2.3 Answers

Answer 1

Statement

For the sequence with the iterative description, $u_1 = 1, \ \ u_{n+1} = 2u_n + 1$

for all positive integers n, the n^{th} term is given by $u_n = 2^n - 1$

Proof by Induction

When $n = 1, \ \ u_1 = 2^1 - 1 = 1$ which is the correct initial term

Assume that when $n = k, \ \ u_k = 2^k - 1$ in which case...

$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2\left(2^k - 1\right) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[5 marks]

Answer 2

Statement

For the sequence with the iterative description, $u_1 = 7, \ \ u_{n+1} = 2u_n - 1$

for all positive integers n, the n^{th} term is given by $u_n = 3 \times 2^n + 1$

Proof by Induction

When $n = 1, \ \ u_1 = 3 \times 2^1 + 1 = 7$ which is the correct initial term

Assume that when $n = k, \ \ u_k = 3 \times 2^k + 1$ in which case...

$$\begin{aligned} u_{k+1} &= 2u_k - 1 \\ &= 2\left(3 \times 2^k + 1\right) - 1 \\ &= 3 \times 2^{k+1} + 2 - 1 \\ &= 3 \times 2^{k+1} + 1 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[5 marks]

Answer 3

Statement

For the sequence with the iterative description, $u_1 = 6, \quad u_{n+1} = u_n + 2^n + 4$

for all positive integers n, the n^{th} term is given by $u_n = 2^n + 4n$

Proof by Induction

When $n = 1, \quad u_1 = 2^1 + 4 \times 1 = 6$ which is the correct initial term

Assume that when $n = k, \; u_k = 2^k + 4k$ in which case...

$$\begin{aligned} u_{k+1} &= u_k + 2^k + 4 \\ &= 2^k + 4k + 2^k + 4 \\ &= 2 \times 2^k + 4k + 4 \\ &= 2^{k+1} + 4(k+1) \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[5 marks]

Answer 4

Statement

For the sequence with the iterative description, $u_1 = 8, \quad u_{n+1} = 4u_n - 9n$

for all positive integers n, the n^{th} term is given by $u_n = 4^n + 3n + 1$

Proof by Induction

When $n = 1, \quad u_1 = 4^1 + 3 \times 1 + 1 = 8$ which is the correct initial term

Assume that when $n = k, \; u_k = 4^k + 3k + 1$ in which case...

$$\begin{aligned} u_{k+1} &= 4u_k - 9n \\ &= 4\left(4^k + 3k + 1\right) - 9k \\ &= 4 \times 4^k + 4 \times 3k + 4 - 9k \\ &= 4^{k+1} + 12k - 9k + 4 \\ &= 4^{k+1} + 3k + 4 \\ &= 4^{k+1} + 3(k+1) + 1 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[5 marks]

Answer 5

Statement

For the sequence with the iterative description, $u_1 = 3, \quad u_{n+1} = u_n + 3n - 2$

for all positive integers n, the n^{th} term is given by $u_n = \frac{3}{2}n^2 - \frac{7}{2}n + 5$

Proof by Induction

When $n = 1, \quad u_1 = \frac{3}{2} \times 1^2 - \frac{7}{2} \times 1 + 5$

$= 3$ which is the correct initial term

Assume that when $n = k, \ u_k = \frac{3}{2}k^2 - \frac{7}{2}k + 5$

in which case...

$$\begin{aligned}
u_{k+1} &= u_k + 3k - 2 \\
&= \frac{3}{2}k^2 - \frac{7}{2}k + 5 + 3k - 2 \\
&= \frac{3}{2}\left[(k+1)^2 - 2k - 1\right] - \frac{7}{2}\left[(k+1) - 1\right] + 5 + 3k - 2 \\
&= \frac{3}{2}(k+1)^2 - 3k - \frac{3}{2} - \frac{7}{2}(k+1) + \frac{7}{2} + 5 + 3k - 2 \\
&= \frac{3}{2}(k+1)^2 - \frac{7}{2}(k+1) - 3k + 3k + 5 - \frac{3}{2} + \frac{7}{2} - 2 \\
&= \frac{3}{2}(k+1)^2 - \frac{7}{2}(k+1) + 5
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that

it is true for all $n \in \mathbb{Z}^+$ by mathematical induction □

[5 marks]

Answer 6

Statement

For the sequence with the iterative description, $u_1 = 1, \quad u_{n+1} = u_n + n(3n + 1)$

for all positive integers n, the n^{th} term is given by $u_n = n^2(n - 1) + 1$

Proof by Induction

When $n = 1, \quad u_1 = 1^2 \times (1 - 1) + 1 = 1$ which is the correct initial term

Assume that when $n = k, \; u_k = k^2(k - 1) + 1$ in which case...

$$\begin{aligned} u_{k+1} &= u_k + n(3n + 1) \\ &= k^2 (k - 1) + 1 + k(3k + 1) \\ &= k^3 - k^2 + 3k^2 + k + 1 \\ &= k^3 + 2k^2 + k + 1 \\ &= k\left(k^2 + 2k + 1\right) + 1 \\ &= k(k + 1)^2 + 1 \\ &= (k + 1)^2(k + 1 - 1) + 1 \\ &= (k + 1)^2 ((k + 1) - 1) + 1 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[5 marks]