1.5 Answers

Answer 1

Statement

For all positive integers n, $n^3 + 2n$ is divisible by 3

Proof by Induction

Let $f(n) = n^3 + 2n, \quad n \in \mathbb{Z}^+$

When $n = 1, \quad f(1) = 1^3 + 2 \times 1$

$= 3$ which is divisible by 3

Assume that when $n = k$, $f(k) = k^3 + 2k$ is divisible by 3
in which case...

$$\begin{aligned} f(k+1) &= (k+1)^3 + 2(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= 3k^2 + 3k + 3 + k^3 + 2k \\ &= 3\left(k^2 + k + 1\right) + f(k) \end{aligned}$$

$\therefore f(k+1)$ is divisible by 3

Therefore, if $f(n)$ is divisible by 3 when $n = k$,

then $f(n)$ is also divisible by 3 when $n = k + 1$

As $f(n)$ is divisible by 3 when $n = 1$, $f(n)$ is also divisible by 3 for all $n \in \mathbb{Z}^+$
by mathematical induction. $\square$

[6 marks]

Answer 2

Statement

For all positive integers n, $3^{2n} - 1$ is divisible by 8

Proof by Induction

Let $f(n) = 3^{2n} - 1, \quad n \in \mathbb{Z}^+$

When $n = 1, \quad f(1) = 3^{2 \times 1} - 1$

$= 8$ which is divisible by 8

Assume that when $n = k, \ f(k) = 3^{2k} - 1$ is divisible by 8
in which case...

$$\begin{aligned} f(k+1) &= 3^{2(k+1)} - 1 \\ &= 3^{2k} \times 3^2 - 1 \\ &= 9 \times 3^{2k} - 1 \\ &= 8\left(3^{2k}\right) + 3^{2k} - 1 \\ &= 8\left(3^{2k}\right) + f(k) \end{aligned}$$

$\therefore f(k+1)$ is divisible by 8

Therefore, if $f(n)$ is divisible by 8 when $n = k$,

then $f(n)$ is also divisible by 8 when $n = k + 1$

As $f(n)$ is divisible by 8 when $n = 1$, $f(n)$ is also divisible by 8 for all $n \in \mathbb{Z}^+$
by mathematical induction. □

[6 marks]

Answer 3

(i) As integers alternate odd, even, odd, even, odd, even...
one of any two consecutive integers must be even.
Therefore the product of any two consecutive integers will divide by 2 □

[1 mark]

(ii) If k is an integer, then $k + 1$ is the next consecutive integer.

The product of any two consecutive integers can be written as $k(k + 1)$

As just explained the product, $k(k + 1)$ is divisible by 2.

Furthermore, the number 51 is divisible by 3.

$\therefore\ 51k(k + 1)$ is divisible by 6

[2 marks]

(iii) **Statement**

For all positive integers n, $17n^3 + 103n$ is divisible by 6

Proof by Induction

Let $f(n) = 17n^3 + 103n, \quad n \in \mathbb{Z}^+$

When $n = 1, \quad f(1) = 17 \times 1^3 + 103 \times 1$

$= 120$ which is divisible by 6

Assume that when $n = k$, $f(k) = 17k^3 + 103k$ is divisible by 6
in which case...

$$\begin{aligned} f(k + 1) &= 17(k + 1)^3 + 103(k + 1) \\ &= 17k^3 + 51k^2 + 51k + 17 + 103k + 103 \\ &= 17k^3 + 103k + 51k^2 + 51k + 120 \\ &= f(k) + 51k(k + 1) + 120 \end{aligned}$$

The three terms on the RHS are all divisible by 6

$\therefore f(k + 1)$ is divisible by 6

Therefore, if $f(n)$ is divisible by 6 when $n = k$,

then $f(n)$ is also divisible by 6 when $n = k + 1$

As $f(n)$ is divisible by 6 when $n = 1$, $f(n)$ is also divisible by 6 for all $n \in \mathbb{Z}^+$ by mathematical induction

□

[6 marks]

