Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

PROOF

BY

INDUCTION

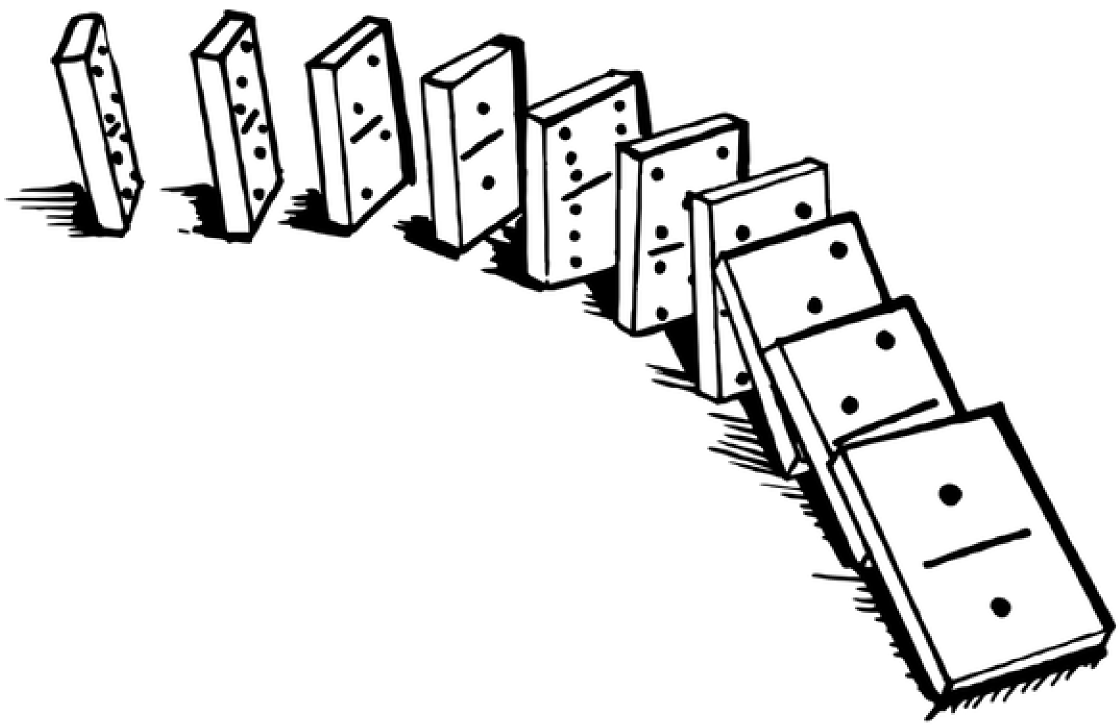
P R O O F
B Y
I N D U C T I O N

Lesson 1

Further A-Level Pure Mathematics : Core 1
Proof by Induction

1.1 The Domino Effect

Proof by induction is often likened to an imaginary line of infinite extent of dominoes. To prove that all of the dominoes will fall over one must prove, firstly, that the first domino in the line will fall over and, secondly, that if any domino in the line falls over it will cause the domino after it to also fall over.



1.2 Example

Suppose that numbers in the sequence $E_n = 3^{2n} + 11$ are to be investigated.

n	1	2	3	4	5	6
E_n	20	92	740	6572	59060	531452
Prime Factorisation	$2^2.5$	$2^2.23$	$2^2.5.37$	$2^2.31.53$	$2^2.5.2953$	$2^2.132863$

From looking at the data in the table, the conjecture could be made that all numbers in this sequence are divisible by 4 which is the most striking feature for the first six terms in the sequence.
However, there is uncertainly as to whether or not the property of being divisible by 4 will continue to hold for all subsequent terms.
To resolve the matter a proof is needed one way or the other.

Statement

For all positive integers n, $3^{2n} + 11$ is divisible by 4

Proof by Induction

Let $f(n) = 3^{2n} + 11, \quad n \in \mathbb{Z}^+$

When $n = 1, \quad f(1) = 3^{2 \times 1} + 11$

$= 20$ which is divisible by 4

Assume that when $n = k$, $f(k) = 3^{2k} + 11$ is divisible by 4
in which case...

$$
\begin{aligned}
f(k+1) &= 3^{2(k+1)} + 11 \\
&= 3^{2k} \times 3^2 + 11 \\
&= 9 \times 3^{2k} + 11 \\
&= 8\left(3^{2k}\right) + 3^{2k} + 11 \\
&= 8\left(3^{2k}\right) + f(k) \\
&= 4\left(2\left(3^{2k}\right)\right) + f(k)
\end{aligned}
$$

$\therefore\ f(k+1)$ is divisible by 4

Therefore, if $f(n)$ is divisible by 4 when $n = k$,

then $f(n)$ is also divisible by 4 when $n = k + 1$

As $f(n)$ is divisible by 4 when $n = 1$, $f(n)$ is also divisible by 4 for all $n \in \mathbb{Z}^+$
by mathematical induction. □

[6 marks]

The Teaching Video will talk you through the above proof.

Teaching Video : http://www.NumberWonder.co.uk/v9096/1.mp4

1.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

Numbers in the sequence $D_n = n^3 + 2n$ are to be investigated.

n	1	2	3	4	5	6
D_n	3	12	33	72	135	228
Prime Factorisation	3	$2^2.3$	3.11	$2^3.3^2$	$3^3.5$	$2^2.3.19$

From looking at the data it would seem that for all positive integers, n, the terms in the sequence D_n are divisible by 3.
Use mathematical induction to prove that this is indeed the case.

[6 marks]

Question 2

Numbers in the sequence $P_n = 3^{2n} - 1$ are to be investigated.

n	1	2	3	4	5	6
P_n	8	80	728	6560	59048	531440
Prime Factorisation	2^3	$2^4.5$	$2^3.7.13$	$2^5.5.41$	$2^3.11^2.61$	$2^4.5.7.13.73$

From looking at the data it would seem that for all positive integers, n, the terms in the sequence P_n are divisible by 8
Use mathematical induction to prove that this is indeed the case.

[6 marks]

Question 3

(i) Explain why the product of any two consecutive integers will divide by 2.

[1 mark]

(ii) Explain why $51k(k + 1)$ is divisible by 6 for any integer k

[2 marks]

(iii) Numbers in the sequence $Z_n = 17n^3 + 103n$ are to be investigated.

n	1	2	3	4	5	6
Z_n	120	342	768	1500	2640	4290
Prime Factorisation	$2^3.3.5$	$2.3^2.19$	$2^8.3$	$2^2.3.5^3$	$2^4.3.3.11$	$2.3.5.11.13$

From looking at the data it would seem that for all positive integers, n, the terms in the sequence Z_n are divisible by 6.
Use mathematical induction to prove that this is indeed the case.

