Answer 1

Statement

If $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$ for all positive integers, n

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ which matches that given for **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$ in which case...

$$\begin{aligned} \mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A} \\ &= \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^k \times 2 & 0 \\ (2^k - 1) \times 2 + 1 & 1 \end{pmatrix} \quad \text{* Must show unsimplified matrix *} \\ &= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 2 + 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix} \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 2

Statement

If $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 2n+1 & -4n \\ n & -2n+1 \end{pmatrix}$ for all positive integers, n

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 2 \times 1 + 1 & -4 \times 1 \\ 1 & -2 \times 1 + 1 \end{pmatrix}$

$= \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ which matches that given for **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 2k+1 & -4k \\ k & -2k+1 \end{pmatrix}$ in which case...

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A}$$

$$= \begin{pmatrix} 2k+1 & -4k \\ k & -2k+1 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} (2k+1) \times 3 - 4k & (2k+1) \times (-4) + (-4k) \times (-1) \\ k \times 3 + (-2k+1) \times 1 & k \times (-4) + (-2k+1) \times (-1) \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 2k + 3 - 4k & -8k - 4 + 4k \\ 3k - 2k + 1 & -4k + 2k - 1 \end{pmatrix}$$ * Must show unsimplified matrix *

$$= \begin{pmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(k+1)+1 & -4(k+1) \\ k+1 & -2(k+1)+1 \end{pmatrix}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 3

Let $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ in which case the question is claiming that the following is true,

Statement

If $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} -3n + 1 & 9n \\ -n & 3n + 1 \end{pmatrix}$ for all $n \in \mathbb{Z}^+$

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} -3 \times 1 + 1 & 9 \times 1 \\ -1 & 3 \times 1 + 1 \end{pmatrix}$

$$= \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$$ which watches that given for **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} -3k + 1 & 9k \\ -k & 3k + 1 \end{pmatrix}$ in which case...

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A}$$

$$= \begin{pmatrix} -3k + 1 & 9k \\ -k & 3k + 1 \end{pmatrix}\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} (-3k + 1) \times (-2) + 9k \times (-1) & (-3k + 1) \times 9 + 9k \times 4 \\ (-k) \times (-2) + (3k + 1) \times (-1) & (-k) \times 9 + (3k + 1) \times 4 \end{pmatrix}$$

$$= \begin{pmatrix} 6k - 2 - 9k & -27k + 9 + 36k \\ 2k - 3k - 1 & -9k + 12k + 4 \end{pmatrix}$$ * Must show unsimplified matrix *

$$= \begin{pmatrix} -3k - 2 & 9k + 9 \\ -k - 1 & 3k + 4 \end{pmatrix}$$

$$= \begin{pmatrix} -3(k + 1) + 1 & 9(k + 1) \\ -(k + 1) & 3(k + 1) + 1 \end{pmatrix}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 4

Let $\mathbf{A} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ in which case the question is claiming that the following is true,

Statement

If $\mathbf{A} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 4n + 1 & -8n \\ 2n & 1 - 4n \end{pmatrix}$ for all $n \in \mathbb{Z}^+$

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 4 \times 1 + 1 & (-8) \times 1 \\ 2 \times 1 & 1 - 4 \times 1 \end{pmatrix}$

$= \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ which matches that given for **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 4k + 1 & -8k \\ 2k & 1 - 4k \end{pmatrix}$ in which case...

$$\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A} \\
&= \begin{pmatrix} 4k + 1 & -8k \\ 2k & 1 - 4k \end{pmatrix}\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \\
&= \begin{pmatrix} (4k + 1) \times 5 + (-8k) \times 2 & (4k + 1) \times (-8) + (-8k) \times (-3) \\ 2k \times 5 + (1 - 4k) \times 2 & 2k \times (-8) + (1 - 4k) \times (-3) \end{pmatrix} \\
&= \begin{pmatrix} 20k + 5 - 16k & -32k - 8 + 24k \\ 10k + 2 - 8k & -16k - 3 + 12k \end{pmatrix} \quad \text{* Must show unsimplified matrix *} \\
&= \begin{pmatrix} 4k + 5 & -8k - 8 \\ 2k + 2 & -4k - 3 \end{pmatrix} \\
&= \begin{pmatrix} 4(k + 1) + 1 & -8(k + 1) \\ 2(k + 1) & 1 - 4(k + 1) \end{pmatrix}
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 5

Statement

If $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2 + 3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$ for all $n \in \mathbb{Z}^+$

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1^2 + 3 \times 1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ which is **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2 + 3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$ in which case...

$$
\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A} \\
&= \begin{pmatrix} 1 & k & \frac{1}{2}(k^2 + 3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 + k & 2 + k + \frac{1}{2}(k^2 + 3k) \\ 0 & 1 & 1 + k \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 + k & \frac{1}{2}(4 + 2k + k^2 + 3k) \\ 0 & 1 & 1 + k \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 + k & \frac{1}{2}(k^2 + 2k + 1 + 3k + 3) \\ 0 & 1 & 1 + k \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 + k & \frac{1}{2}((k + 1)^2 + 3(k + 1)) \\ 0 & 1 & 1 + k \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction $\square$

[6 marks]