mathbf{S}^{-1}$

[1 mark]

(vii) Find, in terms of a, the values of x, y, and z

[2 marks]

(viii) Show that if $a = 3$, the values of x, y and z are integers.

[2 marks]

Question 3
Three planes A, B and C are defined by the following equations;

$$\begin{aligned} A &: \; x + y + z = 3 \\ B &: \; 2x - y - z = 0 \\ C &: \; 3x - 2y + z = -1 \end{aligned}$$

By constructing and solving a suitable matrix equation, show that these three planes intersect at a single point and find the coordinates of that point

[5 marks]

Question 4

Use your calculator to find the inverse of,

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 3 & 5 \end{pmatrix}$$

[4 marks]

Question 5

Find the inverse of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where k is a constant

[3 marks]

4.3 Answers

4.3.1 Solution to 4.1 Example (Teaching Video)

$$\begin{aligned} 2x + 4y - z &= 12 \\ x - y + 4z &= 6 \\ 4x + 5y - z &= 17 \end{aligned}$$

In matrix form these become,

$$\begin{pmatrix} 2 & 4 & -1 \\ 1 & -1 & 4 \\ 4 & 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ 17 \end{pmatrix}$$

Using a calculator to get the inverse matrix, and pre multiplying both sides by it gives,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{21}\begin{pmatrix} -19 & -1 & 15 \\ 17 & 2 & -9 \\ 9 & 6 & -6 \end{pmatrix}\begin{pmatrix} 12 \\ 6 \\ 17 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

$\therefore$ The unique solution is $x = 1, \quad y = 3, \quad z = 2$

[4 marks]

4.3.2 Solutions to 4.2 Exercise

Answer 1

(a) $\quad \mathbf{M} = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -2 & 1 \\ 3 & 5 & -2 \end{pmatrix} \Rightarrow \mathbf{M}^{-1} = \frac{1}{69}\begin{pmatrix} 1 & 13 & 5 \\ -11 & -5 & 14 \\ -26 & 7 & 8 \end{pmatrix}$

[2 marks]

(b) $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{69}\begin{pmatrix} 1 & 13 & 5 \\ -11 & -5 & 14 \\ -26 & 7 & 8 \end{pmatrix}\begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$\therefore$ The unique solution is $x = 2, \quad y = 1, \quad z = 3$

[2 marks]

Answer 2

(i)
$$\begin{pmatrix} 1 & -1 & 1 \\ 4 & 0 & 1 \\ 2 & a & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2a \\ a \end{pmatrix}$$

[1 mark]

(ii)
$$\begin{vmatrix} 1 & -1 & 1 \\ 4 & 0 & 1 \\ 2 & a & 2 \end{vmatrix} = 1 \times \begin{vmatrix} 0 & 1 \\ a & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 4 & 0 \\ 2 & a \end{vmatrix}$$
$$= -a + 6 + 4a$$
$$= 3(a + 2)$$

[2 marks]

(iii)
$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ a & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ 2 & a \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ a & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & a \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} -a & 6 & 4a \\ -2 - a & 0 & a + 2 \\ -1 & -3 & 4 \end{pmatrix}$$

[4 marks]

(iv)
$$\mathbf{C} = \begin{pmatrix} -a & -6 & 4a \\ 2 + a & 0 & -a - 2 \\ -1 & 3 & 4 \end{pmatrix}$$

[1 mark]

(v)
$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -a & 2 + a & -1 \\ -6 & 0 & 3 \\ 4a & -a - 2 & 4 \end{pmatrix}$$

[1 mark]

(vi)
$$\mathbf{S}^{-1} = \frac{1}{3(a + 2)}\begin{pmatrix} -a & 2 + a & -1 \\ -6 & 0 & 3 \\ 4a & -a - 2 & 4 \end{pmatrix}$$

[1 mark]

(vii) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3(a+2)} \begin{pmatrix} -a & 2+a & -1 \\ -6 & 0 & 3 \\ 4a & -a-2 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 2a \\ a \end{pmatrix}$$

$$= \frac{1}{3(a+2)} \begin{pmatrix} -4a + 4a + 2a^2 - a \\ -24 + 0 + 3a \\ 16a - 2a^2 - 4a + 4a \end{pmatrix}$$

$$= \frac{1}{3(a+2)} \begin{pmatrix} a(2a-1) \\ 3(a-8) \\ 2a(8-a) \end{pmatrix}$$

[2 marks]

(viii) With $a = 3$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3(3+2)} \begin{pmatrix} 3(2 \times 3 - 1) \\ 3(3-8) \\ 2 \times 3(8-3) \end{pmatrix}$$

$$= \frac{1}{15} \begin{pmatrix} 15 \\ -15 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\therefore \; x = 1, \quad y = -1, \quad z = 2$$

which shows that x, y and z are integers. as required. □

[2 marks]

Answer 3

The given system of equations can be written in the following matrix form,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

Using a calculator to obtain the inverse matrix, this becomes,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 3 & 3 & 0 \\ 5 & 2 & -3 \\ 1 & -5 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\therefore \; x = 1, \quad y = 2, \quad z = 0$$

[5 marks]

Answer 4

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 3 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 7 & -3 & 0 & 1 \\ -8 & 4 & 1 & -2 \\ 2 & -1 & 1 & 0 \\ 2 & -1 & -1 & 1 \end{pmatrix}$$

[4 marks]

Answer 5

With such a simple matrix the inverse is easiest to find by setting up a general matrix as the inverse and using the fact that a matrix multiplied by its inverse is the identity matrix. Then study the multiplications...

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ kd & ke & kf \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a = 1, \quad b = 0, \quad c = 0$$

$$d = 0, \quad e = \frac{1}{k}, \quad f = 0$$

$$g = 0, \quad h = 0, \quad i = 1$$

Thus, $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{k} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{k}\begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}$$

[3 marks]

5.1 Word Problems

The examination may feature a question in which the matrix calculations can be done on a calculator. Before getting to that point, however, the problem given may be in words, and setting up the matrix equation in the first place becomes a significant part of the problem.

5.2 Example

Miranda has decided to invest £24500 spread between three bank accounts which pay the following interest rates per annum;

Account X : An “Extra Gold” Account ... ... 4 %
Account Y : A “Why settle for less ?” Account ... ... 5.5 %
Account Z ; An “eZee saver” Account ... ... 6 %

The amount of money that Miranda places in the “Extra Gold” account is four times the amount of money that she places in the “Why settle for less ?” account. One year later, she has made £1300 in interest.

Set up a matrix equation from the information provided and then solve that equation to determine how much money Miranda initially placed in each account.

Teaching Video : http://www.NumberWonder.co.uk/v9095/5a.mp4
http://www.NumberWonder.co.uk/v9095/5b.mp4

<== Part 1 Part 2 ==>

Watch the video and then write out a full solution here:

☞

[6 marks]

5.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 30

Question 1

Harry, Boris and Sid went went to the shop, “Play and Party”, to buy decorations for their school's sixth form dance. For £244, Harry purchased three inflatable space aliens, four boxes of indoor fireworks and five bags of party poppers. Boris spent £304 when he bought six inflatable space aliens, five boxes of indoor fireworks and two bags of party poppers. Sid bought three inflatable space aliens, two boxes of indoor fireworks and one bag of party poppers. Sid spent £134.

Form a system of equations in matrix form from the information provided and then solve them to determine the unit cost of each item.

[6 marks]

Question 2

A specialist farmer breeds only Suri Alpacas, Huacaya Alpacas or Domestic Llamas.

The BBC's Kate Humble with a young Suri Alpaca in Peru

Initially the farm has 2810 of these three types of animal.
There were 160 more Suri Alpacas than Huacaya Alpacas.
After one year :

- the number of Suri Alpacas had increased by 5 %
- the number of Huacaya Alpacas had increased by 3 %
- the number of Domestic Llamas had decreased by 4 %
- overall the farm had 46 more of these three types of animal

Form and solve a matrix equation to find out how many Suri Alpacas, how many Huacaya Alpacas and how many Domestic Llamas there were initially.

[6 marks]

Question 3

Oscar is making a tropical fruit punch for his school's sixth form dance using bananas, oranges and papayas. Altogether he uses seventy pieces of fruit, costing £34.20. He uses twice as many oranges as bananas.

Each banana cost 27p, each orange cost 32p and each papaya cost £1.60.

Form and solve a matrix equation to find out how many of each type of fruit were used in Oscar's (rather good) tropical fruit punch. Cheers Oscar !

[6 marks]

Question 4

A colony of bats is made up of brown bats, grey bats and black bats.

Initially there are 2000 bats and there are 250 more brown bats than grey bats.

After one year;

- the number of brown bats had fallen by 1 %
- the number of grey bats had fallen by 2 %
- the number of black bats had increased by 4 %
- overall there were 40 more bats

Form and solve a matrix equation to find out how many of each colour bat there were in the initial colony.

[7 marks]

Question 5

Use a calculator to find the unique solution to this system of equations,

$$\begin{aligned} x + y + z + w &= 13 \\ 2x + 3y + 0z - w &= -1 \\ -3x + 4y + z + 2w &= 10 \\ x + 2y - z + w &= 1 \end{aligned}$$

[5 marks]

5.4 Answers

5.4.1 Solution to 5.2 Example (Teaching Video)

The words give rise to the following three equations;

$$\begin{aligned} x + y + z &= 24500 \\ 0.04x + 0.055y + 0.06z &= 1300 \\ x - 4y + 0z &= 0 \end{aligned}$$

The middle equation can be multiplied through by 100 to make the numbers a little easier to handle,

$$\begin{aligned} x + y + z &= 24500 \\ 4x + 5.5y + 6z &= 130000 \\ x - 4y + 0z &= 0 \end{aligned}$$

Translating this into a matrix equation gives,

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 5.5 & 6 \\ 1 & -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24500 \\ 130000 \\ 0 \end{pmatrix}$$

Using a calculator to get the inverse of the leftmost matrix results in,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{17} \begin{pmatrix} 48 & -8 & 1 \\ 12 & -2 & -4 \\ -43 & 10 & 3 \end{pmatrix} \begin{pmatrix} 24500 \\ 130000 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 8000 \\ 2000 \\ 14500 \end{pmatrix}$$

$\therefore$ Miranda invested £8000 in account X,
£2000 in account Y
and £14500 in account Z

[5 marks]

5.4.2 Solutions to 5.3 Exercise

Answer 1

Let a represent the number of inflatable space aliens.
Let f represent the number of boxes of indoor fireworks.
Let p represent the number of bags of party poppers.
Thus the system of equations becomes,

$$\begin{aligned} 3a + 4f + 5p &= 244 \\ 6a + 5f + 2p &= 304 \\ 3a + 2f + p &= 134 \end{aligned}$$

As a matrix equation this will be,

$$\begin{pmatrix} 3 & 4 & 5 \\ 6 & 5 & 2 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ f \\ p \end{pmatrix} = \begin{pmatrix} 244 \\ 304 \\ 134 \end{pmatrix}$$

Using a calculator to get the inverse of the leftmost matrix results in,

$$\begin{pmatrix} a \\ f \\ p \end{pmatrix} = \frac{1}{12}\begin{pmatrix} -1 & -6 & 17 \\ 0 & 12 & -24 \\ 3 & -6 & 9 \end{pmatrix}\begin{pmatrix} 244 \\ 304 \\ 134 \end{pmatrix}$$

$$= \begin{pmatrix} 17.50 \\ 36.00 \\ 9.50 \end{pmatrix}$$

The unit cost of each item is thus,

1 inflatable space alien : £17.50

1 box of indoor fireworks : £36.00

1 bag of party poppers : £9.50

[6 marks]

Answer 2
Let s represent the initial number of Suri Alpacas.
Let h represent the initial number of Huacaya Alpacas.
Let d represent the initial number of Domestic Llamas.

The words give rise to the following three equations;

$$\begin{aligned} s + h + d &= 2810 \\ 0.05s + 0.03h - 0.04d &= 46 \\ s - h + 0d &= 160 \end{aligned}$$

The middle equation can be multiplied through by 100 to make the numbers easier to handle,

$$\begin{aligned} s + h + d &= 2810 \\ 5s + 3h - 4d &= 4600 \\ s - h + 0d &= 160 \end{aligned}$$

Translating this into a matrix equation gives,

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 3 & -4 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} s \\ h \\ d \end{pmatrix} = \begin{pmatrix} 2810 \\ 4600 \\ 160 \end{pmatrix}$$

Using a calculator to get the inverse of the leftmost matrix results in,

$$\begin{pmatrix} s \\ h \\ d \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 4 & 1 & 7 \\ 4 & 1 & -9 \\ 8 & -2 & 2 \end{pmatrix}\begin{pmatrix} 2810 \\ 4600 \\ 160 \end{pmatrix}$$

$$= \begin{pmatrix} 1060 \\ 900 \\ 850 \end{pmatrix}$$

$\therefore$ Initially there were 1060 Suri Alpacas, 900 Huacaya Alpacas and 850 Domestic Llamas.