[6 marks]

Question 4

Numbers in the sequence $f(n) = 13^n - 6^n$ are to be investigated.

n	1	2	3	4	5	6
$f(n)$	7	133	1981	27265	363517	4780153
Prime Factorisation	7	7.19	7.283	5.7.19.41	7.11.4721	7.19.127.283

From looking at the data it would seem that for all positive integers, n, the terms in the sequence A_n are divisible by 7

(i) Show that $f(k+1) = 6f(k) + 7\left(13^k\right)$

[3 marks]

(ii) Hence, or otherwise, prove by induction that for all positive integers, n, $f(n)$ is divisible by 7

[4 marks]

Question 5

Further A-Level Examination Question from June 2019, Core 1, Q6 (Edexcel)

Prove by induction that for all positive integers n

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

[6 marks]

Question 6

Further A-Level Examination Question from January 2018, IAL, F1, Q8 (ii) (Edexcel)

Prove by induction, for all positive integers n,

$$f(n) = 3^{2n+3} + 40n - 27 \text{ is divisible by } 64$$

[6 marks]

1.5 Answers

Answer 1

Statement

For all positive integers n, $n^3 + 2n$ is divisible by 3

Proof by Induction

Let $f(n) = n^3 + 2n, \quad n \in \mathbb{Z}^+$

When $n = 1, \quad f(1) = 1^3 + 2 \times 1$

$= 3$ which is divisible by 3

Assume that when $n = k$, $f(k) = k^3 + 2k$ is divisible by 3
in which case...

$$\begin{aligned} f(k+1) &= (k+1)^3 + 2(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= 3k^2 + 3k + 3 + k^3 + 2k \\ &= 3\left(k^2 + k + 1\right) + f(k) \end{aligned}$$

$\therefore f(k+1)$ is divisible by 3

Therefore, if $f(n)$ is divisible by 3 when $n = k$,

then $f(n)$ is also divisible by 3 when $n = k + 1$

As $f(n)$ is divisible by 3 when $n = 1$, $f(n)$ is also divisible by 3 for all $n \in \mathbb{Z}^+$
by mathematical induction. $\square$

[6 marks]

Answer 2

Statement

For all positive integers n, $3^{2n} - 1$ is divisible by 8

Proof by Induction

Let $f(n) = 3^{2n} - 1, \quad n \in \mathbb{Z}^+$

When $n = 1, \quad f(1) = 3^{2 \times 1} - 1$

$= 8$ which is divisible by 8

Assume that when $n = k$, $f(k) = 3^{2k} - 1$ is divisible by 8
in which case...

$$\begin{aligned} f(k+1) &= 3^{2(k+1)} - 1 \\ &= 3^{2k} \times 3^2 - 1 \\ &= 9 \times 3^{2k} - 1 \\ &= 8\left(3^{2k}\right) + 3^{2k} - 1 \\ &= 8\left(3^{2k}\right) + f(k) \end{aligned}$$

$\therefore f(k+1)$ is divisible by 8

Therefore, if $f(n)$ is divisible by 8 when $n = k$,

then $f(n)$ is also divisible by 8 when $n = k + 1$

As $f(n)$ is divisible by 8 when $n = 1$, $f(n)$ is also divisible by 8 for all $n \in \mathbb{Z}^+$ by mathematical induction. □

[6 marks]

Answer 3

(i) As integers alternate odd, even, odd, even, odd, even...
one of any two consecutive integers must be even.
Therefore the product of any two consecutive integers will divide by 2 □

[1 mark]

(ii) If k is an integer, then $k + 1$ is the next consecutive integer.

The product of any two consecutive integers can be written as $k(k + 1)$

As just explained the product, $k(k + 1)$ is divisible by 2.

Furthermore, the number 51 is divisible by 3.

$\therefore\ 51k(k + 1)$ is divisible by 6

[2 marks]

(iii) **Statement**

For all positive integers n, $17n^3 + 103n$ is divisible by 6

Proof by Induction

Let $f(n) = 17n^3 + 103n, \quad n \in \mathbb{Z}^+$

When $n = 1, \quad f(1) = 17 \times 1^3 + 103 \times 1$

$= 120$ which is divisible by 6

Assume that when $n = k$, $f(k) = 17k^3 + 103k$ is divisible by 6
in which case...

$$
\begin{aligned}
f(k + 1) &= 17(k + 1)^3 + 103(k + 1) \\
&= 17k^3 + 51k^2 + 51k + 17 + 103k + 103 \\
&= 17k^3 + 103k + 51k^2 + 51k + 120 \\
&= f(k) + 51k(k + 1) + 120
\end{aligned}
$$

The three terms on the RHS are all divisible by 6

$\therefore f(k + 1)$ is divisible by 6

Therefore, if $f(n)$ is divisible by 6 when $n = k$,

then $f(n)$ is also divisible by 6 when $n = k + 1$

As $f(n)$ is divisible by 6 when $n = 1$, $f(n)$ is also divisible by 6 for all $n \in \mathbb{Z}^+$ by mathematical induction

□

[6 marks]

Answer 4

(i)

$$
\begin{aligned}
f(k) &= 13^k - 6^k \\
\therefore\ f(k+1) &= 13^{k+1} - 6^{k+1} \\
&= 13^k \times 13 - 6^k \times 6 \\
&= 6\left(13^k - 6^k\right) + 7\left(13^k\right) \\
&= 6f(k) + 7\left(13^k\right) \qquad \square
\end{aligned}
$$

[3 marks]

(ii) **Statement**

For all positive integers n, $13^n - 6^n$ is divisible by 7

Proof by Induction

Let $f(n) = 13^n - 6^n, \ n \in \mathbb{Z}^+$

When $n = 1, \ f(1) = 13 - 6$

$= 7$ which is divisible by 7

Assume that when $n = k, \ f(k) = 13^k - 6^k$ is divisible by 7

in which case...

$$
\begin{aligned}
f(k+1) &= 13^k - 6^k \\
&= 6f(k) + 7\left(13^k\right) \qquad \text{from part (i)} \\
f(k+1) - 6f(k) &= 7\left(13^k\right)
\end{aligned}
$$

$\therefore f(k+1)$ is divisible by 7

Therefore, if $f(n)$ is divisible by 7 when $n = k$,

then $f(n)$ is also divisible by 7 when $n = k + 1$

As $f(n)$ is divisible by 7 when $n = 1$, $f(n)$ is also divisible by 7 for all $n \in \mathbb{Z}^+$ by mathematical induction

$\square$

[4 marks]

Answer 5

Statement

For all positive integers n, $3^{2n+4} - 2^{2n}$ is divisible by 5

Proof by Induction

Let $f(n) = 3^{2n+4} - 2^{2n}$, $n \in \mathbb{Z}^+$

When $n = 1$, $f(1) = 3^6 - 2^2$

$= 725$ which is divisible by 5

Assume that when $n = k$, $f(k) = 3^{2k+4} - 2^{2k}$ is divisible by 5

in which case...

$$
\begin{aligned}
f(k+1) &= 3^{2(k+1)+4} - 2^{2(k+1)} \\
&= 3^{2k+4} \times 3^2 - 2^{2k} \times 2^2 \\
&= 9 \times 3^{2k+4} - 4 \times 2^{2k} \\
&= 4\left(3^{2k+4} - 2^{2k}\right) + 5 \times 3^{2k+4} \\
&= 4f(k) + 5\left(3^{2k+4}\right) \\
f(k+1) - 4f(k) &= 5\left(3^{2k+4}\right)
\end{aligned}
$$

$\therefore f(k+1)$ is divisible by 5

Therefore, if $f(n)$ is divisible by 5 when $n = k$,

then $f(n)$ is also divisible by 5 when $n = k + 1$

As $f(n)$ is divisible by 5 when $n = 1$, $f(n)$ is also divisible by 5 for all $n \in \mathbb{Z}^+$ by mathematical induction. □

[6 marks]

Answer 6

Statement

For all positive integers n, $3^{2n+3} + 40n - 27$ is divisible by 64

Proof by Induction

Let $f(n) = 3^{2n+3} + 40n - 27, \quad n \in \mathbb{Z}^+$

When $n = 1, \quad f(1) = 3^5 + 40 - 27$

$= 256$ which is divisible by 64

Assume that when $n = k$, $f(k) = 3^{2k+3} + 40k - 27$ is divisible by 64
in which case...