Answer 4

(i)

$$\begin{aligned} f(k) &= 13^k - 6^k \\ \therefore\ f(k+1) &= 13^{k+1} - 6^{k+1} \\ &= 13^k \times 13 - 6^k \times 6 \\ &= 6\left(13^k - 6^k\right) + 7\left(13^k\right) \\ &= 6f(k) + 7\left(13^k\right) \qquad \square \end{aligned}$$

[3 marks]

(ii) **Statement**

For all positive integers n, $13^n - 6^n$ is divisible by 7

Proof by Induction

Let $f(n) = 13^n - 6^n$, $n \in \mathbb{Z}^+$

When $n = 1$, $f(1) = 13 - 6$

$= 7$ which is divisible by 7

Assume that when $n = k$, $f(k) = 13^k - 6^k$ is divisible by 7

in which case...

$$\begin{aligned} f(k+1) &= 13^k - 6^k \\ &= 6f(k) + 7\left(13^k\right) \qquad \text{from part (i)} \\ f(k+1) - 6f(k) &= 7\left(13^k\right) \end{aligned}$$

$\therefore f(k+1)$ is divisible by 7

Therefore, if $f(n)$ is divisible by 7 when $n = k$,

then $f(n)$ is also divisible by 7 when $n = k + 1$

As $f(n)$ is divisible by 7 when $n = 1$, $f(n)$ is also divisible by 7 for all

$n \in \mathbb{Z}^+$ by mathematical induction

$\square$

[4 marks]

Answer 5

Statement

For all positive integers n, $3^{2n+4} - 2^{2n}$ is divisible by 5

Proof by Induction

Let $f(n) = 3^{2n+4} - 2^{2n}, \quad n \in \mathbb{Z}^{+}$

When $n = 1, \quad f(1) = 3^{6} - 2^{2}$

$= 725$ which is divisible by 5

Assume that when $n = k$, $f(k) = 3^{2k+4} - 2^{2k}$ is divisible by 5
in which case...

$$
\begin{aligned}
f(k+1) &= 3^{2(k+1)+4} - 2^{2(k+1)} \\
&= 3^{2k+4} \times 3^{2} - 2^{2k} \times 2^{2} \\
&= 9 \times 3^{2k+4} - 4 \times 2^{2k} \\
&= 4\left(3^{2k+4} - 2^{2k}\right) + 5 \times 3^{2k+4} \\
&= 4f(k) + 5\left(3^{2k+4}\right) \\
f(k+1) - 4f(k) &= 5\left(3^{2k+4}\right)
\end{aligned}
$$

$\therefore f(k+1)$ is divisible by 5

Therefore, if $f(n)$ is divisible by 5 when $n = k$,

then $f(n)$ is also divisible by 5 when $n = k + 1$

As $f(n)$ is divisible by 5 when $n = 1$, $f(n)$ is also divisible by 5 for all $n \in \mathbb{Z}^{+}$
by mathematical induction. □

[6 marks]

Answer 6

Statement

For all positive integers n, $3^{2n+3} + 40n - 27$ is divisible by 64

Proof by Induction

Let $f(n) = 3^{2n+3} + 40n - 27, \quad n \in \mathbb{Z}^+$

When $n = 1, \quad f(1) = 3^5 + 40 - 27$

$\qquad = 256 \qquad$ which is divisible by 64

Assume that when $n = k$, $f(k) = 3^{2k+3} + 40k - 27$ is divisible by 64
in which case...

$$
\begin{aligned}
f(k+1) &= 3^{2(k+1)+3} + 40(k+1) - 27 \\
&= 3^{2k+5} + 40k + 40 - 27 \\
&= \left[9 \times 3^{2k+3} \right] + 40k + 13 \\
&= \left[9\left(3^{2k+3} + 40k - 27 \right) - 9 \times 40k + 9 \times 27 \right] + 40k + 13 \\
&= 9f(k) - 320k + 256 \\
&= 9f(k) + 64(4 - 5k)
\end{aligned}
$$

$\therefore \quad f(k+1)$ is divisible by 64

Therefore, if $f(n)$ is divisible by 64 when $n = k$,

then $f(n)$ is also divisible by 64 when $n = k + 1$

As $f(n)$ is divisible by 64 when $n = 1$, $f(n)$ is also divisible by 64 for all $n \in \mathbb{Z}^+$ by mathematical induction. $\square$

[6 marks]