[5 marks]

Answer 3

Let b represent the number of bananas.
Let o represent the number of oranges.
Let p represent the number of papayas.
Thus the system of equations becomes,

$$\begin{aligned} b + o + p &= 70 \\ 27b + 32o + 160p &= 3420 \\ 2b - o + 0p &= 0 \end{aligned}$$

As a matrix equation this will be,

$$\begin{pmatrix} 1 & 1 & 1 \\ 27 & 32 & 160 \\ 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} b \\ o \\ p \end{pmatrix} = \begin{pmatrix} 70 \\ 3420 \\ 0 \end{pmatrix}$$

Using a calculator to get the inverse of the leftmost matrix results in,

$$\begin{pmatrix} a \\ f \\ p \end{pmatrix} = \frac{1}{389}\begin{pmatrix} 160 & -1 & 128 \\ 320 & -2 & -133 \\ -91 & 3 & 5 \end{pmatrix}\begin{pmatrix} 70 \\ 3420 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 20 \\ 40 \\ 10 \end{pmatrix}$$

The number of each item is thus,

bananas : 20

oranges : 40

papaya : 10

[6 marks]

Answer 4
Let *brn* represent the number of brown bats.
Let *gry* represent the number of grey bats.
Let *blk* represent the number of black bats.
Thus the system of equations becomes,

$$\begin{aligned} brn + \quad gry + \quad blk &= 2000 \\ -0.01brn - 0.02gry + 0.04blk &= 40 \\ brn - \quad gry + \quad 0blk &= 250 \end{aligned}$$

The middle equation can be multiplied through by 100 to make the numbers a little easier to handle,

$$\begin{aligned} brn + gry + blk &= 2000 \\ -brn - 2gry + 4blk &= 4000 \\ brn - gry + 0blk &= 250 \end{aligned}$$

As a matrix equation this will be,

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & 4 \\ 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} brn \\ gry \\ blk \end{pmatrix} = \begin{pmatrix} 2000 \\ 4000 \\ 250 \end{pmatrix}$$

Using a calculator to get the inverse of the leftmost matrix results in,

$$\begin{pmatrix} brn \\ gry \\ blk \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 4 & -1 & 6 \\ 4 & -1 & -5 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2000 \\ 4000 \\ 250 \end{pmatrix}$$

$$= \begin{pmatrix} 500 \\ 250 \\ 1250 \end{pmatrix}$$

The number of each type of bat is thus,

brown : 500

grey : 250

black : 1250

[7 marks]

Answer 5

The system of equations,

$$\begin{aligned} x + y + z + w &= 13 \\ 2x + 3y + 0z - w &= -1 \\ -3x + 4y + z + 2w &= 10 \\ x + 2y - z + w &= 1 \end{aligned}$$

converted into matrix form becomes,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & -1 \\ -3 & 4 & 1 & 2 \\ 1 & 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 13 \\ -1 \\ 10 \\ 1 \end{pmatrix}$$

Using a calculator to get the inverse of the leftmost matrix results in,

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \frac{1}{54}\begin{pmatrix} 15 & 3 & -9 & 6 \\ -4 & 10 & 6 & 2 \\ 25 & 5 & 3 & -26 \\ 18 & -18 & 0 & 18 \end{pmatrix}\begin{pmatrix} 13 \\ -1 \\ 10 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \\ 6 \\ 5 \end{pmatrix}$$

$$\therefore\ x = 2, \quad y = 0, \quad z = 6, \quad w = 5$$

[5 marks]

6.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 4 \\ -4 & 2 & 1 \end{pmatrix}$ and matrix $\mathbf{B} = \begin{pmatrix} 3 & 1 & 2 \\ k & 4 & 5 \\ 0 & 2 & 3 \end{pmatrix}$ where k is a constant.

(i) Evaluate the determinant of **A**

[2 marks]

Given that the determinant of **B** is 2,

(ii) find the value of k

[3 marks]

Question 2

Given that $\mathbf{A} = \begin{pmatrix} a & 2 \\ 3 & 2a \end{pmatrix}$ where a is a real constant

(i) find $\mathbf{A}^{-1}$ in terms of a

[3 marks]

(ii) write down two values of a for which $\mathbf{A}^{-1}$ does not exist

[1 mark]

Question 3

If $\mathbf{A}^{-1} = \mathbf{A}$, then the matrix $\mathbf{A}$ is said to be self-inverse.

(It's its own opposite)

Give a example of a matrix that is self-inverse other than an identity matrix.

[1 mark]

Question 4

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \\ k & 1 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

(i) Show that $det\,\mathbf{A} = 3(k + 1)$

[3 marks]

(ii) Given that $k \neq -1$, find $\mathbf{A}^{-1}$

[4 marks]

Question 5

A colony of 1000 mole-rats is made up of adult males, adult females and youngsters.
Originally there were 100 more adult females than adult males.
After one year;

- the number of adult males had increased by 2 %
- the number of adult females had increased by 3 %
- the number of youngsters had decreased by 4 %
- the total number of mole-rats had decreased by 20

Form and solve a matrix equation to find out how many of each type of mole-rat were in the original colony.

[6 marks]

Question 6

Further A-Level Examination Question, June 2019, Core 2, Q7 (Edexcel)

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \text{ where } k \text{ is a constant}$$

(a) Find the values of k for which **M** has an inverse

[2 marks]

(b) Find, in terms of p, the coordinates of the point where the following planes intersect,

$$\begin{aligned} 2x - y + z &= p \\ 3x - 6y + 4z &= 1 \\ 3x + 2y - z &= 0 \end{aligned}$$

[5 marks]

6.2 Answers

Answer 1

(i) Expanding along the top row,

$$\begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & 4 \\ -4 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ -4 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ -4 & 2 \end{vmatrix}$$

$$= 2 \times (-8) - 1 \times 17 - 1 \times 2$$

$$= -35$$

[2 marks]

(ii)
$$\begin{vmatrix} 3 & 1 & 2 \\ k & 4 & 5 \\ 0 & 2 & 3 \end{vmatrix} = 2$$

$$3\begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} k & 5 \\ 0 & 3 \end{vmatrix} + 2\begin{vmatrix} k & 4 \\ 0 & 2 \end{vmatrix} = 2$$

$$3 \times 2 - 1 \times 3k + 2 \times 2k = 2$$

$$6 - 3k + 4k = 2$$

$$k = -4$$

[3 marks]

Answer 2

(i) Using the result, if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \dfrac{1}{\det \mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$|\mathbf{A}| = \begin{vmatrix} a & 2 \\ 3 & 2a \end{vmatrix}$$

$$= 2a^2 - 6$$

$$= 2(a^2 - 3)$$

$$\mathbf{A}^{-1} = \frac{1}{2(a^2 - 3)} \begin{pmatrix} 2a & -2 \\ -3 & a \end{pmatrix}$$

[3 marks]

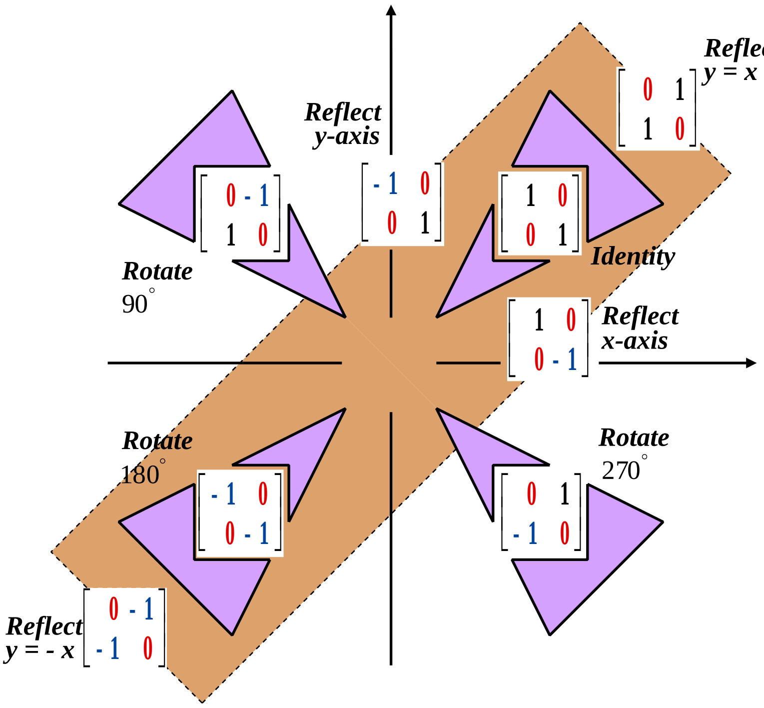
(ii) The inverse matrix does nor exist if the determinant is zero.

That is, when $a = \pm\sqrt{3}$

[1 mark]

Answer 3

The obvious answer is to give a 2×2 reflection matrix or the $180°$ rotation matrix.



Any answer, $\mathbf{A}$, should satisfy $\mathbf{A}^2 = \mathbf{I}$ which can be used to check answers given.

[1 mark]

Answer 4

(i) Expanding along the top row,

$$\begin{vmatrix} 2 & 0 & 3 \\ k & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 2 \times 3 + 3 \times (k - 1)$$

$$= 3(k + 1) \qquad \square$$

[3 marks]

(ii) $$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} k & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ k & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ k & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4k - 1 & k - 1 \\ -3 & 5 & 2 \\ -3 & 2 - 3k & 2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 3 & 1 - 4k & k - 1 \\ 3 & 5 & -2 \\ -3 & 3k - 2 & 2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 3 & 3 & -3 \\ 1 - 4k & 5 & 3k - 2 \\ k - 1 & -2 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{3(k + 1)}\begin{pmatrix} 3 & 3 & -3 \\ 1 - 4k & 5 & 3k - 2 \\ k - 1 & -2 & 2 \end{pmatrix}$$

[4 marks]

Answer 5
Let m represent the number of adult male mole-rats.
Let f represent the number of adult female mole-rats.
Let y represent the number of youngster mole-rats.
Thus the system of equations becomes,

$$\begin{aligned} m + f + y &= 1000 \\ 0.02m + 0.03f - 0.04y &= -20 \\ -m + f + 0y &= 100 \end{aligned}$$

The middle equation can be multiplied through by 100 to make the numbers a little easier to handle,

$$\begin{aligned} m + f + y &= 1000 \\ 2m + 3f - 4y &= -2000 \\ -m + f + 0y &= 100 \end{aligned}$$

As a matrix equation this will be,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -4 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} m \\ f \\ y \end{pmatrix} = \begin{pmatrix} 1000 \\ -2000 \\ 100 \end{pmatrix}$$

Using a calculator to get the inverse of the leftmost matrix results in,

$$\begin{aligned} \begin{pmatrix} m \\ f \\ y \end{pmatrix} &= \frac{1}{13} \begin{pmatrix} 4 & 1 & -7 \\ 4 & 1 & 6 \\ 5 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1000 \\ -2000 \\ 100 \end{pmatrix} \\ &= \begin{pmatrix} 100 \\ 200 \\ 700 \end{pmatrix} \end{aligned}$$

The initial number of each type of mole-rat is thus,

adult male : 100

adult female : 200

youngsters : 700

[6 marks]

Answer 6

(a) Finding the determinant by expanding along the top row,

$$\begin{vmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{vmatrix} = 2\begin{vmatrix} k & 4 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & k \\ 3 & 2 \end{vmatrix}$$

$$= 2\,(-k - 8) + 1 \times (-15) + 1 \times (6 - 3k)$$

$$= -2k - 16 - 15 + 6 - 3k$$

$$= -5k - 25$$

$$= (-5)(k + 5)$$

$\therefore$ **M** has an inverse for all real values of k, except $k = -5$

(b) The given equations can be written in the matrix form,

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & -6 & 4 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix}$$

The leftmost matrix is the part (a) matrix with $k = -6$

Using a calculator to get the inverse of the leftmost matrix,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 & 1 & 2 \\ 15 & -5 & -5 \\ 24 & -7 & -9 \end{pmatrix}\begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -2p + 1 \\ 15p - 5 \\ 24p - 7 \end{pmatrix}$$

The point of intersection is thus,

$$\left(\frac{1 - 2p}{5},\ 3p - 1,\ \frac{24p - 7}{5} \right)$$

[5 marks]

7.1 Describing a Three Dimensional Plane

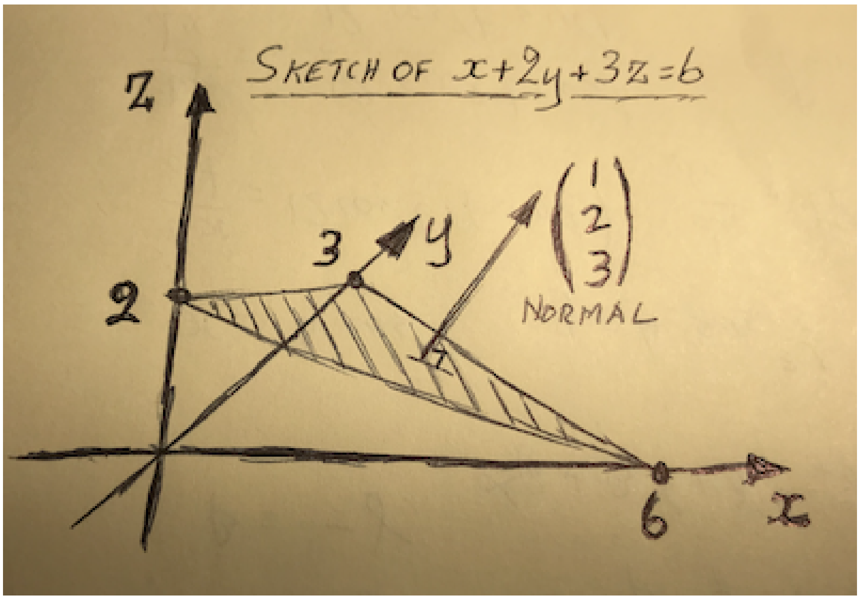
In three dimensional Euclidean space, a plane has an equation of the form,

$$ax + by + cz = d \text{ where } a,\ b,\ c \text{ and } d \text{ are constants}$$

For example the equation $x + 2y + 3z = 6$ describes a plane.
It can be sketched by realising that,

- it will cross the x-axis when y and z are zero; (6, 0, 0)
- it will cross the y-axis when x and z are zero; (0, 3, 0)
- it will cross the z-axis when x and y are zero; (0, 0, 2)

which leads to,



A plane can be considered to "look" in the direction given by its normal, which gives us a means of detecting parallel planes; they will have normals that look in the same direction (or the 180° opposite direction).