$$\begin{aligned}
f(k+1) &= 3^{2(k+1)+3} + 40(k+1) - 27 \\
&= 3^{2k+5} + 40k + 40 - 27 \\
&= \left[9 \times 3^{2k+3}\right] + 40k + 13 \\
&= \left[9\left(3^{2k+3} + 40k - 27\right) - 9 \times 40k + 9 \times 27\right] + 40k + 13 \\
&= 9f(k) - 320k + 256 \\
&= 9f(k) + 64(4 - 5k)
\end{aligned}$$

$\therefore \quad f(k+1)$ is divisible by 64

Therefore, if $f(n)$ is divisible by 64 when $n = k$,

then $f(n)$ is also divisible by 64 when $n = k + 1$

As $f(n)$ is divisible by 64 when $n = 1$, $f(n)$ is also divisible by 64 for all $n \in \mathbb{Z}^+$
by mathematical induction. □

[6 marks]

2.1 Iterative Relations and Induction

A sequence is defined iteratively by the term-to-term formula,

$$u_1 = 1, \quad u_{n+1} = 3u_n + 4$$

The first few terms of this sequence are,

n	1	2	3	4	5	6	7	8
u_n	1	7	25	79	241	727	2185	6559

It is thought that a position-to-term formula is, $u_n = 3^n - 2$
It works for the eight terms listed, but to ensure it will always work a proof is needed.

Statement

For the sequence with the iterative definition, $u_1 = 1, \; u_{n+1} = 3u_n + 4$

for all positive integers n, the n^{th} term is given by $u_n = 3^n - 2$

Proof by Induction

When $n = 1, \; u_1 = 3^1 - 2$

$= 1$ which is the correct initial term

Assume that when $n = k, \; u_k = 3^k - 2$ in which case...

$$\begin{aligned} u_{k+1} &= 3u_k + 4 \\ &= 3\left(3^k - 2\right) + 4 \\ &= 3^{k+1} - 6 + 4 \\ &= 3^{k+1} - 2 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[5 marks]

The Teaching Video will talk you through the above proof.

Teaching Video : http://www.NumberWonder.co.uk/v9096/2.mp4

2.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

A sequence is defined iteratively by the term-to-term formula,

$$u_1 = 1, \quad u_{n+1} = 2u_n + 1$$

The first few terms of this sequence are,

n	1	2	3	4	5	6	7	8
u_n	1	3	7	15	31	63	127	255

Prove by induction that the position-to-term formula is $u_n = 2^n - 1$

[5 marks]

Did you know ?

Numbers of the form $2^n - 1$ that are prime are called Mersenne Primes

Question 2

A sequence is defined iteratively by the term-to-term formula,

$$u_1 = 7, \quad u_{n+1} = 2u_n - 1$$

The first few terms of this sequence are,

n	1	2	3	4	5	6	7	8
u_n	7	13	25	49	97	193	385	769

Prove by induction that the position-to-term formula is $u_n = 3 \times 2^n + 1$

[5 marks]

Question 3

A sequence is defined iteratively by the hybrid formula,

$$u_1 = 6, \quad u_{n+1} = u_n + 2^n + 4 \qquad n \geqslant 1$$

The first few terms of this sequence are,

n	1	2	3	4	5	6	7	8
u_n	6	12	20	32	52	88	156	288

Prove by induction that the position-to-term formula is $u_n = 2^n + 4n$

[5 marks]

Question 4

Further A-Level Examination Question from June 2013, IAL, FP1, Q9 (a)

A sequence of numbers is defined by

$$u_1 = 8$$

$$u_{n+1} = 4u_n - 9n \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 4^n + 3n + 1$$

[5 marks]

Question 5

Further A-Level Examination Question from January 2018, IAL, F1, Q8 (i) (Edexcel)

A sequence of numbers is defined by,

$$u_1 = 3$$

$$u_{n+1} = u_n + 3n - 2 \qquad n \geqslant 1$$

Prove by induction, for all positive integers n

$$u_n = \frac{3}{2}n^2 - \frac{7}{2}n + 5$$

[5 marks]

Question 6

Further A-Level Examination Question, January 2013, IAL, FP1, Q8 (b) (Edexcel)

A sequence of positive integers is defined by,

$$u_1 = 1$$

$$u_{n+1} = u_n + n(3n+1) \qquad n \in \mathbb{Z}^+$$

Prove by induction that,

$$u_n = n^2(n-1) + 1 \qquad n \in \mathbb{Z}^+$$

[5 marks]

2.3 Answers

Answer 1

Statement

For the sequence with the iterative description, $u_1 = 1, \quad u_{n+1} = 2u_n + 1$

for all positive integers n, the n^{th} term is given by $u_n = 2^n - 1$

Proof by Induction

When $n = 1, \quad u_1 = 2^1 - 1 = 1$ which is the correct initial term

Assume that when $n = k, \; u_k = 2^k - 1$ in which case...

$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2\left(2^k - 1\right) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[5 marks]

Answer 2

Statement

For the sequence with the iterative description, $u_1 = 7, \quad u_{n+1} = 2u_n - 1$

for all positive integers n, the n^{th} term is given by $u_n = 3 \times 2^n + 1$

Proof by Induction

When $n = 1, \quad u_1 = 3 \times 2^1 + 1 = 7$ which is the correct initial term

Assume that when $n = k, \; u_k = 3 \times 2^k + 1$ in which case...

$$\begin{aligned} u_{k+1} &= 2u_k - 1 \\ &= 2\left(3 \times 2^k + 1\right) - 1 \\ &= 3 \times 2^{k+1} + 2 - 1 \\ &= 3 \times 2^{k+1} + 1 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[5 marks]

Answer 3

Statement

For the sequence with the iterative description, $u_1 = 6, \quad u_{n+1} = u_n + 2^n + 4$

for all positive integers *n*, the n^{th} term is given by $u_n = 2^n + 4n$

Proof by Induction

When $n = 1, \quad u_1 = 2^1 + 4 \times 1 = 6$ which is the correct initial term

Assume that when $n = k, \; u_k = 2^k + 4k$ in which case...

$$\begin{aligned} u_{k+1} &= u_k + 2^k + 4 \\ &= 2^k + 4k + 2^k + 4 \\ &= 2 \times 2^k + 4k + 4 \\ &= 2^{k+1} + 4(k+1) \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[5 marks]

Answer 4

Statement

For the sequence with the iterative description, $u_1 = 8, \quad u_{n+1} = 4u_n - 9n$

for all positive integers *n*, the n^{th} term is given by $u_n = 4^n + 3n + 1$

Proof by Induction

When $n = 1, \quad u_1 = 4^1 + 3 \times 1 + 1 = 8$ which is the correct initial term

Assume that when $n = k, \; u_k = 4^k + 3k + 1$ in which case...

$$\begin{aligned} u_{k+1} &= 4u_k - 9n \\ &= 4\left(4^k + 3k + 1\right) - 9k \\ &= 4 \times 4^k + 4 \times 3k + 4 - 9k \\ &= 4^{k+1} + 12k - 9k + 4 \\ &= 4^{k+1} + 3k + 4 \\ &= 4^{k+1} + 3(k+1) + 1 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[5 marks]

Answer 5

Statement

For the sequence with the iterative description, $u_1 = 3, \quad u_{n+1} = u_n + 3n - 2$

for all positive integers n, the n^{th} term is given by $u_n = \frac{3}{2} n^2 - \frac{7}{2} n + 5$

Proof by Induction

When $n = 1, \quad u_1 = \frac{3}{2} \times 1^2 - \frac{7}{2} \times 1 + 5$

$= 3$ which is the correct initial term

Assume that when $n = k, \; u_k = \frac{3}{2} k^2 - \frac{7}{2} k + 5$

in which case...

