Answer 1

Statement

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} r^2 = 1^2 = 1$

RHS $= \frac{1}{6} \times 1 \times (1+1)(2 \times 1 + 1) = 1$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{1}^{k} r^2 = \frac{1}{6} k(k+1)(2k+1)$

With $n = k + 1$ the summation formula becomes,

$$\begin{aligned}
\sum_{1}^{k+1} r^2 &= \sum_{1}^{k} r^2 + (k+1)^2 \\
&= \frac{1}{6} k(k+1)(2k+1) + (k+1)^2 \\
&= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\
&= \frac{1}{6}(k+1)\left(2k^2 + k + 6k + 6\right) \\
&= \frac{1}{6}(k+1)\left(2k^2 + 7k + 6\right) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 2

Statement

For all positive integers n, $f(n) = 2^{3n+1} + 3\left(5^{2n+1}\right)$ is divisible by 17

Proof by Induction

$$\begin{aligned}
\text{When } n = 1, \quad f(1) &= 2^{3\times1+1} + 3\left(5^{2\times1+1}\right) \\
&= 2^4 + 3 \times 5^3 \\
&= 16 + 375 \\
&= 391 \\
&= 17 \times 23 \qquad \text{which is divisible by 17}
\end{aligned}$$

Assume that when $n = k$, $f(k) = 2^{3k+1} + 3 \times 5^{2n+1}$ is divisible by 17
in which case...

$$\begin{aligned}
f(k+1) &= 2^{3(k+1)+1} + 3 \times 5^{2(k+1)+1} \\
&= 2^{3k+4} + 3 \times 5^{2k+3} \\
&= 2^{3k+1}2^3 + 3 \times \left(5^{2k+1} \times 5^2\right) \\
&= 8 \times 2^{3k+1} + 75 \times 5^{2k+1} \\
&= 8\left(2^{3k+1} + 3 \times 5^{2k+1}\right) + 51 \times 5^{2k+1} \\
&= 8f(k) + 17\left(3 \times 5^{2k+1}\right) \\
&\therefore f(k+1) \text{ is divisible by 17}
\end{aligned}$$

Therefore, if $f(n)$ is divisible by 17 when $n = k$,

then $f(n)$ is also divisible by 17 when $n = k + 1$

As $f(n)$ is divisible by 17 when $n = 1$, $f(n)$ is also divisible by 17 for all $n \in \mathbb{Z}^+$ by mathematical induction. □

[6 marks]

Answer 3
(i)

Statement

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

Proof by Induction

When $n = 1$, LHS $= \sum_{r=1}^{1} \frac{2r+1}{r^2(r+1)^2} = \frac{2 \times 1 + 1}{1^2\,(1+1)^2} = \frac{3}{4}$

RHS $= 1 - \frac{1}{(1+1)^2} = 1 - \frac{1}{4} = \frac{3}{4}$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{r=1}^{k} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(k+1)^2}$

With $n = k + 1$ the summation formula becomes,

$$\sum_{r=1}^{k+1} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{k} \frac{2r+1}{r^2(r+1)^2} + \frac{2(k+1)+1}{(k+1)^2((k+1)+1)^2}$$

$$= 1 - \frac{1}{(k+1)^2} + \frac{2k+3}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{1}{(k+1)^2} \times \frac{(k+2)^2}{(k+2)^2} + \frac{2k+3}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{k^2 + 4k + 4 - 2k - 3}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{k^2 + 2k + 1}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{(k+1)^2}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{1}{(k+2)^2}$$

$$= 1 - \frac{1}{((k+1)+1)^2}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction ☐

[5 marks]

(ii)

Statement

For the sequence with the iterative description, $u_1 = 3, \quad u_{n+1} = \frac{1}{3} u_n + \frac{8}{9}$

for all positive integers n, the n^{th} term is given by $u_n = 5 \times \left(\frac{1}{3} \right)^n + \frac{4}{3}$

Proof by Induction

When $n = 1, \quad u_1 = 5 \times \left(\frac{1}{3} \right)^1 + \frac{4}{3} = 3$ which is the correct initial term

Assume that when $n = k, \; u_k = 5 \times \left(\frac{1}{3} \right)^k + \frac{4}{3}$ in which case...

$$
\begin{aligned}
u_{k+1} &= \frac{1}{3} u_k + \frac{8}{9} \\
&= \frac{1}{3} \left(5 \times \left(\frac{1}{3} \right)^k + \frac{4}{3} \right) + \frac{8}{9} \\
&= 5 \times \left(\frac{1}{3} \right)^{k+1} + \frac{4}{9} + \frac{8}{9} \\
&= 5 \times \left(\frac{1}{3} \right)^{k+1} + \frac{4}{3}
\end{aligned}
$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction ☐

[5 marks]

Answer 4

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} a^1 & 0 \\ \dfrac{a^1 - b^1}{a - b} & b^1 \end{pmatrix}$

$= \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$ which matches that given for **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} a^k & 0 \\ \dfrac{a^k - b^k}{a - b} & b^k \end{pmatrix}$ in which case...

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A}$$

$$= \begin{pmatrix} a^k & 0 \\ \dfrac{a^k - b^k}{a - b} & b^k \end{pmatrix}\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^k \times a + 0 \times 1 & a^k \times 0 + 0 \times b \\ \dfrac{a^k - b^k}{a - b} \times a + b^k \times 1 & \dfrac{a^k - b^k}{a - b} \times 0 + b^k \times b \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1} & 0 \\ \dfrac{a^{k+1} - a\,b^k}{a - b} + b^k \times \dfrac{a - b}{a - b} & b^{k+1} \end{pmatrix}$$ * Must show unsimplified matrix *

$$= \begin{pmatrix} a^{k+1} & 0 \\ \dfrac{a^{k+1} - a\,b^k + a\,b^k - b^{k+1}}{a - b} & b^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1} & 0 \\ \dfrac{a^{k+1} - b^{k+1}}{a - b} & b^{k+1} \end{pmatrix}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[6 marks]

Answer 5

Statement

For all positive integers n, $f(n) = 5^{2n} + 3n - 1$ is divisible by 9

Proof by Induction

When $n = 1$, $f(1) = 5^{2 \times 1} + 3 \times 1 - 1$

$= 27$ which is divisible by 9

Assume that when $n = k$, $f(k) = 5^{2k} + 3k - 1$ is divisible by 9
in which case...

$$
\begin{aligned}
f(k+1) &= 5^{2(k+1)} + 3(k+1) - 1 \\
&= 5^{2k+2} + 3k + 3 - 1 \\
&= 25 \times 5^{2k} + 3k + 2 \\
&= 25\left(5^{2k} + 3k - 1\right) - 25 \times 3k + 25 + 3k + 2 \\
&= 25f(k) - 72k + 27 \\
f(k+1) &= 25f(k) + 9(-8k + 3)
\end{aligned}
$$

$\therefore f(k+1)$ is divisible by 9

Therefore, if $f(n)$ is divisible by 9 when $n = k$,

then $f(n)$ is also divisible by 9 when $n = k + 1$

As $f(n)$ is divisible by 9 when $n = 1$, $f(n)$ is also divisible by 9 for all $n \in \mathbb{Z}^+$
by mathematical induction. □

[6 marks]