The Three-Dimensional Plane

A Cartesian equation of a plane in three dimensions can be written in the form

$$ax + by + cz = d \text{ where } a,\ b,\ c \text{ and } d \text{ are constants.}$$

The vector $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal to the plane and, intuitively, gives the

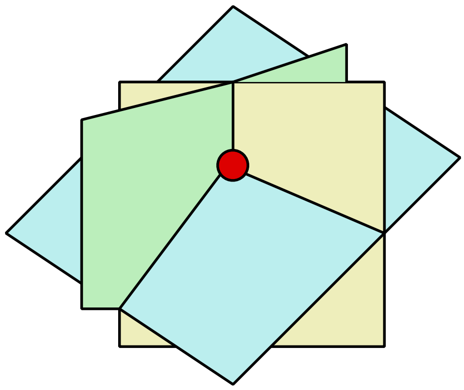
in which the plane is "looking".
Parallel planes look either in the same direction, or 180° opposite directions.
The *unit normal* is a vector in the normal's direction of length 1 unit.

7.2 Point Intersection of Three Planes

Suppose that we have the following three planes,

$$\begin{aligned} 2x - 3y + z &= -6 \\ x + 2y - 4z &= 12 \\ 3x + y + 2z &= 7 \end{aligned}$$

Typically, three planes chosen at random will intersect in a single common point and such planes are said to be "in general position" as illustrated below.



The point of intersection is most easily found by setting up and solving a matrix system of equations. Use a calculator if all of the entries in the matrix are numbers.

For the above three planes we'd proceed as follows;

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & -4 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 12 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{53}\begin{pmatrix} 8 & 7 & 10 \\ -14 & 1 & 9 \\ -5 & -11 & 7 \end{pmatrix}\begin{pmatrix} -6 \\ 12 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

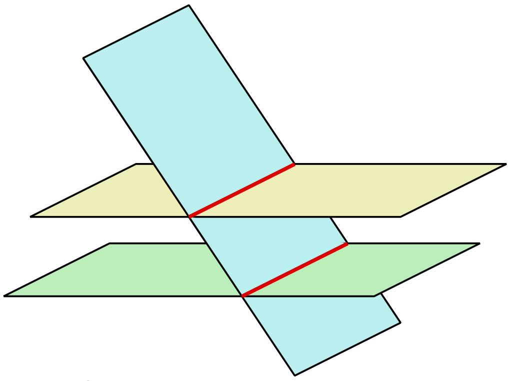
So the three planes intersect at the point $(2, 3, -1)$

Notice that the intermediate line of working, showing the inverse matrix, should be given. However, most of the calculation is all in the calculator with nothing written down.

7.3 Special Case Three Plane Intersections

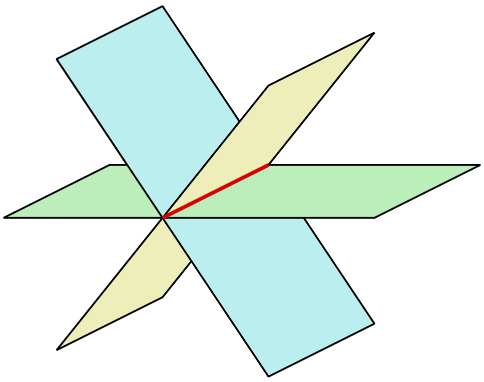
There are only two types of configuration with two planes; either the planes are parallel and they do not intersect, or else they intersect in a common line.

With three planes, there are a total of five situations that can arise. The first of these, where the the planes are in general position, has been dealt with but there are four special cases to consider, as illustrated below.



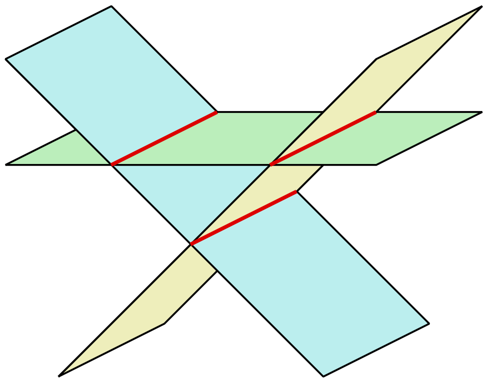
Two of the planes (only) are parallel

All three planes are parallel



The planes form a sheaf

They share a common line



The planes form a prism

They meet in pairs in parallel lines

With all of these different possibilities to worry about, we need a general strategy that will work through then in a brisk and efficient fashion. This is in three steps and is presented next.

Step 1 : Test for Planes in General Position

Unless you have reason to believe one of the other configurations is in play, test first to see if the three planes are in general position with a single point of intersection through which all three planes pass. Do this by expressing the equations of the three planes as one 3×3 matrix **S** and consider it to be a system of equations with a solution that is the point of intersection (x, y, z). If the determinant of **S** is not zero then **this is good news**; there is a unique solution, a single point, and the planes are in general position. Solve the system of equations to find this point.

If the determinant is zero then one of the other four situations applies and it is necessary to proceed to step 2.

Step 2 : Test for Parallel Planes

Having found that the determinant of the system of equations is zero, the next test is to find out if two or three of the planes are parallel. Do this by looking at the directions of the normal vectors. The number of normal vectors in the same direction (or a 180° opposite direction match) gives the number of parallel planes. The 'raw' normal vectors will likely be of different lengths (unless you process them to make them all of unit length) so take care not miss any that are in the same direction. Unfortunately, the A-Level course does not teach you how to do this using matrices. Beyond A-Level, you calculate something called the rank of a matrix and you may like to explore Wolfram Alpha's **MatrixRank** command to help determine the number of normal vectors that are in the same direction.

If no parallel planes are detected one of the remaining two situations applies and it is necessary to proceed to step 3

Step 3 : Testing for a Sheaf or Prism

Having worked through steps 1 and 2 without identifying the configuration, only two possibilities remain. We can test for which in one go. Combine one of the three equations separately with each the other two. For example, say the first is added to the second and, separately, the first is added to the third. If a consistency is obtained then a sheaf has been detected. If an inconsistency is obtained then a prism has been detected. Alas, once again the A-Level course does not develop the topic of matrices far enough to do this by “Gaussian elimination” so you need to do it using algebraic wits.

7.4 An Example

Consider the three planes with equations

$$x + 2y + 3z = 13$$

$$-x - 3y + 2z = 2$$

$$-x - 4y + 7z = 9$$

(i) Set up a matrix system of equations and then show that the 3×3 matrix involved has a determinant of zero. State what this tells you about the configuration of the planes.

[4 marks]

(ii) Write down the normals for each plane.
Explain how these tell you none of the planes are parallel.

[4 marks]

(iii) Add together the upper and middle equations.
Separately, add together the upper and lower equations.
Show that an inconsistency results.
In which configuration are the three planes ?

[4 marks]

7.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 45

Question 1

By setting up a matrix system of equations, find the point of intersection of the following three planes using the matrix facility on your calculator.

$$\begin{aligned} x + 4y - 5z &= 8 \\ 2x + 3y + z &= -1 \\ 3x - y + 2z &= 3 \end{aligned}$$

[4 marks]

Question 2

Three planes have the equations,

$$\begin{aligned} x - 2y + z &= 5 \\ 3x - 6y + 3z &= 12 \\ x + 2y - z &= -2 \end{aligned}$$

Kevin has worked out the the associated matrix has a determinant of zero.
What is the configuration of these planes ?
Give a reason for your answer.

[4 marks]

Question 3

FM A-Level Examination Question from October 2020, Paper Core Pure, Q15 (MEI)

(a) Show that the three planes with equations,

$$x + \lambda y + 3z = -12$$
$$2x + y + 5z = -11$$
$$x - 2y + 2z = -9$$

where λ is a constant, meet at a unique point except for one value of λ which is to be determined.

[3 marks]

(b) In the case $\lambda = -2$, use matrices to find the point of intersection of the planes, showing your method clearly.

[3 marks]

Question 4

Determine the configuration of the following three planes,

$$
\begin{aligned}
x + 4y - 5z &= 8 \\
2x + 3y + z &= -1 \\
3x + 7y - 4z &= 5
\end{aligned}
$$

[4 marks]

Question 5

FM A-Level Examination Question from October 2021, Paper Core Pure, Q15 (MEI)

The equations of three planes are,

$$-4x + py + 7z = 4$$

$$x - 2y + 5z = q$$

$$2x + 3y + z = 2$$

Given that the planes form a sheaf, determine the values of p and q

[6 marks]

Question 6

FM A-Level Examination Question from October 2020, Paper Core Pure, Q9 (MEI)

Three planes have equations,

$$\begin{aligned} kx + y - 2z &= 0 \qquad \text{where } k \text{ is a constant} \\ 2x + 3y - 6z &= -5 \\ 3x - 2y + 5z &= 1 \end{aligned}$$

Investigate the arrangement of the planes for each of the following cases.

If in either case the planes meet at a unique point, find the coordinates of that point.

(a) $k = -1$

[3 marks]

(b) $k = \frac{2}{3}$

[4 marks]

Question 7

FM A-Level Examination Question from June 2019, Paper Core Pure, Q14 (MEI)

Three planes have equations,

$$\begin{aligned} -x + ay \quad &= 2 \\ 2x + 3y + z &= -3 \\ x + by + z &= c \end{aligned}$$

where a, b and c are constants.

(a) In the case where the planes **do not** intersect at a unique point,

(i) find b in terms of a

[4 marks]

(ii) find the value of c for which the planes form a sheaf.

[3 marks]

(b) In the case where $b = a$ and $c = 1$, find the coordinates of the point of intersection of the planes in terms of a.

[6 marks]

Question 8

FM A-Level Examination Question from June 2021, Paper 1, Q12 (AQA)

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 5 & 3 \\ 4 & -2 & p \\ 8 & 5 & -11 \end{pmatrix}$, where p is a constant.

(a) Given that $\mathbf{A}$ is a non-singular matrix, find $\mathbf{A}^{-1}$ in terms of p.

State any restrictions on the value of p.

[6 marks]

(b) The equations below represent three planes,

$$\begin{aligned} x + 5y + 3z &= 5 \\ 4x - 2y + pz &= 24 \\ 8x + 5y - 11z &= -30 \end{aligned}$$

(i) Find, in terms of p, the coordinates of the point of intersection of the three planes.

[4 marks]

(ii) In the case where $p = 2$, show that the planes are mutually perpendicular.

[4 marks]

7.6 Answers

7.6.1 Solution to 7.4 Example

(i) In matrix form the system of equations will be...

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -3 & 2 \\ -1 & -4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 \\ 2 \\ 9 \end{pmatrix}$$

Expanding along the upper row to obtain the determinant...

$$1\,((-3)(7) - (-4)(2)) - 2\,(-1)(7) - (-1)(2)) + 3\,((-1)(-4) - (-1)(-3))$$

$$= (-21 + 8) - 2\,(-7 + 2) + 3\,(4 - 3)$$

$$= -13 + 10 + 3$$

$$= 0 \text{ as anticipated}$$

This tells us that the three planes do not meet is a single point.