$$
\begin{aligned}
u_{k+1} &= u_k + 3k - 2 \\
&= \frac{3}{2} k^2 - \frac{7}{2} k + 5 + 3k - 2 \\
&= \frac{3}{2} \left[(k+1)^2 - 2k - 1 \right] - \frac{7}{2} \left[(k+1) - 1 \right] + 5 + 3k - 2 \\
&= \frac{3}{2} (k+1)^2 - 3k - \frac{3}{2} - \frac{7}{2} (k+1) + \frac{7}{2} + 5 + 3k - 2 \\
&= \frac{3}{2} (k+1)^2 - \frac{7}{2} (k+1) - 3k + 3k + 5 - \frac{3}{2} + \frac{7}{2} - 2 \\
&= \frac{3}{2} (k+1)^2 - \frac{7}{2} (k+1) + 5
\end{aligned}
$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that

it is true for all $n \in \mathbb{Z}^+$ by mathematical induction $\square$

[5 marks]

Answer 6

Statement

For the sequence with the iterative description, $u_1 = 1, \quad u_{n+1} = u_n + n(3n+1)$

for all positive integers n, the n^{th} term is given by $u_n = n^2(n-1)+1$

Proof by Induction

When $n = 1, \quad u_1 = 1^2 \times (1-1) + 1 = 1$ which is the correct initial term

Assume that when $n = k, \ u_k = k^2(k-1)+1$ in which case...

$$\begin{aligned} u_{k+1} &= u_k + n(3n+1) \\ &= k^2(k-1) + 1 + k(3k+1) \\ &= k^3 - k^2 + 3k^2 + k + 1 \\ &= k^3 + 2k^2 + k + 1 \\ &= k\left(k^2 + 2k + 1\right) + 1 \\ &= k(k+1)^2 + 1 \\ &= (k+1)^2(k+1-1) + 1 \\ &= (k+1)^2((k+1)-1) + 1 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k+1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[5 marks]

3.1 Matrices and Induction

This lesson focuses on matrix based statements that can be proved using induction.
The overall structure of the proofs are the same.

3.2 Example

Prove by induction the following statement,

Statement

If $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 1 & 0 \\ 5n & 1 \end{pmatrix}$ for all positive integers, n

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 1 & 0 \\ 5 \times 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ which matches that given for $\mathbf{A}$

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 1 & 0 \\ 5k & 1 \end{pmatrix}$ in which case...

$$\begin{aligned} \mathbf{A}^{k+1} &= \mathbf{A}^k \mathbf{A} \\ &= \begin{pmatrix} 1 & 0 \\ 5k & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 5k + 5 & 1 \end{pmatrix} \qquad \text{* Must show unsimplified matrix *} \\ &= \begin{pmatrix} 1 & 0 \\ 5(k + 1) & 1 \end{pmatrix} \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction □

[6 marks]

The Teaching Video will talk you through the above proof.

Teaching Video : http://www.NumberWonder.co.uk/v9096/3.mp4

3.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Prove by induction the following statement,

Statement

If $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$ for all positive integers, n

[6 marks]

Question 2
Prove by induction the following statement,

Statement
If $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 2n + 1 & -4n \\ n & -2n + 1 \end{pmatrix}$ for all positive integers, n

[6 marks]

Question 3

Prove by induction that for all positive integers n,

$$\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} -3n+1 & 9n \\ -n & 3n+1 \end{pmatrix}$$

Hint : One way to do this question would be to let $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ and observe the question is claiming that the following statement is true,

Statement

If $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} -3n+1 & 9n \\ -n & 3n+1 \end{pmatrix}$ for all $n \in \mathbb{Z}^+$

[6 marks]

Question 4

Further A-Level Examination Question from May 2018, Core 1, Q8 (Edexcel)

Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

[6 marks]

Question 5

Further A-Level Examination Question from 2018, Mock Core 2, Q6 (Edexcel)

Prove by induction, that for all positive integers n,

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}\left(n^2 + 3n\right) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

[6 marks]

Answer 1

Statement

If $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$ for all positive integers, n

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ which matches that given for **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$ in which case...

$$\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A} \\
&= \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^k \times 2 & 0 \\ (2^k - 1) \times 2 + 1 & 1 \end{pmatrix} \qquad \text{* Must show unsimplified matrix *} \\
&= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 2 + 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 2

Statement

If $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 2n + 1 & -4n \\ n & -2n + 1 \end{pmatrix}$ for all positive integers, n

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 2 \times 1 + 1 & -4 \times 1 \\ 1 & -2 \times 1 + 1 \end{pmatrix}$

$= \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ which matches that given for **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 2k + 1 & -4k \\ k & -2k + 1 \end{pmatrix}$ in which case...

$$\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A} \\
&= \begin{pmatrix} 2k + 1 & -4k \\ k & -2k + 1 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} (2k + 1) \times 3 - 4k & (2k + 1) \times (-4) + (-4k) \times (-1) \\ k \times 3 + (-2k + 1) \times 1 & k \times (-4) + (-2k + 1) \times (-1) \end{pmatrix} \\
&= \begin{pmatrix} 3 \times 2k + 3 - 4k & -8k - 4 + 4k \\ 3k - 2k + 1 & -4k + 2k - 1 \end{pmatrix} \quad \text{* Must show unsimplified matrix *} \\
&= \begin{pmatrix} 2k + 3 & -4k - 4 \\ k + 1 & -2k - 1 \end{pmatrix} \\
&= \begin{pmatrix} 2(k + 1) + 1 & -4(k + 1) \\ k + 1 & -2(k + 1) + 1 \end{pmatrix}
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 3

Let $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ in which case the question is claiming that the following is true,

Statement

If $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} -3n + 1 & 9n \\ -n & 3n + 1 \end{pmatrix}$ for all $n \in \mathbb{Z}^+$

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} -3 \times 1 + 1 & 9 \times 1 \\ -1 & 3 \times 1 + 1 \end{pmatrix}$

$= \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ which watches that given for **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} -3k + 1 & 9k \\ -k & 3k + 1 \end{pmatrix}$ in which case...

$$\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A} \\
&= \begin{pmatrix} -3k + 1 & 9k \\ -k & 3k + 1 \end{pmatrix}\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} (-3k + 1) \times (-2) + 9k \times (-1) & (-3k + 1) \times 9 + 9k \times 4 \\ (-k) \times (-2) + (3k + 1) \times (-1) & (-k) \times 9 + (3k + 1) \times 4 \end{pmatrix} \\
&= \begin{pmatrix} 6k - 2 - 9k & -27k + 9 + 36k \\ 2k - 3k - 1 & -9k + 12k + 4 \end{pmatrix} \quad \text{* Must show unsimplified matrix *} \\
&= \begin{pmatrix} -3k - 2 & 9k + 9 \\ -k - 1 & 3k + 4 \end{pmatrix} \\
&= \begin{pmatrix} -3(k + 1) + 1 & 9(k + 1) \\ -(k + 1) & 3(k + 1) + 1 \end{pmatrix}
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 4

Let $\mathbf{A} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ in which case the question is claiming that the following is true,

Statement

If $\mathbf{A} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 4n + 1 & -8n \\ 2n & 1 - 4n \end{pmatrix}$ for all $n \in \mathbb{Z}^+$

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 4 \times 1 + 1 & (-8) \times 1 \\ 2 \times 1 & 1 - 4 \times 1 \end{pmatrix}$

$= \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ which matches that given for $\mathbf{A}$

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 4k + 1 & -8k \\ 2k & 1 - 4k \end{pmatrix}$ in which case...

