Answer 1

Statement

For all positive integers n, $f(n) = 7^{2n-1} + 5$ is divisible by 12

Proof by Induction

When $n = 1$, $f(1) = 7^{2 \times 1 - 1} + 5$

$= 12$ which is divisible by 12

Assume that when $n = k$, $f(k) = 7^{2k-1} + 5$ is divisible by 12
in which case...

$$
\begin{aligned}
f(k+1) &= 7^{2(k+1)-1} + 5 \\
&= 7^{2k+1} + 5 \\
&= 49 \times 7^{2k-1} + 5 \\
&= (1 + 48) \times 7^{2k-1} + 5 \\
&= 7^{2k-1} + 5 + 48 \times 7^{2k-1} \\
&= f(k) + 48 \times 7^{2k-1} \\
f(k+1) &= f(k) + 12\left(4 \times 7^{2k-1}\right)
\end{aligned}
$$

$\therefore$ $f(k+1)$ is divisible by 12

Therefore, if $f(n)$ is divisible by 12 when $n = k$,

then $f(n)$ is also divisible by 12 when $n = k + 1$

As $f(n)$ is divisible by 12 when $n = 1$, $f(n)$ is also divisible by 12 for all $n \in \mathbb{Z}^{+}$ by mathematical induction. $\square$

[6 marks]

Answer 2

Let $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ in which case the claim is that the following is true,

Statement

If $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$ for all positive integers, n

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix}$

$= \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ which matches that given for **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$ in which case...

$$\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A} \\
&= \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3^k \times 3 + 0 \times 6 & 3^k \times 0 + 0 \times 1 \\ 3(3^k - 1) \times 3 + 1 \times 6 & 3(3^k - 1) \times 0 + 1 \times 1 \end{pmatrix} \\
&= \begin{pmatrix} 3^{k+1} & 0 \\ 3^{k+2} - 9 + 6 & 1 \end{pmatrix} \quad \text{* Must show unsimplified matrix *} \\
&= \begin{pmatrix} 3^{k+1} & 0 \\ 3^{k+2} - 3 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{pmatrix}
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 3

Statement

For all positive integers n, $f(n) = 4^{n+1} + 5^{2n-1}$ is divisible by 21

Proof by Induction

When $n = 1$, $f(1) = 4^{1+1} + 5^{2\times 1-1}$

$= 16 + 5$

$= 21$ which is divisible by 21

Assume that when $n = k$, $f(k) = 4^{k+1} + 5^{2k-1}$ is divisible by 21
in which case...

$$f(k+1) = 4^{(k+1)+1} + 5^{2(k+1)-1}$$

$$= 4 \times 4^{k+1} + 5^2 \times 5^{2k-1}$$

$$= 4 \times 4^{k+1} + 25 \times 5^{2k-1}$$

$$= 4 \times 4^{k+1} + (4 + 21) \times 5^{2k-1}$$

$$= 4\left(4^{k+1} + 5^{2k-1}\right) + 21 \times 5^{2k-1}$$

$$f(k+1) = 4f(k) + 21\left(5^{2k-1}\right)$$

$\therefore f(k+1)$ is divisible by 21

Therefore, if $f(n)$ is divisible by 21 when $n = k$,

then $f(n)$ is also divisible by 21 when $n = k + 1$

As $f(n)$ is divisible by 21 when $n = 1$, $f(n)$ is also divisible by 21 for all $n \in \mathbb{Z}^+$
by mathematical induction. □

[6 marks]

Answer 4

Statement

$$\sum_{r=1}^{n} (2r - 1)^2 = \frac{1}{3} n\left(4n^2 - 1\right)$$

Proof by Induction

When $n = 1$, LHS $= \sum_{r=1}^{1} (2r - 1)^2 = (2 \times 1 - 1)^2 = 1$

RHS $= \frac{1}{3} \times 1\left(4 \times 1^2 - 1\right) = 1$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{r=1}^{k} (2r - 1)^2 = \frac{1}{3} k\left(4k^2 - 1\right)$

With $n = k + 1$ the summation formula becomes,

$$\begin{aligned}
\sum_{r=1}^{k+1} (2r - 1)^2 &= \sum_{r=1}^{k} (2r - 1)^2 + (2(k + 1) - 1)^2 \\
&= \frac{1}{3} k\left(4k^2 - 1\right) + (2k + 1)^2 \\
&= \frac{1}{3} k (2k + 1)(2k - 1) + (2k + 1)^2 \\
&= \frac{1}{3} (2k + 1)(k(2k - 1) + 3(2k + 1)) \\
&= \frac{1}{3} (2k + 1)\left(2k^2 - k + 6k + 3\right) \\
&= \frac{1}{3} (2k + 1)\left(2k^2 + 5k + 3\right) \\
&= \frac{1}{3} (2k + 1)(2k + 3)(k + 1) \\
&= \frac{1}{3} (k + 1) (2(k + 1) - 1)(2(k + 1) + 1) \\
&= \frac{1}{3} \left(k + 1\right)\left(4 (k + 1)^2 - 1\right)
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 5

Statement

For the sequence with the iterative description, $u_1 = 6, \quad u_{n+1} = 6\,u_n - 5,$

for all positive integers n, the n^{th} term is given by $u_n = 5 \times 6^{n-1} + 1$

Proof by Induction

When $n = 1, \quad u_1 = 5 \times 6^{1-1} + 1 = 6$ which is the correct initial term

Assume that when $n = k, \; u_k = 5 \times 6^{k-1} + 1$ in which case...

$$\begin{aligned} u_{k+1} &= 6\,u_k - 5 \\ &= 6 \times \left(5 \times 6^{k-1} + 1\right) - 5 \\ &= 6 \times 5 \times 6^{k-1} + 6 - 5 \\ &= 5 \times 6^{k} + 1 \\ &= 5 \times 6^{(k+1)-1} + 1 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that

it is true for all positive integers by mathematical induction ☐

[5 marks]

Answer 6

Statement

For all positive integers n, $f(n) = 5^n + 8n + 3$ is divisible by 4

Proof by Induction

When $n = 1$, $f(1) = 5^1 + 8 \times 1 + 3$

$= 16$ which is divisible by 4

Assume that when $n = k$, $f(k) = 5^k + 8k + 3$ is divisible by 4
in which case...

$$\begin{aligned} f(k+1) &= 5^{k+1} + 8(k+1) + 3 \\ &= 5 \times 5^k + 8k + 8 + 3 \\ &= (1 + 4) \times 5^k + 8k + 8 + 3 \\ &= 5^k + 8k + 3 + 4 \times 5^k + 8 \\ &= \left(5^k + 8k + 3\right) + 4\left(5^k + 2\right) \\ f(k+1) &= f(k) + 4\left(5^k + 2\right) \end{aligned}$$

$\therefore f(k+1)$ is divisible by 4

Therefore, if $f(n)$ is divisible by 4 when $n = k$,

then $f(n)$ is also divisible by 4 when $n = k + 1$

As $f(n)$ is divisible by 4 when $n = 1$, $f(n)$ is also divisible by 4 for all $n \in \mathbb{Z}^+$
by mathematical induction. □

[7 marks]

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