[4 marks]

(ii) The normals are $\boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\boldsymbol{m} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$ and $\boldsymbol{l} = \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix}$

None of these are parallel because there is no constants k_{um}, k_{ul} or k_{ml} for which $\boldsymbol{u} = k_{um}\,\boldsymbol{m}$, $\boldsymbol{u} = k_{ul}\,\boldsymbol{l}$ or $\boldsymbol{m} = k_{ml}\,\boldsymbol{l}$

As none of the normals are in the same direction (parallel), none of the planes are parallel.

[4 marks]

(iii) Adding together the upper and middle equations...

$$\begin{aligned} x + 2y + 3z &= 13 \\ -x - 3y + 2z &= 2 \\ \hline -y + 5z &= 15 \end{aligned}$$

Adding together upper and lower equations...

$$\begin{aligned} x + 2y + 3z &= 13 \\ -x - 4y + 7z &= 9 \\ \hline -2y + 10z &= 22 \\ -y + 5z &= 11 \end{aligned}$$

The inconsistency shows that the three planes form a prism.

[4 marks]

7.6.2 Solutions to 7.5 Exercise

Answer 1

$$\begin{pmatrix} 1 & 4 & -5 \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{58} \begin{pmatrix} 7 & -3 & 19 \\ -1 & 17 & -11 \\ -11 & 13 & -5 \end{pmatrix} \begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

The point of intersection is $(2, -1, -2)$

[4 marks]

Answer 2

As Kevin has worked out that the determinant is zero we know the three planes do not intersect in a unique point (Thanks, Kevin).

So, we now look at the normals which are,

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad \boldsymbol{m} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} \qquad \boldsymbol{l} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

As $\boldsymbol{u} = \frac{1}{3} \boldsymbol{m}$ there are at least two parallel planes.

As there are no constants k_{ul} or k_{ml} for which $\boldsymbol{u} = k_{ul}\, \boldsymbol{l}$ or $\boldsymbol{m} = k_{ml}\, \boldsymbol{l}$

the remaining plane is not also parallel to the other two.

So the configuration is “exactly two parallel planes”.

[4 marks]

Answer 3

(a) In matrix form the system of equations will be...

$$\begin{pmatrix} 1 & \lambda & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$$

Expanding along the upper row to obtain the determinant...

$$\begin{aligned} & 1\,((1)(2) - (-2)(5)) - \lambda\,((2)(2) - (1)(5)) + 3\,((2)(-2) - (1)(1)) \\ &= (2 + 10) - \lambda(4 - 5) + 3(-4 - 1) \\ &= 12 + \lambda - 15 \\ &= \lambda - 3 \end{aligned}$$

The determinant will be zero if $\lambda = 3$.

So the planes will meet in a unique point except when $\lambda = 3$.

[3 marks]

(b) With $\lambda = -2$ we have,

$$\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -12 & 2 & 13 \\ -1 & 1 & -1 \\ 5 & 0 & -5 \end{pmatrix}\begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

Thus, the point of intersection is $(\,1,\ 2, -3\,)$

[3 marks]

Answer 4

$$\begin{pmatrix} 1 & 4 & -5 \\ 2 & 3 & 1 \\ 3 & 7 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 5 \end{pmatrix}$$

Determinant :

$$1((3)(-4) - (7)(1)) - 4((2)(-4) - (3)(1)) - 5((2)(7) - (3)(3))$$

$$= (-12 - 7) - 4(-8 - 3) - 5(14 - 9)$$

$$= -19 + 44 - 25$$

$$= 0$$ So planes do not meet in a single point

The normals are $\boldsymbol{u} = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}$, $\boldsymbol{m} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{l} = \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$

None of these are parallel because there is no constants k_{um}, k_{ul} or k_{ml} for which $\boldsymbol{u} = k_{um}\,\boldsymbol{m}$, $\boldsymbol{u} = k_{ul}\,\boldsymbol{l}$ or $\boldsymbol{m} = k_{ml}\,\boldsymbol{l}$

As none of the normals are in the same direction (parallel), none of the planes are parallel.

Let the equations be upper, U, middle, M, and lower, L, as follows,

$$\begin{array}{rcll} x + 4y - 5z &=& 8 & U \\ 2x + 3y + z &=& -1 & M \\ 3x + 7y - 4z &=& 5 & L \end{array}$$

Working out $U + 5M$

$$\begin{array}{rcl} x + 4y - 5z &=& 8 \\ 10x + 15y + 5z &=& -5 \\ \hline 11x + 19y &=& 3 \end{array}$$

Working out $L + 4M$

$$\begin{array}{rcl} 3x + 7y - 4z &=& 5 \\ 8x + 12y + 4z &=& -4 \\ \hline 11x + 19y &=& 1 \end{array}$$

As these two equations for $11x + 19y$ are inconsistent, the three planes form a prism.

[4 marks]

Answer 5

In matrix form the planes can be expressed as the system of equations given by,

$$\begin{pmatrix} -4 & p & 7 \\ 1 & -2 & 5 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ q \\ 2 \end{pmatrix}$$

To form a sheaf, the determinant must be zero.

$$\begin{aligned} -4((-2)(1) - (3)(5)) - p((1)(1) - (2)(5)) + 7((1)(3) - (2)(-2)) &= 0 \\ -4(-17) - p(-9) + 7 \times (7) &= 0 \\ 68 + 9p + 49 &= 0 \\ 9p &= -117 \\ p &= -13 \end{aligned}$$

Substituting in the value for p gives the equations as,

$$\begin{aligned} -4x - 13y + 7z &= 4 \qquad U \\ x - 2y + 5z &= q \qquad M \\ 2x + 3y + z &= 2 \qquad L \end{aligned}$$

where the equations have be labelled upper, U, middle, M, and lower, L

Working out $7L - U$

$$\begin{aligned} 14x + 21y + 7z &= 14 \\ -4x - 13y + 7z &= 4 \end{aligned}$$

$$\begin{aligned} 18x + 34y &= 10 \\ 9x + 17y &= 5 \end{aligned}$$

Working out $5L - M$

$$\begin{aligned} 10x + 15y + 5z &= 10 \\ x - 2y + 5z &= q \end{aligned}$$

$$9x + 17y = 10 - q$$

For a sheaf these two equations for $9x + 17y$ must be consistent.

Thus, $q = 5$

[6 marks]

Answer 6

(a) With $k = -1$ the system of equations is,

$$\begin{pmatrix} -1 & 1 & -2 \\ 2 & 3 & -6 \\ 3 & -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}$$

Determinant:

$$(-1)((3)(5) - (-2)(-6)) - 1((2)(5) - (3)(-6)) - 2((2)(-2) - (3)(3))$$

$$= (-1)(15 - 12) - 1(10 + 18) - 2(-4 - 9)$$

$$= -3 - 28 + 26$$

$= -5$ so the three planes will intersect in a unique point.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 1 & 0 \\ 28 & -1 & 10 \\ 13 & -1 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

Thus, there is a point of intersection at $(-1,\ 3,\ 2)$

[3 marks]

(b) With $k = \dfrac{2}{3}$ the system of equations is,

$$\begin{pmatrix} \frac{2}{3} & 1 & -2 \\ 2 & 3 & -6 \\ 3 & -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}$$

Determinant:

$$\left(\frac{2}{3}\right)((3)(5) - (-2)(-6)) - 1((2)(5) - (3)(-6)) - 2((2)(-2) - (3)(3))$$

$$= \left(\frac{2}{3}\right)(15 - 12) - 1(10 + 18) - 2(-4 - 9)$$

$$= 2 - 28 + 26$$

$= 0$ So planes do not meet in a single point

So, we now look at the normals which are,

$$\boldsymbol{u} = \begin{pmatrix} \frac{2}{3} \\ 1 \\ -2 \end{pmatrix} \qquad \boldsymbol{m} = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} \qquad \boldsymbol{l} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

As $\boldsymbol{u} = \dfrac{1}{3}\boldsymbol{m}$ there are at least two parallel planes.

As there are no constants k_{ul} or k_{ml} for which $\boldsymbol{u} = k_{ul}\,\boldsymbol{l}$ or $\boldsymbol{m} = k_{ml}\,\boldsymbol{l}$

the remaining plane is not also parallel to the other two.

So the configuration is “exactly two parallel planes” with the third separately intersecting each in a line.

[4 marks]

Answer 7

In matrix form the planes can be expressed as the system of equations given by,

$$\begin{pmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & b & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ c \end{pmatrix}$$

(a) **(i)** To **not** intersect at a unique point, the determinant must be zero.

$$(-1)((3)(1) - b(1)) - a((2)(1) - (1)(1)) = 0$$

$$-3 + b - a = 0$$

$$b = a + 3$$

[4 marks]

(ii) With $b = a + 3$ (because a sheaf has determinant zero) the equations are,

$$\begin{aligned} -x + \quad & ay & &= 2 & & U \\ 2x + \quad & 3y + z & &= -3 & & M \\ x + \quad & (a+3)y + z & &= c & & L \end{aligned}$$

are labelled upper, U, middle, M, and lower, L

Working out $L - M$

$$\begin{aligned} x + \quad & (a+3)y + z &= c \\ 2x + \quad & 3y + z &= -3 \\ \hline -x + \quad & ay &= c + 3 \end{aligned}$$

For a sheaf this result must be consistent with the upper equation.

Thus, $c = -1$

[3 marks]

(b) For $b = a$ and $c = 1$ the system of equations becomes,

$$\begin{pmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & a & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Let $\mathbf{A} = \begin{pmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & a & 1 \end{pmatrix}$

$$|\mathbf{A}| = \begin{vmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & a & 1 \end{vmatrix}$$

$$= (-1) \times \begin{vmatrix} 3 & 1 \\ a & 1 \end{vmatrix} - a \times \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= (-1)(3 - a) - a$$

$$= -3$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 3 & 1 \\ a & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & a \end{vmatrix} \\ \begin{vmatrix} a & 0 \\ a & 1 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -1 & a \\ 1 & a \end{vmatrix} \\ \begin{vmatrix} a & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & a \\ 2 & 3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3-a & 1 & 2a-3 \\ a & -1 & -2a \\ a & -1 & -3-2a \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 3-a & -1 & 2a-3 \\ -a & -1 & 2a \\ a & 1 & -3-2a \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 3-a & -a & a \\ -1 & -1 & 1 \\ 2a-3 & 2a & -3-2a \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{(-3)} \begin{pmatrix} 3-a & -a & a \\ -1 & -1 & 1 \\ 2a-3 & 2a & -3-2a \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{(-3)} \begin{pmatrix} 3-a & -a & a \\ -1 & -1 & 1 \\ 2a-3 & 2a & -3-2a \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$= \frac{1}{(-3)} \begin{pmatrix} 6-2a+3a+a \\ -2+3+1 \\ 4a-6-6a-3-2a \end{pmatrix}$$

$$= \frac{1}{(-3)} \begin{pmatrix} 2a+6 \\ 2 \\ -4a-9 \end{pmatrix}$$

The point of intersection is $\left(\dfrac{2a+6}{(-3)}, \dfrac{2}{(-3)}, \dfrac{-4a-9}{(-3)} \right)$

$$= \left(-\frac{2a+6}{3}, -\frac{2}{3}, \frac{4a-9}{3} \right)$$

[6 marks]

Answer 8

(a) $$\begin{vmatrix} 1 & 5 & 3 \\ 4 & -2 & p \\ 8 & 5 & -11 \end{vmatrix} = 1 \times \begin{vmatrix} -2 & p \\ 5 & -11 \end{vmatrix} - 5 \times \begin{vmatrix} 4 & p \\ 8 & -11 \end{vmatrix} + 3 \times \begin{vmatrix} 4 & -2 \\ 8 & 5 \end{vmatrix}$$

$$= 22 - 5p + 5 \times 44 + 40p + 60 + 48$$

$$= 350 + 35p$$

$$= 35(10 + p) \quad \text{restriction on } p \text{ is that } p \neq -10$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -2 & p \\ 5 & -11 \end{vmatrix} & \begin{vmatrix} 4 & p \\ 8 & -11 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ 8 & 5 \end{vmatrix} \\ \begin{vmatrix} 5 & 3 \\ 5 & -11 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 8 & -11 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 8 & 5 \end{vmatrix} \\ \begin{vmatrix} 5 & 3 \\ -2 & p \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 4 & p \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 4 & -2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 22 - 5p & -44 - 8p & 36 \\ -70 & -35 & -35 \\ 5p + 6 & p - 12 & -22 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 22 - 5p & 44 + 8p & 36 \\ 70 & -35 & 35 \\ 5p + 6 & -p + 12 & -22 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 22 - 5p & 70 & 5p + 6 \\ 44 + 8p & -35 & -p + 12 \\ 36 & 35 & -22 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{35(10 + p)} \begin{pmatrix} 22 - 5p & 70 & 5p + 6 \\ 44 + 8p & -35 & -p + 12 \\ 36 & 35 & -22 \end{pmatrix}$$

[6 marks]

(b) **(i)**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{35(10+p)} \begin{pmatrix} 22-5p & 70 & 5p+6 \\ 44+8p & -35 & -p+12 \\ 36 & 35 & -22 \end{pmatrix} \begin{pmatrix} 5 \\ 24 \\ -30 \end{pmatrix}$$

$$= \frac{1}{35(10+p)} \begin{pmatrix} 110-25p+1680-150p-180 \\ 220+40p-840+30p-360 \\ 180+840+660 \end{pmatrix}$$

$$= \frac{1}{35(10+p)} \begin{pmatrix} 1610-175p \\ -980+70p \\ 1680 \end{pmatrix}$$

$$= \frac{1}{10+p} \begin{pmatrix} 46-5p \\ -28+2p \\ 48 \end{pmatrix}$$

The point of intersection is $\left(\dfrac{46-5p}{10+p}, \dfrac{-28+2p}{10+p}, \dfrac{48}{10+p} \right)$

[4 marks]

(ii) With $p = 2$ the equations become,

$$x + 5y + 3z = 5$$

$$4x - 2y + 2z = 24$$

$$8x + 5y - 11z = -30$$

Showing 1st and 2nd are mutually perpendicular;

$$\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} = 1 \times 4 + 5 \times -2 + 3 \times 2 = 0$$

Showing 1st and 3rd are mutually perpendicular;

$$\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 8 \\ 5 \\ -11 \end{pmatrix} = 1 \times 8 + 5 \times 5 + 3 \times (-11) = 0$$

Showing 2nd and 3rd are mutually perpendicular;

$$\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 8 \\ 5 \\ -11 \end{pmatrix} = 4 \times 8 + (-2) \times 5 + 2 \times (-11) = 0$$

Thus, via the scalar products of all possible pairs of the planes' normal vectors it is shown that the planes are mutually perpendicular.