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A}$$

$$= \begin{pmatrix} 4k + 1 & -8k \\ 2k & 1 - 4k \end{pmatrix}\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} (4k + 1) \times 5 + (-8k) \times 2 & (4k + 1) \times (-8) + (-8k) \times (-3) \\ 2k \times 5 + (1 - 4k) \times 2 & 2k \times (-8) + (1 - 4k) \times (-3) \end{pmatrix}$$

$$= \begin{pmatrix} 20k + 5 - 16k & -32k - 8 + 24k \\ 10k + 2 - 8k & -16k - 3 + 12k \end{pmatrix}$$ * Must show unsimplified matrix *

$$= \begin{pmatrix} 4k + 5 & -8k - 8 \\ 2k + 2 & -4k - 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4(k + 1) + 1 & -8(k + 1) \\ 2(k + 1) & 1 - 4(k + 1) \end{pmatrix}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 5

Statement

If $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2 + 3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$ for all $n \in \mathbb{Z}^+$

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1^2 + 3 \times 1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ which is $\mathbf{A}$

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2 + 3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$ in which case...

$$\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A} \\
&= \begin{pmatrix} 1 & k & \frac{1}{2}(k^2 + 3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 + k & 2 + k + \frac{1}{2}(k^2 + 3k) \\ 0 & 1 & 1 + k \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 + k & \frac{1}{2}(4 + 2k + k^2 + 3k) \\ 0 & 1 & 1 + k \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 + k & \frac{1}{2}(k^2 + 2k + 1 + 3k + 3) \\ 0 & 1 & 1 + k \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 + k & \frac{1}{2}((k + 1)^2 + 3(k + 1)) \\ 0 & 1 & 1 + k \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction $\square$

[6 marks]

Lesson 4

4.1 Summation Formulae and Induction

Here is a number pattern,

$$\frac{1}{2} = \frac{1}{2}$$
$$\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$
$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$
$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$$
$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6}$$

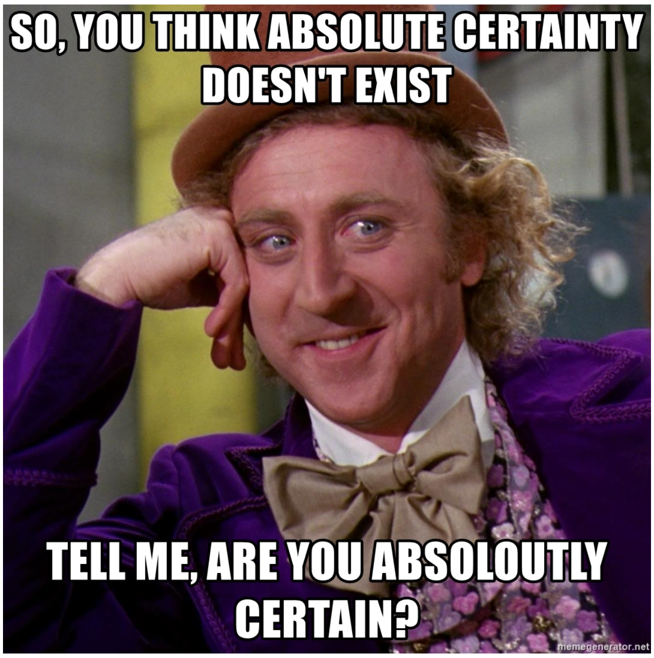
Here is a "reveal" of how the next line of this pattern would be constructed

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \frac{1}{5 \times 6} + \frac{1}{6 \times 7} = \frac{6}{7}$$

The pattern suggests the relationship,

$$\sum_{1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1} \quad \text{for} \quad n \geqslant 1$$

Further exploration may well convince most sceptics that this relationship will continue to hold an the number of terms continues to be increased but, as always, what is really sought is a proof such that the relationship is established with absolute certainty.



4.2 Example
Prove by induction the following statement,
Statement

$$\sum_{1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1} \quad \text{for all positive integers, } n$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} \frac{1}{r(r+1)} = \frac{1}{1 \times 2} = \frac{1}{2}$

RHS $= \frac{1}{1+1} = \frac{1}{2}$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

With $n = k + 1$ the summation formula becomes,

$$\begin{aligned}
\sum_{1}^{k+1} \frac{1}{r(r+1)} &= \sum_{1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+1+1)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} \times \frac{(k+2)}{(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{(k+1)}{(k+2)} \\
&= \frac{(k+1)}{(k+1)+1}
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction □

[6 marks]

Teaching Video : http://www.NumberWonder.co.uk/v9096/4.mp4

<== The Teaching Video will talk through the proof

4.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

Consider the following first four lines of a number pattern,

$$\frac{1}{3} = \frac{1}{3}$$

$$\frac{1}{3} + \frac{1}{15} = \frac{2}{5}$$

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} = \frac{3}{7}$$

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} = \frac{4}{9}$$

The left hand side is being constructed from the reciprocals of product pairs of consecutive odd numbers.

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots + \frac{1}{(2r - 1)(2r + 1)}$$

The left hand side is conjectured to sum to $\dfrac{n}{2n + 1}$

(i) Write down the fifth line of the number pattern and verify that it too has the sum suggested.

[2 marks]

(ii) Assuming that the sum of the series is indeed $\dfrac{n}{2n + 1}$ what is the sum of the series as the number of terms tends towards infinity ?

That is, what is,

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots$$

[2 mark]

(iii) The pattern suggests the relationship,

$$\sum_{1}^{n} \frac{1}{(2r - 1)(2r + 1)} = \frac{n}{2n + 1} \quad \text{for} \quad n \geqslant 1$$

Prove by induction the suggested relationship.

[7 marks]

Question 2

Prove by induction the well known result that,

$$\sum_{1}^{n} r = \frac{1}{2} n(n+1)$$

[6 marks]

Question 3

Further A-Level Examination Question from January 2013, FP1, Q8 (a) (Edexcel)

Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$$

[6 marks]

Question 4

Further A-Level Examination Question from January 2017, IAL, F1, Q9 (i) (Edexcel)

Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} \left(4r^3 - 3r^2 + r\right) = n^3(n+1)$$

[6 marks]

Question 5

Further A-Level Examination Question from January 2012, FP1, Q6 (Edexcel)

(a) Prove by induction

$$\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n + 1)^2$$

[5 marks]

(b) Using the result in part (a), show that

$$\sum_{r=1}^{n} (r^3 - 2) = \frac{1}{4} n \left(n^3 + 2n^2 + n - 8 \right)$$

[3 marks]

(c) Calculate the exact value of $\displaystyle\sum_{r=20}^{50} \left(r^3 - 2 \right)$

[3 marks]

4.4 Answers

Answer 1

(i) The fifth line in the pattern is,

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \frac{1}{99} = \frac{5}{11}$$

Easily verified using a calculator.

