4.4 Answers

Further A-Level Pure Mathematics : Core 1

Proof by Induction

Answer 1

(i) The fifth line in the pattern is,

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \frac{1}{99} = \frac{5}{11}$$

Easily verified using a calculator.

[2 marks]

(ii)

$$\begin{aligned} \lim_{n \to \infty} \frac{n}{2n + 1} &= \lim_{n \to \infty} \frac{n}{2n} \\ &= \lim_{n \to \infty} \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

[1 mark]

(iii) **Statement**

$$\sum_{1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1} \quad \text{for all positive integers, } n$$

Proof by Induction

When $n = 1$, $\text{LHS} = \sum_{1}^{1} \frac{1}{(2r-1)(2r+1)} = \frac{1}{(2\times 1 - 1)(2\times 1 + 1)}$

$$= \frac{1}{1\times 3}$$

$$= \frac{1}{3}$$

$$\text{RHS} = \frac{1}{2\times 1 + 1} = \frac{1}{3}$$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$

With $n = k + 1$ the summation formula becomes,

$$\sum_{1}^{k+1} \frac{1}{(2r-1)(2r+1)}$$

$$= \sum_{1}^{k} \frac{1}{(2r-1)(2r+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)\ (2k+3)}$$

$$= \frac{k}{2k+1} \times \frac{(2k+3)}{(2k+3)} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)}$$

$$= \frac{(k+1)(2k+1)}{(2k+1)\ (2k+3)}$$

$$= \frac{(k+1)}{(2k+3)}$$

$$= \frac{(k+1)}{2(k+1)+1}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 2

Statement

$$\sum_{1}^{n} r = \frac{1}{2} n(n+1)$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} r = 1$

RHS $= \frac{1}{2} \times 1 \times (1+1) = 1$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{1}^{k} r = \frac{1}{2} k(k+1)$

With $n = k+1$ the summation formula becomes,

$$\sum_{1}^{k+1} r = \sum_{1}^{k} r + (k+1)$$

$$= \frac{1}{2} k(k+1) + (k+1)$$

$$= (k+1)\left(\frac{1}{2}k + 1\right)$$

$$= (k+1)\frac{1}{2}(k+2)$$

$$= \frac{1}{2}(k+1)((k+1)+1)$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction □

[6 marks]

Answer 3

Statement

$$\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} 1 \times (1+3) = 4$

RHS $= \frac{1}{3} \times 1 \times (1+1) \times (1+5) = 4$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{1}^{k} r(r+3) = \frac{1}{3}k(k+1)(k+5)$

With $n = k+1$ the summation formula becomes,

$$\begin{aligned}
\sum_{1}^{k+1} r(r+3) &= \sum_{1}^{k} r(r+3) + (k+1)(k+1+3) \\
&= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4) \\
&= \frac{1}{3}(k+1)\big(k(k+5) + 3(k+4)\big) \\
&= \frac{1}{3}\big(k+1\big)\big(k^2 + 5k + 3k + 12\big) \\
&= \frac{1}{3}\big(k+1\big)\big(k^2 + 8k + 12\big) \\
&= \frac{1}{3}(k+1)(k+2)(k+6) \\
&= \frac{1}{3}(k+1)((k+1)+1)((k+1)+5)
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction □

[6 marks]

Answer 4

Statement

$$\sum_{r=1}^{n}\left(4r^3 - 3r^2 + r\right) = n^3(n+1)$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1}\left(4 \times 1^3 - 3 \times 1^2 + 1\right) = 2$

RHS $= 1^3 \times (1+1) = 2$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{1}^{k}\left(4r^3 - 3r^2 + r\right) = k^3(k+1)$

With $n = k + 1$ the summation formula becomes,

$$\sum_{1}^{k+1}\left(4r^3 - 3r^2 + r\right)$$

$$= \sum_{1}^{k}\left(4r^3 - 3r^2 + r\right) + 4(k+1)^3 - 3(k+1)^2 + (k+1)$$

$$= k^3(k+1) + 4(k+1)^3 - 3(k+1)^2 + (k+1)$$

$$= (k+1)\left(k^3 + 4(k+1)^2 - 3(k+1) + 1\right)$$

$$= (k+1)\left(k^3 + 4k^2 + 8k + 4 - 3k - 3 + 1\right)$$

$$= (k+1)\left(k^3 + 4k^2 + 5k + 2\right)$$

$(k+2)$ has to be a factor if the statement is true

so force factor it out and see if the other factor is then what is required

$$\begin{array}{r|l}
 & k^2 + 2k + 1 \\ \hline
k + 2 & k^3 + 4k^2 + 5k + 2 \\
 & \underline{k^3 + 2k^2} \\
 & \quad 2k^2 + 5k \\
 & \quad \underline{2k^2 + 4k} \\
 & \qquad k + 2 \\
 & \qquad \underline{k + 2} \\
 & \qquad\quad 0 \leftarrow \text{ as expected}
\end{array}$$

$$= (k+1)(k+2)(k+1)^2 = (k+1)^3((k+1)+1)$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction □

[6 marks]

Answer 5
(a)

Statement

$$\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} r^3 = 1$

RHS $= \frac{1}{4} \times 1^2 \times (1+1)^2 = 1$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{r=1}^{k} r^3 = \frac{1}{4} k^2 (k+1)^2$

With $n = k + 1$ the summation formula becomes,

$$\begin{aligned}
\sum_{1}^{k+1} r^3 &= \sum_{1}^{k} r^3 + (k+1)^3 \\
&= \frac{1}{4} k^2 (k+1)^2 + (k+1)^3 \\
&= \frac{1}{4} \left(k+1\right)^2 \left(k^2 + 4(k+1)\right) \\
&= \frac{1}{4} \left(k+1\right)^2 \left(k^2 + 4k + 4\right) \\
&= \frac{1}{4} (k+1)^2 (k+2)^2 \\
&= \frac{1}{4} (k+1)^2 ((k+1)+1)^2
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction □

[5 marks]

(b)

$$\sum_{r=1}^{n}\left(r^3 - 2\right) = \sum_{r=1}^{n} r^3 - 2\sum_{r=1}^{n} 1$$

$$= \frac{1}{4}n^2(n+1)^2 - 2n$$

$$= \frac{1}{4}n\left(n(n+1)^2 - 8\right)$$

$$= \frac{1}{4}n\left(n\left(n^2 + 2n + 1\right) - 8\right)$$

$$= \frac{1}{4}n\left(n^3 + 2n^2 + n - 8\right)$$

[3 marks]

(c)

$$\sum_{r=20}^{50}\left(r^3 - 2\right) = \sum_{r=1}^{50}\left(r^3 - 2\right) - \sum_{r=1}^{19}\left(r^3 - 2\right)$$

$$= \frac{1}{4}\times 50 \times\left(50^3 + 2\times 50^2 + 50 - 8\right)$$

$$- \frac{1}{4}\times 19 \times\left(19^3 + 2\times 19^2 + 19 - 8\right)$$

$$= 1625525 - 36062$$

$$= 1589463$$

[3 marks]