[4 marks]

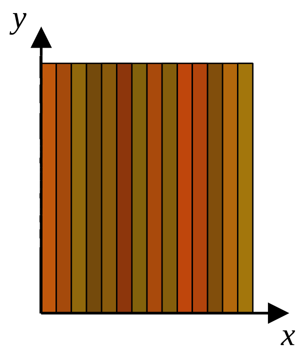
8.1 In The Corner Of A House

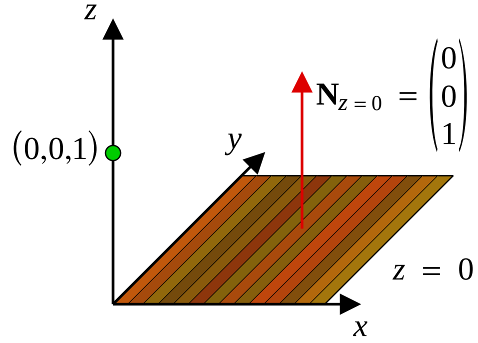
Traditionally, in moving from two dimensions into three, the x-axis stays where is is and the old wall made by the x-axis and the y-axis falls backward to become the floor, with the newly introduced z-axis pointing skyward.

The floor is a surface, and points on that surface have no height.

In other words, the floor is a plane with equation $z = 0$ and the z-axis gives the direction of that plane's normal. The point (0, 0, 1) is on the z-axis and the displacement vector from the origin to that point is a handy version of the (floor) plane's normal. That is,

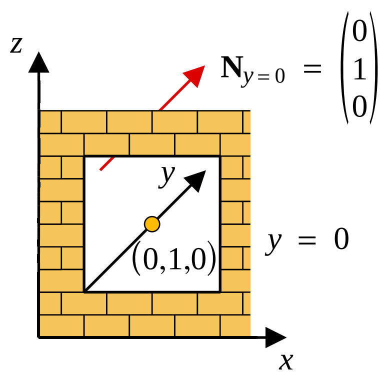
$$\mathbf{N}_{z=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

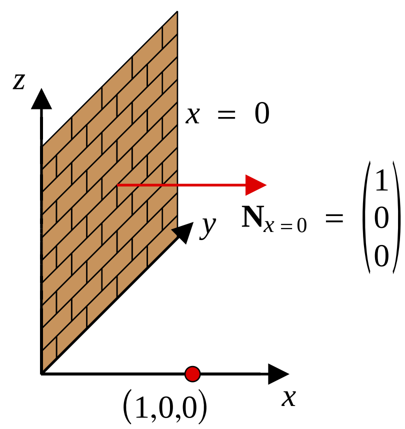




Likewise, the front wall (with a window in it) is the plane with equation $y = 0$. It has normal, $\mathbf{N}_{y=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Finally, the left side wall has equation $x = 0$ and normal $\mathbf{N}_{x=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.





8.2 The Matrix Of Normals

The three normals can be used as the columns of a 3×3 matrix,

$$\mathbf{N}_{x=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{N}_{y=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{N}_{z=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix formed is the 3×3 identity matrix. If any three dimensional point (x, y, z) is multiplied by this matrix it will remain where it is. Now for the clever bit: for a transformation of interest, T, ask “what will T do to the standard three normals”? Then, write down the corresponding matrix of (transformed) normals. You will then have a matrix that will T transform any points you feed it.

8.3 Reflection in $z = 0$ (Example)

Suppose that it is desired to reflect points in the floor, the plane $z = 0$. To work out the matrix that will do this note that the side wall and the front (with a window) walls normal vectors need to be left alone, but the floor's normal vector, instead of pointing upward, will (after the reflection) point downward.

The matrix to reflect in the plane $z = 0$ is now formed like so;

$$\mathbf{N}_{x=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{N}_{y=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{Ref}_{z=0} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

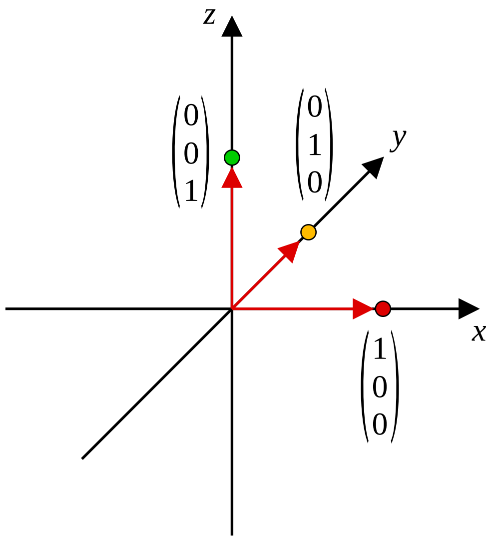
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

8.4 Asleep On The Floor

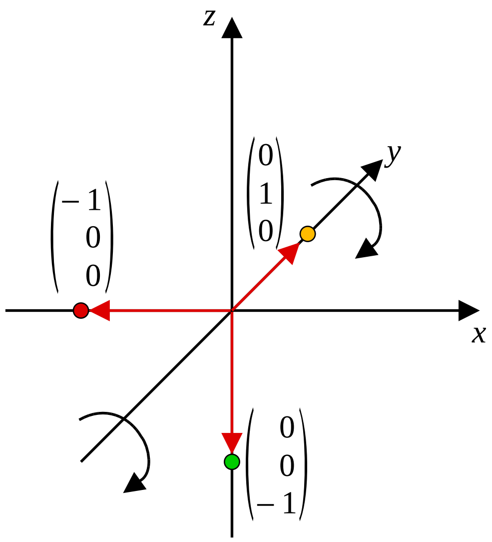
Sleep on the floor : Sleep is $z\ z\ z\ Z\ Z$: The floor is $z = 0$...

8.5 Rotation of 180° about the y-axis

To reduce clutter in diagrams, regard the axes as the normal vectors and view the points $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$ and $(0,\ 0,\ 1)$ as displacement vectors that are the normals to the planes $x = 0$, $y = 0$ and $z = 0$ respectively.



Suppose that it is required that the matrix representing a 180° rotation about the y-axis is determined. Think about where this would send the three normal vectors associated with the red, amber, and green points in the above diagram. After a pause for thought, a diagram like the one below will be in mind.



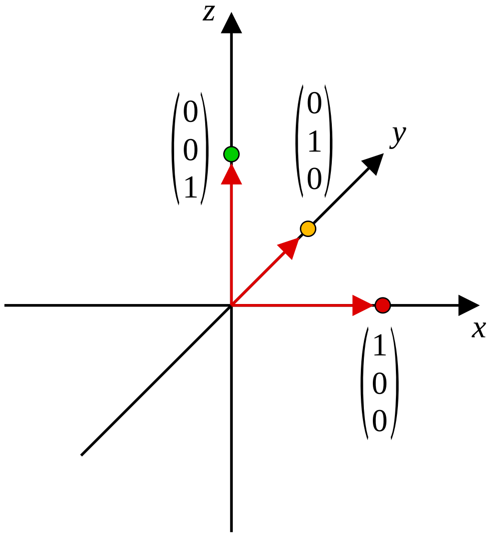
Writing down the matrix of normals in red, amber, green order: $\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$

Multiplying any points by this matrix will now rotate them by 180° about the y-axis. Note that the rotation is anticlockwise when looking down the positive y-axis towards the origin. For 180° is does not matter if you went the wrong way but for other angles, say 90°, it would be important to get that correct.

8.6 Example

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

(i) With the aid of the following diagram, or otherwise, determine the single transformation represented by the matrix **M**.



[3 marks]

(ii) The point $A(3, -1, 4)$ is transformed using this matrix.
Find the coordinates of the image of A.

[1 mark]

(iii) The point $B(a, -a, 2a - 1)$ is transformed to the point with coordinates $(a, a - 5, -a)$ using matrix **M**.
Find the value of a.

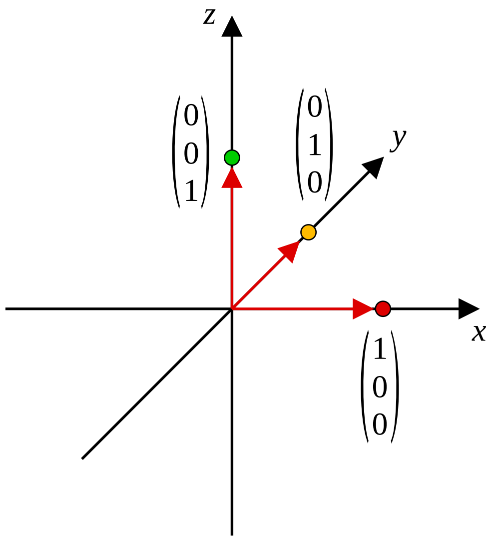
[3 marks]

8.7 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1



With the aid of the above diagram, or otherwise, write down the matrix that will represent,

(i) reflection in the plane $x = 0$

[2 marks]

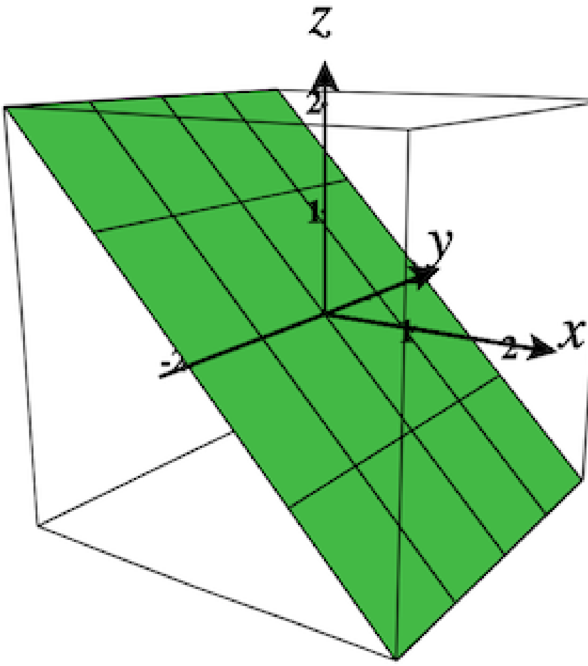
(ii) rotation of 180° about the x-axis

[2 marks]

(iii) rotation of 90° about the y-axis

[2 marks]

Question 2
The three dimensional plot is of the plane with equation $z = -y$



Write down the matrix that will reflect points in the plane $z = -y$

[2 marks]

Question 3
Describe the transformations represented by the following matrices,

(i) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[2 marks]

(ii) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

[2 marks]

(iii) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[2 marks]

Question 4

Further A-Level Examination Question from October 2020, Paper 1, Q3 (OCR)

Your are given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

(a) Find $\mathbf{A}^4$

[1 mark]

(b) Describe the transformation that **A** represents.

[2 marks]

The matrix **B** represents a reflection in the plane $x = 0$

(c) Write down the matrix **B**

[1 mark]

The point P has coordinates $(2,\ 3,\ 4)$.