[2 marks]

(ii)

$$\begin{aligned} \lim_{n \to \infty} \frac{n}{2n + 1} &= \lim_{n \to \infty} \frac{n}{2n} \\ &= \lim_{n \to \infty} \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

[1 mark]

(iii) **Statement**

$$\sum_{1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1} \quad \text{for all positive integers, } n$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} \frac{1}{(2r-1)(2r+1)} = \frac{1}{(2 \times 1 - 1)(2 \times 1 + 1)}$

$$= \frac{1}{1 \times 3}$$

$$= \frac{1}{3}$$

RHS $= \frac{1}{2 \times 1 + 1} = \frac{1}{3}$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$

With $n = k + 1$ the summation formula becomes,

$$\sum_{1}^{k+1} \frac{1}{(2r-1)(2r+1)}$$

$$= \sum_{1}^{k} \frac{1}{(2r-1)(2r+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)\ (2k+3)}$$

$$= \frac{k}{2k+1} \times \frac{(2k+3)}{(2k+3)} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)}$$

$$= \frac{(k+1)(2k+1)}{(2k+1)\ (2k+3)}$$

$$= \frac{(k+1)}{(2k+3)}$$

$$= \frac{(k+1)}{2(k+1)+1}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[6 marks]

Answer 2

Statement

$$\sum_{1}^{n} r = \frac{1}{2} n(n+1)$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} r = 1$

RHS $= \frac{1}{2} \times 1 \times (1+1) = 1$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{1}^{k} r = \frac{1}{2} k(k+1)$

With $n = k+1$ the summation formula becomes,

$$\sum_{1}^{k+1} r = \sum_{1}^{k} r + (k+1)$$

$$= \frac{1}{2} k(k+1) + (k+1)$$

$$= (k+1)\left(\frac{1}{2}k + 1\right)$$

$$= (k+1)\frac{1}{2}(k+2)$$

$$= \frac{1}{2}(k+1)((k+1)+1)$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction □

[6 marks]

Answer 3

Statement

$$\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} 1 \times (1+3) = 4$

RHS $= \frac{1}{3} \times 1 \times (1+1) \times (1+5) = 4$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{1}^{k} r(r+3) = \frac{1}{3}k(k+1)(k+5)$

With $n = k + 1$ the summation formula becomes,

$$\begin{aligned}
\sum_{1}^{k+1} r(r+3) &= \sum_{1}^{k} r(r+3) + (k+1)(k+1+3) \\
&= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4) \\
&= \frac{1}{3}(k+1)\big(k(k+5) + 3(k+4)\big) \\
&= \frac{1}{3}\big(k+1\big)\big(k^2 + 5k + 3k + 12\big) \\
&= \frac{1}{3}\big(k+1\big)\big(k^2 + 8k + 12\big) \\
&= \frac{1}{3}(k+1)(k+2)(k+6) \\
&= \frac{1}{3}(k+1)((k+1)+1)((k+1)+5)
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[6 marks]

Answer 4

Statement

$$\sum_{r=1}^{n} \left(4r^3 - 3r^2 + r \right) = n^3(n + 1)$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} \left(4 \times 1^3 - 3 \times 1^2 + 1 \right) = 2$

RHS $= 1^3 \times (1 + 1) = 2$

As LHS = RHS the summation formula is true for n = 1

Assume that when $n = k$, $\sum_{1}^{k} \left(4r^3 - 3r^2 + r \right) = k^3(k + 1)$

With $n = k + 1$ the summation formula becomes,

$$\sum_{1}^{k+1} \left(4r^3 - 3r^2 + r \right)$$

$$= \sum_{1}^{k} \left(4r^3 - 3r^2 + r \right) + 4(k + 1)^3 - 3(k + 1)^2 + (k + 1)$$

$$= k^3(k + 1) + 4(k + 1)^3 - 3(k + 1)^2 + (k + 1)$$

$$= \left(k + 1 \right)\left(k^3 + 4(k + 1)^2 - 3(k + 1) + 1 \right)$$

$$= \left(k + 1 \right)\left(k^3 + 4k^2 + 8k + 4 - 3k - 3 + 1 \right)$$

$$= \left(k + 1 \right)\left(k^3 + 4k^2 + 5k + 2 \right)$$

$(k + 2)$ has to be a factor if the statement is true

so force factor it out and see if the other factor is then what is required

$$
\begin{array}{r|rrrrrrr}
 & & & k^2 & + & 2k & + & 1 \\
\hline
k + 2 & k^3 & + & 4k^2 & + & 5k & + & 2 \\
 & \underline{k^3} & \underline{+} & \underline{2k^2} & & & & \\
 & & & 2k^2 & + & 5k & & \\
 & & & \underline{2k^2} & \underline{+} & \underline{4k} & & \\
 & & & & & k & + & 2 \\
 & & & & & \underline{k} & \underline{+} & \underline{2} \\
 & & & & & & & 0
\end{array}
\leftarrow \text{ as expected}
$$

$$= (k + 1)\,(k + 2)(k + 1)^2 = (k + 1)^3(\,(k + 1) + 1)$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction $\square$

[6 marks]

Answer 5
(a)

Statement

$$\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n + 1)^2$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} r^3 = 1$

RHS $= \frac{1}{4} \times 1^2 \times (1 + 1)^2 = 1$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{r=1}^{k} r^3 = \frac{1}{4} k^2 (k + 1)^2$

With $n = k + 1$ the summation formula becomes,

$$\begin{aligned}
\sum_{1}^{k+1} r^3 &= \sum_{1}^{k} r^3 + (k + 1)^3 \\
&= \frac{1}{4} k^2 (k + 1)^2 + (k + 1)^3 \\
&= \frac{1}{4} \left(k + 1 \right)^2 \left(k^2 + 4 (k + 1) \right) \\
&= \frac{1}{4} \left(k + 1 \right)^2 \left(k^2 + 4k + 4 \right) \\
&= \frac{1}{4} (k + 1)^2 (k + 2)^2 \\
&= \frac{1}{4} (k + 1)^2 ((k + 1) + 1)^2
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction ☐

[5 marks]

(b)

$$\sum_{r=1}^{n}\left(r^3 - 2\right) = \sum_{r=1}^{n} r^3 - 2\sum_{r=1}^{n} 1$$

$$= \frac{1}{4}n^2(n+1)^2 - 2n$$

$$= \frac{1}{4}n\left(n(n+1)^2 - 8\right)$$

$$= \frac{1}{4}n\left(n\left(n^2 + 2n + 1\right) - 8\right)$$

$$= \frac{1}{4}n\left(n^3 + 2n^2 + n - 8\right)$$

[3 marks]

(c)

$$\sum_{r=20}^{50}\left(r^3 - 2\right) = \sum_{r=1}^{50}\left(r^3 - 2\right) - \sum_{r=1}^{19}\left(r^3 - 2\right)$$

$$= \frac{1}{4} \times 50 \times \left(50^3 + 2 \times 50^2 + 50 - 8\right)$$

$$- \frac{1}{4} \times 19 \times \left(19^3 + 2 \times 19^2 + 19 - 8\right)$$

$$= 1625525 - 36062$$

$$= 1589463$$

[3 marks]

Lesson 5

Further A-Level Pure Mathematics : Core 1
Proof by Induction

5.1 From the Examination Hall

Summary
Proof by induction is used to prove that a statement is true for all positive integers.
A typical proof by induction will have four main steps;

• Basis	The statement is proved true for $n = 1$
• Assumption	The statement is assumed to be true for $n = k$
• Inductive	The statement is shown, in consequence, to be true for $n = k + 1$
• Conclusion	The statement is consequently deduced true for all positive integers n

Cases Considered
- Proving divisibility by an integer
- Proving a position-to-term formula for a sequence defined by a term to term formula
- Proving a relationship about a matrix raised to the power n
- Proving a formula for the sum of a series, often involving sigma notation

As with all proofs, what examiners are looking for is the quality of the logical reasoning and an attention to the detail.