The point P' is the image of P under the transformation represented by **B**

(d) Find the coordinates of P'

[1 mark]

Question 5

Further A-Level Examination Question from Practice Paper Set 1, Q5 (OCR)

(a) Write down the 3×3 matrix $\mathbf{M}_1$ that represents a reflection in the plane $y = 0$

[1 mark]

(b) Write down the single transformation represented by the matrix $\mathbf{M}_2$

$$\mathbf{M}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

[1 mark]

(c) **(i)** Find the determinants of $\mathbf{M}_1$ and $\mathbf{M}_2$

[2 marks]

(ii) Explain how the signs and magnitudes of these determinants relate to the transformations represented by $\mathbf{M}_1$ and $\mathbf{M}_2$

[2 marks]

(d) **(i)** Find the matrix $\mathbf{M}_3$ where $\mathbf{M}_3 = \mathbf{M}_1\,\mathbf{M}_2$

[1 mark]

(ii) Describe the single transformation represented by $\mathbf{M}_3$

[2 marks]

Question 6
A is the matrix representing a reflection in the plane $x = 0$ and **B** is the matrix representing a reflection in the plane $y = 0$

(i) Write down the matrices **A** and **B**

[2 marks]

(ii) The point $P\,(a,\ b,\ c)$ is transformed using matrix **A**.
Find the coordinates of P' in terms of a, b and c

[2 marks]

(iii) P' is transformed using matrix **B**.
Find the coordinates of the image of P' in terms of a, b and c.

[2 marks]

Question 7
Further A-Level Examination Question from May 2020, Paper 1, Q3 (AQA)
Which one of the matrices below represents a rotation of 90° about the x-axis ?
Circle your answer.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

[2 marks]

Question 8

Further AS-Level examination Question from October 2020, Q4, (OCR)

The matrix **M** is $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(a) **(i)** Calculate det **M**

[1 mark]

(ii) State two geometrical consequences of this value for the transformation associated with **M**.

[2 marks]

(b) Describe fully the transformation associated with **M**.

[1 mark]

8.8 Answers

8.8.1 Solution to 8.6 Example

(i) Rotation of 90° (anticlockwise) about the x-axis.

[3 marks]

(ii) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$$

$$\therefore\ (3, -1, 4) \rightarrow (3, -4, -1)$$

[1 mark]

(iii) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ -a \\ 2a-1 \end{pmatrix} = \begin{pmatrix} a \\ a-5 \\ -a \end{pmatrix}$$

$$\begin{pmatrix} a \\ -2a+1 \\ -a \end{pmatrix} = \begin{pmatrix} a \\ a-5 \\ -a \end{pmatrix}$$

$$-2a + 1 = a - 5$$

$$a = 2$$

[3 marks]

8.8.1 Solutions to 8.7 Exercise

Answer 1

(i) reflection in the plane $x = 0$

$x = 0$ is the (left hand) side wall

$$\mathbf{Ref}_{x=0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[2 marks]

(ii) rotation of 180° about the x-axis

$$\mathbf{Rot}_{180x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

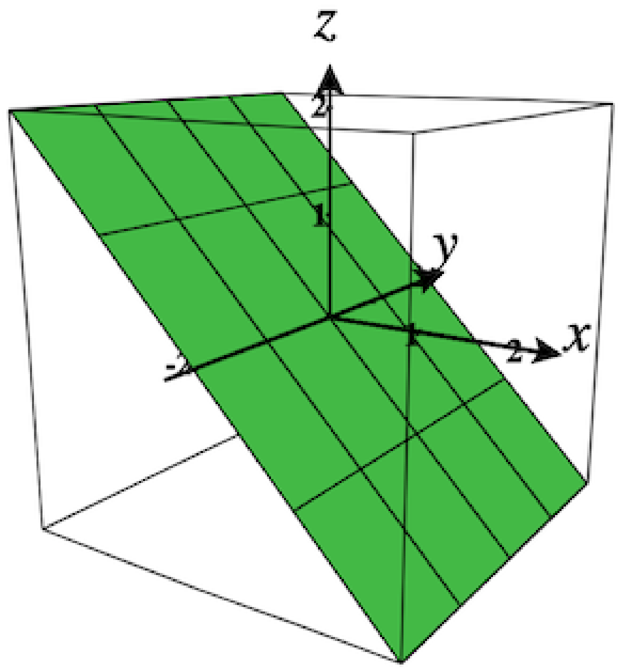
[2 marks]

(iii) rotation of 90° about the y-axis

$$\mathbf{Rot}_{90y} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

[2 marks]

Answer 2



$$\mathbf{Ref}_{z=-y} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

[2 marks]

Answer 3

Describe the transformations represented by the following matrices,

(i) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: Reflection in the plane $y = 0$

[2 marks]

(ii) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$: Reflection in the plane $x = z$

[2 marks]

(iii) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: Rotation of 180° about the z-axis

[2 marks]

Answer 4

(a) $\mathbf{A}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \mathbf{I}$

[1 mark]

(b) A rotation of 270° about the *x*-axis

[2 marks]

(c) $\mathbf{B} = \mathbf{Ref}_{x=0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[2 marks]

(d) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$

[1 mark]

Answer 5

(a) $\quad \mathbf{M}_1 = \mathbf{Ref}_{y=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[1 mark]

(b) $\quad \mathbf{M}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$: Rotation 180° about the x-axis

[1 mark]

(c) **(i)** $\quad |\mathbf{M}_1| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1)(-1)(1) = -1$

$$|\mathbf{M}_2| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = (1)(-1)(-1) = 1$$

[2 marks]

(ii) The magnitude of the determinant has the geometric interpretation of "area scale factor". Both of these matrices would not change the area of a shape being transformed by them.
The sign of the determinant gives the orientation of the image and so $\mathbf{M}_1$ will flip the orientation and $\mathbf{M}_2$ will not.

[2 marks]

(d) **(i)** $\quad \mathbf{M}_3 = \mathbf{M}_1\mathbf{M}_2$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

[1 mark]

(ii) M_3 represents reflection in the plane $z = 0$

[2 marks]

Answer 6

(i) $\mathbf{A} = \mathbf{Ref}_{x=0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\mathbf{B} = \mathbf{Ref}_{y=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[2 marks]

(ii) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ b \\ c \end{pmatrix}$

[2 marks]

(iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ c \end{pmatrix}$

[2 marks]

Answer 7

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

[2 marks]

Answer 8

(a) **(i)** $\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -(-1)(1)(1) = 1$

[1 mark]

(ii) The magnitude of the determinant has the geometric interpretation of "area scale factor". This matrix will not change the area of a shape being transformed by it. The sign of the determinant gives the orientation of the image and this matrix will preserve the orientation.

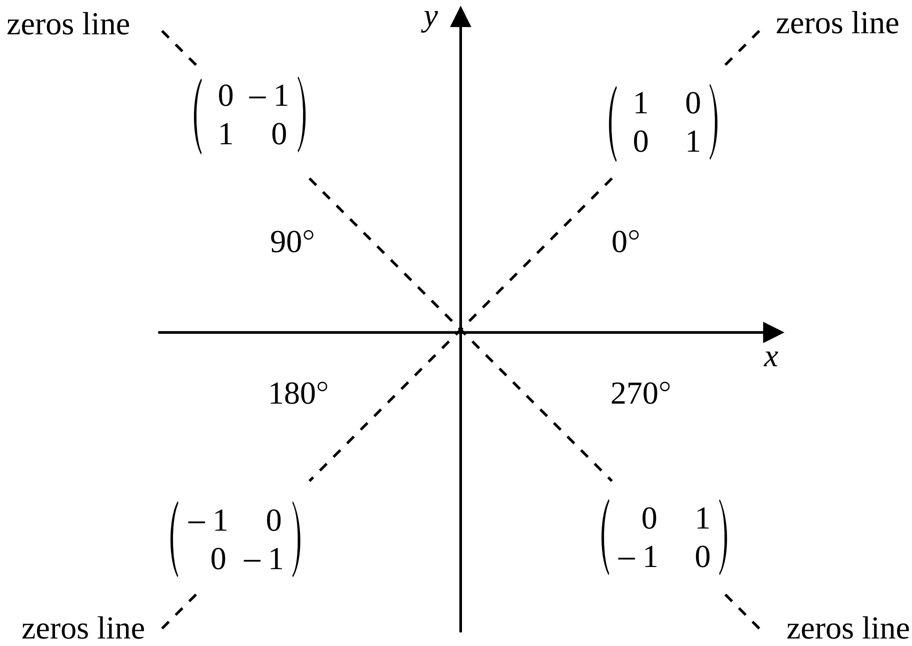
[2 marks]

(b) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: Rotation of $90°$ about the z-axis.

[1 mark]

9.1 Rotations In Two and Three Dimensions

Consider the following diagram. It is an aid to remembering the two dimensional rotation matrices for rotations (anticlockwise) of 0°, 90°, 180° and 270°.



$$\mathbf{I} = \mathbf{R}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{R}_{90} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{R}_{180} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbf{R}_{270} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

These matrices are special cases of the following general result,

---

Rotation through angle θ about (0, 0)

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

This generalised result was proven in *Matrix Transformations*, Lesson 6.

Proof: https://www.NumberWonder.co.uk/Download/PD9090/9090(6).pdf

The two dimensional rotation matrix, has a centre of rotation about the origin. To rotate about an other point would involve translating the other point to the origin, doing the rotation there, and then translating the origin back to the other point. In three dimensions, the rotation is naturally about either the x, the y or the z axis. For each of the three axes, we'll now give the special rotations of 0°, 90°, 180° and 270° about the axis, followed by the general rotation matrix for that axis. Keep in mind the generalised matrix for two dimensional rotation because, in moving up a dimension, it is this 2 x 2 matrix that is embedded with in the more general 3 x 3 rotation matrix.

- Rotation by θ about the x-axis;

$$\mathbf{R}_{x,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_{x,90} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{R}_{x,180} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \mathbf{R}_{x,270} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\mathbf{R}_{x,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

- Rotation by θ about the y-axis;

$$\mathbf{R}_{y,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_{y,90} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_{y,180} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \mathbf{R}_{y.270} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

- Rotation by θ about the z-axis;

$$\mathbf{R}_{z,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_{z,90} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_{z,180} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_{z,270} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9.2 Example

Further A-Level Examination Question from June 2020, Paper 2, Q9 (AQA)

The matrix $\mathbf{C} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where a and b are positive real numbers, and

$$\mathbf{C}^2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Use **C** to show that $cos\left(\frac{\pi}{12}\right)$ can be written in the form $\frac{\sqrt{\sqrt{m} + n}}{2}$, where m and n are integers.

[7 marks]

9.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Further AS-Level Examination Question from November 2021, Q3

The matrix **M** represents a rotation about the x-axis.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & \frac{\sqrt{3}}{2} \\ 0 & b & -\frac{1}{2} \end{pmatrix}$$

Determine the value of a and the value of b

[2 marks]

Question 2

Further AS-Level Examination Question from November 2020, Q6

Anna has been asked to describe the transformation given by the matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

She writes her answer as follows:

The transformation is a rotation about the x-axis through an angle of θ, where

$$\sin\theta = \frac{1}{2} \quad \text{and} \quad -\sin\theta = -\frac{1}{2}$$

$$\theta = 30°$$

Identify and correct the error in Anna's work.

[2 marks]

Question 3

Further AS-Level Examination Question from June 2018, Q5 (AQA)

Describe fully the transformation given by the matrix $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[3 marks]

Question 4

Further A-Level Examination Question from June 2019, Paper 1, Q7 (AQA)

Three non-singular square matrices, **A**, **B** and **R** are such that

$$\mathbf{AR} = \mathbf{B}$$

The matrix **R** represents a rotation about the z-axis through an angle θ and

$$\mathbf{B} = \begin{pmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Show that **A** is independent of the value of θ

[3 marks]

(b) Give a full description of the single transformation represented **A**

[1 mark]

Question 5

$$\mathbf{M} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Describe the transformation represented by **M**

[3 marks]

(b) Find the image of the point with coordinates $(-1, -2, 1)$ under the transformation represented by **M**

[2 marks]

Question 6

P is the matrix representing a rotation of 120° anticlockwise about the z-axis.

(a) Write down the matrix **P**

[1 mark]

(b) A point $Q = (3, -1, 0)$ is transformed using the matrix **P**.
Find the coordinates of the image of Q.

[1 mark]

(c) A point $R = (k, 0, k)$ is transformed using matrix **P**.
Find, in terms of k, the exact coordinates of the image of R.

[3 marks]

Question 7

The matrix $\mathbf{C} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where a and b are positive real numbers, and

$$\mathbf{C}^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Use $\mathbf{C}$ to show that $cos\left(\frac{\pi}{8}\right)$ can be written in the form $\frac{\sqrt{\sqrt{m} + n}}{2}$, where m and n are integers.

[7 marks]

Question 8

$$\mathbf{M} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Find the transformation represented by **M**

[3 marks]

(b) The point with coordinates (k, $-k$. 0) is transformed using **M**.
Find, in terms of k, the exact coordinates of the image of this point.

[3 marks]

Question 9

The matrix $\mathbf{M} = \begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix}$ represents a rotation followed by an enlargement.

(a) Find the scale factor of the enlargement.

[2 marks]

(b) Find the angle of rotation.

[3 marks]

A point P is mapped onto a point P' under **M**.
Given that the coordinates of P' are (a, b),

(c) find, in terms of a and b, the coordinates of P.