5.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 33

Question 1

Further AS-Level Sample Assessment Materials, Core 1, Q6 (a) (Edexcel)

Prove by induction that for all positive integers n,

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$$

[6 marks]

Question 2

Further A-Level Sample Assessment Materials, Core 1, Q2 (Edexcel)

Prove by induction that for all positive integers n

$$f(n) = 2^{3n+1} + 3\left(5^{2n+1}\right)$$

is divisible by 17

[6 marks]

Question 3

Further A-Level Examination Question from May 2016, Q8 (Edexcel)

(i) Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} \frac{2r + 1}{r^2(r + 1)^2} = 1 - \frac{1}{(n + 1)^2}$$

[5 marks]

(ii) A sequence of positive rational numbers is defined by,

$$u_1 = 3$$

$$u_{n+1} = \frac{1}{3} u_n + \frac{8}{9} \qquad n \in \mathbb{Z}^+$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 5 \times \left(\frac{1}{3} \right)^n + \frac{4}{3}$$

[5 marks]

Question 4

Further A-Level Examination Question from May 2018, IAL F1, Q8 (Edexcel)

Prove by induction that, for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ \dfrac{a^n - b^n}{a - b} & b^n \end{pmatrix}$$

where a and b are constants and $a \neq b$

[5 marks]

Question 5

Further A-Level Examination Question, January 2017, IAL, F1, Q9 (ii) (Edexcel)

Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 5^{2n} + 3n - 1$$

is divisible by 9

[6 marks]

5.2 Answers

Answer 1

Statement

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} r^2 = 1^2 = 1$

RHS $= \frac{1}{6} \times 1 \times (1+1)(2 \times 1 + 1) = 1$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{1}^{k} r^2 = \frac{1}{6} k(k+1)(2k+1)$

With $n = k + 1$ the summation formula becomes,

$$\begin{aligned}
\sum_{1}^{k+1} r^2 &= \sum_{1}^{k} r^2 + (k+1)^2 \\
&= \frac{1}{6} k(k+1)(2k+1) + (k+1)^2 \\
&= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\
&= \frac{1}{6}(k+1)\left(2k^2 + k + 6k + 6\right) \\
&= \frac{1}{6}(k+1)\left(2k^2 + 7k + 6\right) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 2

Statement

For all positive integers n, $f(n) = 2^{3n+1} + 3\left(5^{2n+1}\right)$ is divisible by 17

Proof by Induction

$$\begin{aligned}
\text{When } n = 1, \quad f(1) &= 2^{3\times 1+1} + 3\left(5^{2\times 1+1}\right) \\
&= 2^4 + 3 \times 5^3 \\
&= 16 + 375 \\
&= 391 \\
&= 17 \times 23 \qquad \text{which is divisible by 17}
\end{aligned}$$

Assume that when $n = k$, $f(k) = 2^{3k+1} + 3 \times 5^{2n+1}$ is divisible by 17
in which case...

$$\begin{aligned}
f(k+1) &= 2^{3(k+1)+1} + 3 \times 5^{2(k+1)+1} \\
&= 2^{3k+4} + 3 \times 5^{2k+3} \\
&= 2^{3k+1}2^3 + 3 \times \left(5^{2k+1} \times 5^2\right) \\
&= 8 \times 2^{3k+1} + 75 \times 5^{2k+1} \\
&= 8\left(2^{3k+1} + 3 \times 5^{2k+1}\right) + 51 \times 5^{2k+1} \\
&= 8f(k) + 17\left(3 \times 5^{2k+1}\right)
\end{aligned}$$

$\therefore f(k+1)$ is divisible by 17

Therefore, if $f(n)$ is divisible by 17 when $n = k$,

then $f(n)$ is also divisible by 17 when $n = k + 1$

As $f(n)$ is divisible by 17 when $n = 1$, $f(n)$ is also divisible by 17 for all $n \in \mathbb{Z}^+$ by mathematical induction. □

[6 marks]

Answer 3
(i)

Statement

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

Proof by Induction

When $n = 1$, LHS $= \displaystyle\sum_{r=1}^{1} \frac{2r+1}{r^2(r+1)^2} = \frac{2 \times 1 + 1}{1^2\,(1+1)^2} = \frac{3}{4}$

RHS $= 1 - \dfrac{1}{(1+1)^2} = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\displaystyle\sum_{r=1}^{k} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(k+1)^2}$

With $n = k + 1$ the summation formula becomes,

$$\sum_{r=1}^{k+1} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{k} \frac{2r+1}{r^2(r+1)^2} + \frac{2(k+1)+1}{(k+1)^2((k+1)+1)^2}$$

$$= 1 - \frac{1}{(k+1)^2} + \frac{2k+3}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{1}{(k+1)^2} \times \frac{(k+2)^2}{(k+2)^2} + \frac{2k+3}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{k^2 + 4k + 4 - 2k - 3}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{k^2 + 2k + 1}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{(k+1)^2}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{1}{(k+2)^2}$$

$$= 1 - \frac{1}{((k+1)+1)^2}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction □

[5 marks]

(ii)

Statement

For the sequence with the iterative description, $u_1 = 3, \quad u_{n+1} = \frac{1}{3} u_n + \frac{8}{9}$

for all positive integers n, the n^{th} term is given by $u_n = 5 \times \left(\frac{1}{3} \right)^n + \frac{4}{3}$

Proof by Induction

When $n = 1, \quad u_1 = 5 \times \left(\frac{1}{3} \right)^1 + \frac{4}{3} = 3$ which is the correct initial term

Assume that when $n = k, \; u_k = 5 \times \left(\frac{1}{3} \right)^k + \frac{4}{3}$ in which case...

$$\begin{aligned} u_{k+1} &= \frac{1}{3} u_k + \frac{8}{9} \\ &= \frac{1}{3} \left(5 \times \left(\frac{1}{3} \right)^k + \frac{4}{3} \right) + \frac{8}{9} \\ &= 5 \times \left(\frac{1}{3} \right)^{k+1} + \frac{4}{9} + \frac{8}{9} \\ &= 5 \times \left(\frac{1}{3} \right)^{k+1} + \frac{4}{3} \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction $\square$

[5 marks]

Answer 4

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} a^1 & 0 \\ \dfrac{a^1 - b^1}{a - b} & b^1 \end{pmatrix}$

$= \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$ which matches that given for **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} a^k & 0 \\ \dfrac{a^k - b^k}{a - b} & b^k \end{pmatrix}$ in which case...

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A}$$

$$= \begin{pmatrix} a^k & 0 \\ \dfrac{a^k - b^k}{a - b} & b^k \end{pmatrix}\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^k \times a + 0 \times 1 & a^k \times 0 + 0 \times b \\ \dfrac{a^k - b^k}{a - b} \times a + b^k \times 1 & \dfrac{a^k - b^k}{a - b} \times 0 + b^k \times b \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1} & 0 \\ \dfrac{a^{k+1} - a\,b^k}{a - b} + b^k \times \dfrac{a - b}{a - b} & b^{k+1} \end{pmatrix}$$ * Must show unsimplified matrix *

$$= \begin{pmatrix} a^{k+1} & 0 \\ \dfrac{a^{k+1} - a\,b^k + a\,b^k - b^{k+1}}{a - b} & b^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1} & 0 \\ \dfrac{a^{k+1} - b^{k+1}}{a - b} & b^{k+1} \end{pmatrix}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction □

[6 marks]

Answer 5

Statement

For all positive integers n, $f(n) = 5^{2n} + 3n - 1$ is divisible by 9

Proof by Induction

When $n = 1$, $f(1) = 5^{2\times 1} + 3 \times 1 - 1$

$= 27$ which is divisible by 9

Assume that when $n = k$, $f(k) = 5^{2k} + 3k - 1$ is divisible by 9
in which case...