[4 marks]

Question 10

The formula for the volume of a tetrahedron is,

$$V_{TET} = \frac{1}{3} \times \textit{base area} \times \textit{height}$$

(a) Write down the matrix representing a rotation of 315° anticlockwise about the *y*-axis.

[2 marks]

A tetrahedron *T* has vertices at $(1,\ 0,\ -1)$, $(1,\ 1,\ -1)$, $(3,\ 2,\ 3)$ and $(0,\ 0,\ 0)$.

(b) Find the images of the vertices of the tetrahedron under the transformation described in part (a).

[3 marks]

(c) Hence find the volume of *T*.

[2 marks]

9.4.1 Solution to 9.2 Example

$\mathbf{C}^2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ is in the form of a rotation matrix, $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

with $\cos\theta = \dfrac{\sqrt{3}}{2} \quad \Rightarrow \theta = \dfrac{\pi}{6}, \dfrac{11\pi}{6}$

and $\sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\therefore$ $\mathbf{C}^2$ represents a rotation of $\dfrac{\pi}{6}$ and so $\mathbf{C}$ must represent a rotation of $\dfrac{\pi}{12}$

$$\mathbf{C}^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{pmatrix}$$

$$a^2 - b^2 = \frac{\sqrt{3}}{2} \qquad \text{and} \qquad 2ab = \frac{1}{2}$$

$$b = \frac{1}{4a}$$

$$a^2 - \left(\frac{1}{4a}\right)^2 = \frac{\sqrt{3}}{2}$$

$$a^2 - \frac{1}{16a^2} = \frac{\sqrt{3}}{2}$$

$$16a^4 - 1 = 8\sqrt{3}\,a^2$$

$$16a^4 - 8\sqrt{3}\,a^2 - 1 = 0$$

$$a^2 = \frac{8\sqrt{3} \pm \sqrt{64 \times 3 - 4 \times 16 \times (-1)}}{32}$$

$$= \frac{8\sqrt{3} \pm 16}{32}$$

$$= \frac{\sqrt{3} + 2}{4} \qquad \text{as } a > 0$$

$$a = \frac{\sqrt{\sqrt{3} + 2}}{2}$$

$$\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{\sqrt{3} + 2}}{2} \quad \text{which is in the requested form}$$

[7 marks]

9.4.2 Solutions to 9.3 Exercise

Answer 1

Compare, $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & \frac{\sqrt{3}}{2} \\ 0 & b & -\frac{1}{2} \end{pmatrix}$ with $\mathbf{R}_{x,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

$$a = -\frac{1}{2} \text{ and } b = -\frac{\sqrt{3}}{2}$$

[2 marks]

Answer 2

Anna should be making a comparison between the matrix in the question,

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$ and $\mathbf{R}_{x,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

Anna should have deduced that $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2}$

The first gives $\theta = 30°, 150°$ and the second gives $\theta = 150°, 210°$

So the value of θ that satisfies both equations is $\theta = 150°$

[2 marks]

Answer 3

The matrix given is recognized as being a rotation about the *z*-axis.

So, compare $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\mathbf{R}_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

from which is deduced that, $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = -\frac{1}{2}$

The first gives $\theta = 60°, 120°$ and the second gives $\theta = 120°, 240°$

So the value of θ that satisfies both equations is $\theta = 120°$

The matrix is a rotation about the *z*-axis of 120°

[3 marks]

Answer 4

(a) **R** must be of the form $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The inverse of **R** must be a rotation about the *z*-axis through an angle $(-\theta)$

$$\therefore \mathbf{R}^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(As $\cos\theta$ is an even function, and $\sin\theta$ is odd)

$$\mathbf{AR} = \mathbf{B}$$

$$\mathbf{AR}\,\mathbf{R}^{-1} = \mathbf{B}\,\mathbf{R}^{-1}$$

$$\mathbf{A} = \mathbf{B}\,\mathbf{R}^{-1}$$

$$= \begin{pmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\cos^2\theta - \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ which shows **A** is independent of θ

[3 marks]

(b) **A** is a reflection in the plane $x = 0$

[1 mark]

Answer 5

(a) The matrix given is recognized as being a rotation about the y-axis.

So, compare $\begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$ with $\mathbf{R}_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

from which is deduced that, $\sin\theta = \dfrac{1}{2}$ and $\cos\theta = \dfrac{\sqrt{3}}{2}$

The first gives $\theta = 30°, 150°$ and the second gives $\theta = 30°, 330°$

So the value of θ that satisfies both equations is $\theta = 30°$

The matrix is a rotation about the y-axis of 30°

[3 marks]

(b) $$\begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} + \frac{1}{2} \\ -2 \\ \frac{1}{2} + \frac{\sqrt{3}}{2} \end{pmatrix}$$

The image of the point $(-1, -2, 1)$ is $\left(\dfrac{1-\sqrt{3}}{2}, -2, \dfrac{1+\sqrt{3}}{2}\right)$

[2 marks]

Answer 6

(a) $$\mathbf{P} = \mathbf{R}_{z,120} = \begin{pmatrix} \cos 120 & -\sin 120 & 0 \\ \sin 120 & \cos 120 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[1 mark]

(b) $$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} + \frac{\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} + \frac{1}{2} \\ 0 \end{pmatrix}$$

The image of the point $(3, -1, 0)$ is $\left(\dfrac{-3+\sqrt{3}}{2}, \dfrac{3\sqrt{3}+1}{2}, 0\right)$

[1 mark]

(c) $$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} k \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} -\frac{k}{2} \\ \frac{k\sqrt{3}}{2} \\ k \end{pmatrix}$$

The image of the point $(k, 0, k)$ is $\left(-\dfrac{k}{2}, \dfrac{\sqrt{3}k}{2}, k\right)$

[1 mark]

Answer 7

$\mathbf{C}^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is in the form of a rotation matrix, $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

with $\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}, \frac{7\pi}{4}$

and $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$\therefore$ $\mathbf{C}^2$ represents a rotation of $\frac{\pi}{4}$ and so $\mathbf{C}$ must represent a rotation of $\frac{\pi}{8}$

$$\mathbf{C}^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{pmatrix}$$

$$a^2 - b^2 = \frac{1}{\sqrt{2}} \quad \text{and} \quad 2ab = \frac{1}{\sqrt{2}}$$

$$b = \frac{1}{2\sqrt{2}\,a}$$

$$a^2 - \left(\frac{1}{2\sqrt{2}a}\right)^2 = \frac{1}{\sqrt{2}}$$

$$a^2 - \frac{1}{8a^2} = \frac{\sqrt{2}}{2}$$

$$8a^4 - 1 = 4\sqrt{2}\,a^2$$

$$8a^4 - 4\sqrt{2}\,a^2 - 1 = 0$$

$$a^2 = \frac{4\sqrt{2} \pm \sqrt{16 \times 2 - 4 \times 8 \times (-1)}}{16}$$

$$= \frac{4\sqrt{2} \pm 8}{16}$$

$$= \frac{\sqrt{2} + 2}{16} \quad \text{as } a > 0$$

$$a = \frac{\sqrt{\sqrt{2} + 2}}{4}$$

$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{\sqrt{2} + 2}}{4} \text{ which is in the requested form}$$

[7 marks]

Answer 8

(a) The matrix given is recognized as being a rotation about the y-axis.

So, compare $\begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}$ with $\mathbf{R}_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

from which is deduced that, $\sin\theta = -\dfrac{1}{2}$ and $\cos\theta = -\dfrac{\sqrt{3}}{2}$

The first gives $\theta = 210°, 330°$ and the second gives $\theta = 150°, 210°$

So the value of θ that satisfies both equations is $\theta = 210°$

The matrix is a rotation about the y-axis of $210°$

[3 marks]

(b) $$\begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} k \\ -k \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}\,k}{2} \\ -k \\ \frac{k}{2} \end{pmatrix}$$

The image of the point $(k, -k, 0)$ is $\left(\dfrac{-\sqrt{3}\,k}{2}, -k, \dfrac{k}{2}\right)$

[3 marks]

Answer 9

(a) $$|\mathbf{M}| = \begin{vmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{vmatrix} = 2\sqrt{3} \times 2\sqrt{3} - 2 \times (-2) = 16$$

The determinant is the area scale factor so the length scale factor is 4

[2 marks]

(b) Thus we have that,

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{R}_\theta = \begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix}$$

The opposite of an enlargement by 4 is an enlargement by $\dfrac{1}{4}$

$$\mathbf{R}_\theta = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \text{ compared with } \mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

from which is deduced that, $\sin\theta = \dfrac{1}{2}$ and $\cos\theta = \dfrac{\sqrt{3}}{2}$

So the value of θ that satisfies both equations is $\theta = 30°$

[3 marks]

(c) Let the coordinates of P be (x, y) in which case...

$$\begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 330 & -\sin 330 \\ \sin 330 & \cos 330 \end{pmatrix}\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{\sqrt{3}}{8} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} \sqrt{3}\,a + b \\ -a + \sqrt{3}\,b \end{pmatrix}$$

So P has the coordinates $\left(\dfrac{\sqrt{3}\,a + b}{8}, \dfrac{-a + \sqrt{3}\,b}{8} \right)$

[4 marks]

Answer 10

(a) $\mathbf{R}_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

So $\mathbf{R}_{y,315} = \begin{pmatrix} \cos 315 & 0 & \sin 315 \\ 0 & 1 & 0 \\ -\sin 315 & 0 & \cos 315 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$

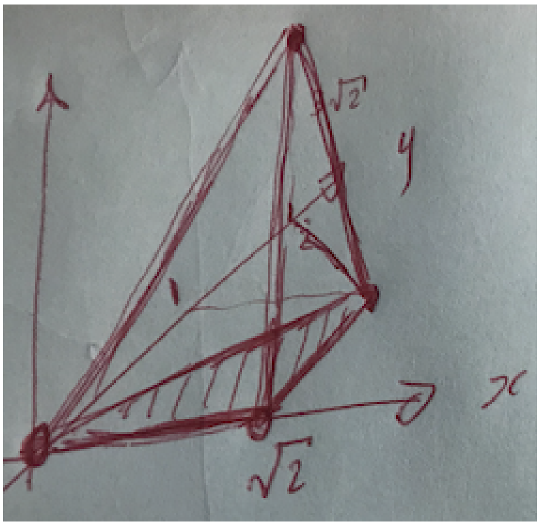
[2 marks]

(b) $\begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & -1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3\sqrt{2} & 0 \end{pmatrix}$

That is,
$$(1,\ 0,\ -1) \to \left(\sqrt{2},\ 0,\ 0\right)$$
$$(1,\ 1,\ -1) \to \left(\sqrt{2},\ 1,\ 0\right)$$
$$(3,\ 2,\ 3) \to \left(0,\ 2,\ 3\sqrt{2}\right)$$
$$(0,\ 0,\ 0) \to (0,\ 0,\ 0)$$

[3 marks]

(c)

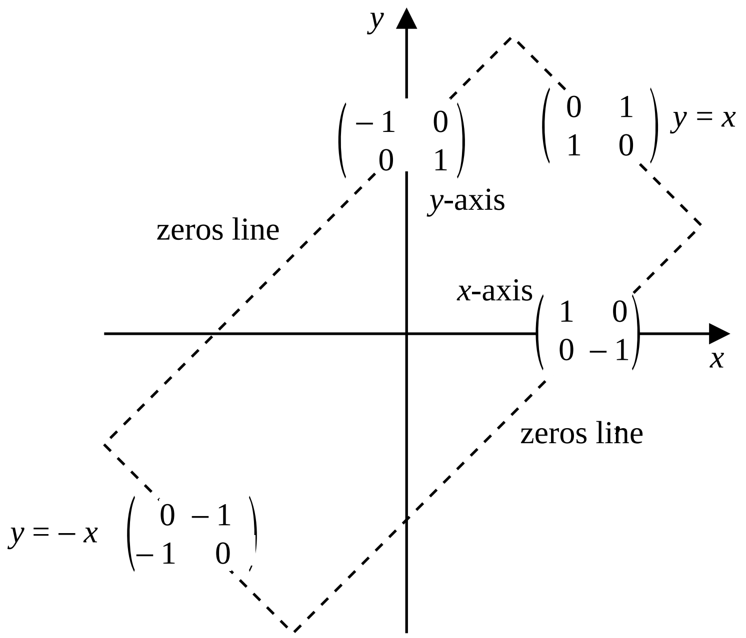


$$V_{TET} = \frac{1}{3} \times \textit{base area} \times \textit{height}$$
$$= \frac{1}{3} \times \frac{1}{2} \times \sqrt{2} \times 1 \times 3\sqrt{2}$$
$$= 1$$

[2 marks]

10.1 Reflections In Two and Three Dimensions

Consider the following diagram. It is an aid to remembering the two dimensional reflection matrices for reflections in the lines $y = x$, $y = -x$, the x-axis and y-axis.



$$\mathbf{M}_{y=x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{M}_{y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{M}_{y=-x} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{M}_{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices are special cases of the following general result,

---

Reflection in the mirror line through (0, 0) at angle θ

$$\mathbf{M}_{\theta} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

The following video briefly shows one way of proving this result, leaving it to the interested viewer to properly work through the steps shown.

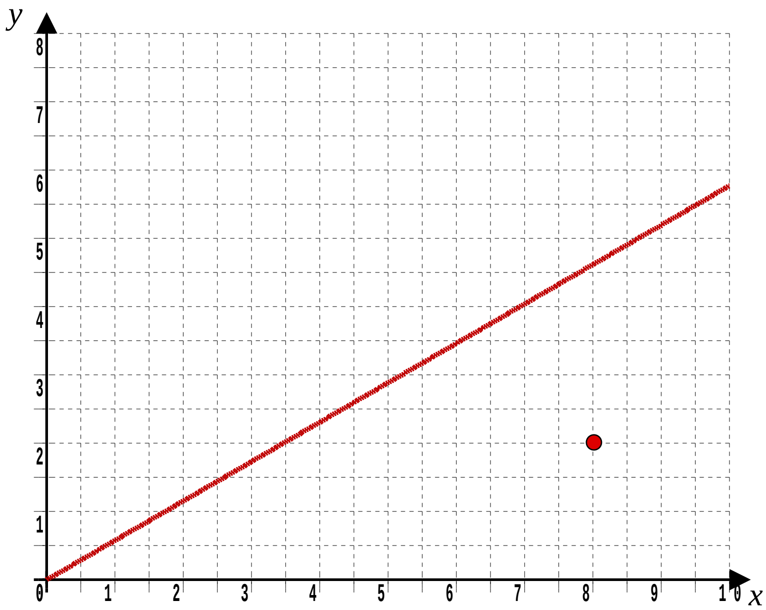
https://www.NumberWonder.co.uk/v9095/10.mp4

10.2 Example

(a) Use matrix methods to work out the exact coordinates of the image when the point (8, 2) is reflected in the line with equation $y = \dfrac{1}{\sqrt{3}}\,x$

[4 marks]

(b) Mark the image of *P* on the diagram below.



[1 mark]

10.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 58

Question 1

$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(i) Describe fully the transformations represented by the matrices **A** and **B**

[4 marks]

(ii) The point (p, q) is transformed by the matrix product **AB**
Give the coordinates of the image of this point in terms of p and q.

[2 marks]

Question 2

(i) Calculate $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}^2$

[2 marks]

(ii) Give a geometric interpretation of your part (i) answer.

[1 mark]

(iii) Hence, or otherwise, determine, $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^{11}$

[2 marks]

Question 3

Further AS-Level Specimen Examination Paper, 2017, Q4 (AQA)

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(a) Find the value of k for which matrix **A** is singular.

[1 mark]

(b) Describe the transformation represented by matrix **B**

[1 mark]

(c) **(i)** Given that **A** and **B** are both non-singular, verify that,

$$\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$$

[4 marks]

(ii) Prove the result $\mathbf{M}^{-1}\mathbf{N}^{-1} = (\mathbf{NM})^{-1}$ for all non-singular square matrices **M** and **N** of the same size.

[4 marks]

Question 4

Advanced Higher Examination Question from May 2015, Q11 (SQA)

(i) Write down the 2×2 matrix, $\mathbf{M}_1$, associated with a reflection in the y-axis.

[1 mark]

(ii) Write down a second 2×2 matrix, $\mathbf{M}_2$, associated with an anticlockwise rotation through an angle of $\frac{\pi}{2}$ radians about the origin.

[1 mark]

(iii) Find the 2×2 matrix, $\mathbf{M}_3$, associated with an anticlockwise rotation through $\frac{\pi}{2}$ radians about the origin followed by a reflection in the y-axis.

[1 mark]

(iv) What single transformation is associated with $\mathbf{M}_3$?

[1 mark]

Question 5

Find the 3×3 matrix representing the single transformation that is equivalent to a reflection in the plane $x = 0$, followed by a rotation of 270° about the y-axis, followed by a reflection in the plane $y = 0$

[4 marks]

Question 6

Further AS-Level Examination Question from October 2020, Q8 (MEI)

(a) The matrix **M** is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(i) Find $\mathbf{M}^2$

[1 mark]

(ii) Write down the transformation represented by **M**

[1 mark]

(iii) Hence state the geometrical significance of the result of part (i)

[1 mark]

(b) The matrix **N** is $\begin{pmatrix} k+1 & 0 \\ k & k+2 \end{pmatrix}$, where k is a constant.

Using determinants, investigate whether **N** can represent a reflection.

[4 marks]

Question 7

Further AS-Level Examination Question from May 2018, Q5 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(a) Describe fully the single geometric transformation U represented by $\mathbf{A}$

[3 marks]

The transformation V, represented by the 2×2 matrix $\mathbf{B}$, is a reflection in $y = -x$

(b) Write down the matrix $\mathbf{B}$

[1 mark]

Given that U followed by V is the transformation T, represented by the matrix $\mathbf{C}$,

(c) find the matrix C

[2 marks]

(d) Show that there is a real number k for which $(1, k)$ is invariant under T

[4 marks]

Question 8

Further A-Level Examination Question from June 2019, Q11 (MEI)

(a) Specify fully the transformation represented by the following matrices,

$$\mathbf{M}_1 = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \qquad \mathbf{M}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[4 marks]

(b) Find the equation of the mirror line of the reflection *R* represented by the matrix $\mathbf{M}_3 = \mathbf{M}_1\,\mathbf{M}_2$

[5 marks]

(c) It is claimed the reflection represented by the matrix $\mathbf{M}_4 = \mathbf{M}_2\,\mathbf{M}_1$ has the same mirror line as *R*. Explain whether or not this claim is correct.

[3 marks]

10.4 Answers

10.4.1 Solution to 10.2 Example

(a) The angle of the line with the positive x-axis is given by,

$$\theta = arctan\left(\frac{1}{\sqrt{3}}\right)$$

$$= 30° \qquad \text{or } \frac{\pi}{6} \text{ radians}$$

The matrix to reflect in the line $y = \frac{1}{\sqrt{3}} x$ is thus,

$$\mathbf{M} = \begin{pmatrix} cos\, 60 & sin\, 60 \\ sin\, 60 & -cos\, 60 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Using this matrix to move the point,

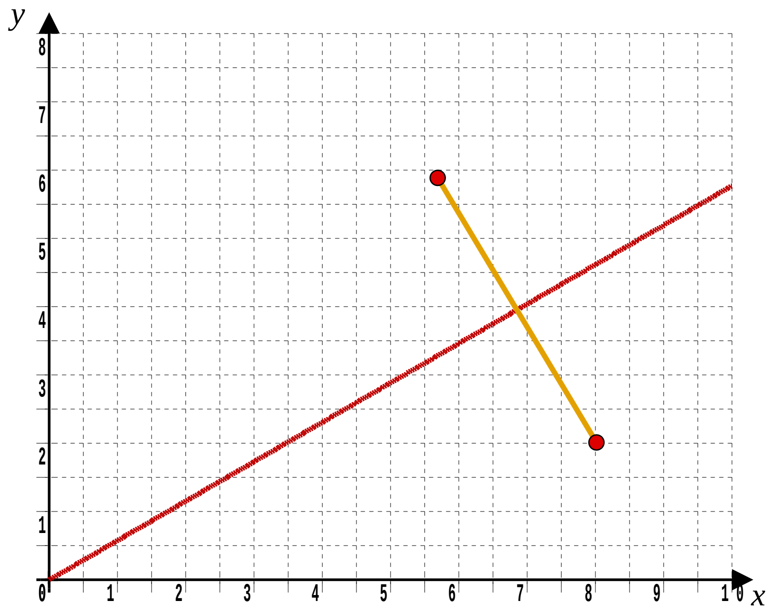
$$\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 + \sqrt{3} \\ 4\sqrt{3} - 1 \end{pmatrix}$$

$$(8,\ 2) \rightarrow (4 + \sqrt{3},\ 4\sqrt{3} - 1)$$

[4 marks]

(b) (8, 2) → (5.7, 5.9) for plotting on the graph.



[1 mark]

10.4.2 Solutions to 10.3 Exercise

Answer 1

(i) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: Reflection in the line $y = x$

$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: Rotation of 270° about (0, 0)

[4 marks]

(ii)

$$\mathbf{AB}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$= \begin{pmatrix} -p \\ q \end{pmatrix}$$

$$(p,\ q) \rightarrow (-p,\ q)$$

[2 marks]

Answer 2

(i)

$$\begin{pmatrix} cos\ 2\theta & sin\ 2\theta \\ sin\ 2\theta & -cos\ 2\theta \end{pmatrix}\begin{pmatrix} cos\ 2\theta & sin\ 2\theta \\ sin\ 2\theta & -cos\ 2\theta \end{pmatrix}$$

$$= \begin{pmatrix} cos^2\ 2\theta + sin^2\ 2\theta & cos\ 2\theta\ sin\ 2\theta - sin\ 2\theta\ cos\ 2\theta \\ sin\ 2\theta\ cos\ 2\theta - cos\ 2\theta\ sin\ 2\theta & sin^2\ 2\theta + cos^2\ 2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

[2 marks]

(ii) Reflecting any point in a mirror, and then reflecting it back is the same as doing nothing, as indicated by the identity matrix answer in part (i)

[1 mark]

(iii)

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^{11} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

[2 marks]

Answer 3

(a) A matrix is singular when its determinant is zero.

$$|\mathbf{A}| = 0$$

$$\begin{vmatrix} 1 & 2 \\ 1 & k \end{vmatrix} = 0$$

$$k - 2 = 0$$

$$k = 2$$

[1 mark]

(b) $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$: Reflection in the y-axis

[1 mark]

(c) **(i)** Verify that $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$

$$\text{LHS} = \mathbf{A}^{-1}\mathbf{B}^{-1}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

$$= \frac{1}{k-2}\begin{pmatrix} k & -2 \\ -1 & 1 \end{pmatrix} \frac{1}{(-1)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{k-2}\begin{pmatrix} k & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{showing B is self inverse}$$

$$= \frac{1}{k-2}\begin{pmatrix} -k & -2 \\ 1 & 1 \end{pmatrix} \qquad k \neq 2$$

$$\text{RHS} = (\mathbf{BA})^{-1}$$

$$= \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} -1 & -2 \\ 1 & k \end{pmatrix}^{-1}$$

$$= \frac{1}{-k+2}\begin{pmatrix} k & 2 \\ -1 & -1 \end{pmatrix}$$

$$= \frac{1}{k-2}\begin{pmatrix} -k & -2 \\ 1 & 1 \end{pmatrix} \qquad k \neq 2$$

$$= \text{LHS}$$

Thus, for matrices **A** and **B** the claimed result is true.

[4 marks]

(ii) If it is true that $\mathbf{M}^{-1}\mathbf{N}^{-1} = (\mathbf{NM})^{-1}$

then $\mathbf{NM} \times \mathbf{M}^{-1}\mathbf{N}^{-1}$ will yield the identity matrix, **I**

$$\begin{aligned} \text{LHS} &= \mathbf{NM} \times \mathbf{M}^{-1}\mathbf{N}^{-1} \\ &= \mathbf{N}\left(\mathbf{M}\ \mathbf{M}^{-1}\right)\mathbf{N}^{-1} \\ &= \mathbf{N}\,\mathbf{I}\,\mathbf{N}^{-1} \\ &= \mathbf{N}\,\mathbf{N}^{-1} \\ &= \mathbf{I} \\ &= \text{RHS} \end{aligned}$$

So the claimed result is indeed true.

[4 marks]

Answer 4

(i) Reflection in the y-axis : $\mathbf{M}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

[1 mark]

(ii) Rotation of $\frac{\pi}{2}$ radians about the origin : $\mathbf{M}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

[1 mark]

(iii) $\mathbf{M}_3 = \mathbf{M}_1\,\mathbf{M}_2$ (Notice the order)

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

[1 mark]

(iv) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: Reflection in the line $y = x$

[1 mark]

Answer 5

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

[4 marks]

Answer 6

(a) **(i)** $\mathbf{M}^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

[1 mark]

(ii) $\mathbf{M} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$: Reflection in the line $y = -x$

[1 mark]

(iii) The opposite of a reflection is the same reflection.
For example, a point is reflected, and then reflected back to its original location.

[1 mark]

(b) $|\mathbf{N}| = \begin{vmatrix} k+1 & 0 \\ k & k+2 \end{vmatrix}$

$= (k+1)(k+2) - k \times 0$

$= (k+1)(k+2)$

If **N** represents a reflection then $|\mathbf{N}| = -1$

$\therefore$ we require $(k+1)(k+2) = -1$

$k^2 + 3k + 2 = -1$

$k^2 + 3k + 3 = 0$

But this has no solutions (it has a negative discriminant)

So **N** can never represent a reflection.

[4 marks]