$$
\begin{aligned}
f(k+1) &= 5^{2(k+1)} + 3(k+1) - 1 \\
&= 5^{2k+2} + 3k + 3 - 1 \\
&= 25 \times 5^{2k} + 3k + 2 \\
&= 25\left(5^{2k} + 3k - 1\right) - 25 \times 3k + 25 + 3k + 2 \\
&= 25f(k) - 72k + 27 \\
f(k+1) &= 25f(k) + 9(-8k + 3)
\end{aligned}
$$

$\therefore f(k+1)$ is divisible by 9

Therefore, if $f(n)$ is divisible by 9 when $n = k$,

then $f(n)$ is also divisible by 9 when $n = k + 1$

As $f(n)$ is divisible by 9 when $n = 1$, $f(n)$ is also divisible by 9 for all $n \in \mathbb{Z}^+$
by mathematical induction. $\square$

[6 marks]

6.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 36

Question 1

Further A-Level Examination Question, June 2011, IAL, FP1, Q9 (b) (Edexcel)

Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 7^{2n-1} + 5$$

is divisible by 12

[6 marks]

Question 2

Further A-Level Sample Assessment Materials, Core 2, Q3 (ii) (Edexcel)

Prove by induction that for all positive integers *n*

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

[6 marks]

Question 3

Further AS-Level Examination Question, May 2018, Core 1, Q8 (ii) (Edexcel)

Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 4^{n+1} + 5^{2n-1}$$

is divisible by 21

[6 marks]

Question 4

Further A-Level Examination Question, May 2015, IAL, FP1, Q6 (ii) (Edexcel)

Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} (2r - 1)^2 = \frac{1}{3} n\left(4n^2 - 1\right)$$

[6 marks]

Question 5

Further A-Level Examination Question, January 2009, IAL, FP1, Q6 (Edexcel)

A series of positive integers u_1, u_2, u_3, ... is defined by

$$u_1 = 6 \quad \text{and} \quad u_{n+1} = 6u_n - 5, \quad \text{for} \quad n \geqslant 1$$

Prove by induction that $u_n = 5 \times 6^{n-1} + 1$ for $n \geqslant 1$

[5 marks]

Question 6

Further A-Level Examination Question, June 2009, IAL, FP1, Q8 (a) (Edexcel)

Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 5^n + 8n + 3 \text{ is divisible by 4}$$

[7 marks]

6.2 Answers

Answer 1

Statement

For all positive integers n, $f(n) = 7^{2n-1} + 5$ is divisible by 12

Proof by Induction

When $n = 1$, $f(1) = 7^{2 \times 1 - 1} + 5$
$= 12$ which is divisible by 12

Assume that when $n = k$, $f(k) = 7^{2k-1} + 5$ is divisible by 12
in which case...

$$\begin{aligned} f(k+1) &= 7^{2(k+1)-1} + 5 \\ &= 7^{2k+1} + 5 \\ &= 49 \times 7^{2k-1} + 5 \\ &= (1 + 48) \times 7^{2k-1} + 5 \\ &= 7^{2k-1} + 5 + 48 \times 7^{2k-1} \\ &= f(k) + 48 \times 7^{2k-1} \\ f(k+1) &= f(k) + 12\left(4 \times 7^{2k-1}\right) \end{aligned}$$

$\therefore$ $f(k+1)$ is divisible by 12

Therefore, if $f(n)$ is divisible by 12 when $n = k$,

then $f(n)$ is also divisible by 12 when $n = k + 1$

As $f(n)$ is divisible by 12 when $n = 1$, $f(n)$ is also divisible by 12 for all $n \in \mathbb{Z}^+$ by mathematical induction. $\square$

[6 marks]

Answer 2

Let $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ in which case the claim is that the following is true,

Statement

If $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$ for all positive integers, n

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix}$

$= \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ which matches that given for $\mathbf{A}$

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$ in which case...

$$\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A} \\
&= \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3^k \times 3 + 0 \times 6 & 3^k \times 0 + 0 \times 1 \\ 3(3^k - 1) \times 3 + 1 \times 6 & 3(3^k - 1) \times 0 + 1 \times 1 \end{pmatrix} \\
&= \begin{pmatrix} 3^{k+1} & 0 \\ 3^{k+2} - 9 + 6 & 1 \end{pmatrix} \quad \text{* Must show unsimplified matrix *} \\
&= \begin{pmatrix} 3^{k+1} & 0 \\ 3^{k+2} - 3 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{pmatrix}
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 3

Statement

For all positive integers n, $f(n) = 4^{n+1} + 5^{2n-1}$ is divisible by 21

Proof by Induction

When $n = 1$, $f(1) = 4^{1+1} + 5^{2\times1-1}$

$= 16 + 5$

$= 21$ which is divisible by 21

Assume that when $n = k$, $f(k) = 4^{k+1} + 5^{2k-1}$ is divisible by 21
in which case...

$$\begin{aligned} f(k+1) &= 4^{(k+1)+1} + 5^{2(k+1)-1} \\ &= 4 \times 4^{k+1} + 5^2 \times 5^{2k-1} \\ &= 4 \times 4^{k+1} + 25 \times 5^{2k-1} \\ &= 4 \times 4^{k+1} + (4 + 21) \times 5^{2k-1} \\ &= 4\left(4^{k+1} + 5^{2k-1}\right) + 21 \times 5^{2k-1} \\ f(k+1) &= 4f(k) + 21\left(5^{2k-1}\right) \end{aligned}$$

$\therefore f(k+1)$ is divisible by 21

Therefore, if $f(n)$ is divisible by 21 when $n = k$,

then $f(n)$ is also divisible by 21 when $n = k + 1$

As $f(n)$ is divisible by 21 when $n = 1$, $f(n)$ is also divisible by 21 for all $n \in \mathbb{Z}^+$ by mathematical induction. $\square$

[6 marks]

Answer 4

Statement

$$\sum_{r=1}^{n} (2r - 1)^2 = \frac{1}{3} n\left(4n^2 - 1\right)$$

Proof by Induction

When $n = 1$, LHS $= \sum_{r=1}^{1} (2r - 1)^2 = (2 \times 1 - 1)^2 = 1$

RHS $= \frac{1}{3} \times 1\left(4 \times 1^2 - 1\right) = 1$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{r=1}^{k} (2r - 1)^2 = \frac{1}{3} k\left(4k^2 - 1\right)$

With $n = k + 1$ the summation formula becomes,

$$\begin{aligned}
\sum_{r=1}^{k+1} (2r - 1)^2 &= \sum_{r=1}^{k} (2r - 1)^2 + (2(k + 1) - 1)^2 \\
&= \frac{1}{3} k\left(4k^2 - 1\right) + (2k + 1)^2 \\
&= \frac{1}{3} k(2k + 1)(2k - 1) + (2k + 1)^2 \\
&= \frac{1}{3}(2k + 1)(k(2k - 1) + 3(2k + 1)) \\
&= \frac{1}{3}(2k + 1)\left(2k^2 - k + 6k + 3\right) \\
&= \frac{1}{3}(2k + 1)\left(2k^2 + 5k + 3\right) \\
&= \frac{1}{3}(2k + 1)(2k + 3)(k + 1) \\
&= \frac{1}{3}(k + 1)(2(k + 1) - 1)(2(k + 1) + 1) \\
&= \frac{1}{3}\left(k + 1\right)\left(4(k + 1)^2 - 1\right)
\end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that
it is true for all positive integers by mathematical induction ☐

[6 marks]

Answer 5

Statement

For the sequence with the iterative description, $u_1 = 6, \quad u_{n+1} = 6\,u_n - 5,$

for all positive integers n, the n^{th} term is given by $u_n = 5 \times 6^{n-1} + 1$

Proof by Induction

When $n = 1, \quad u_1 = 5 \times 6^{1-1} + 1 = 6$ which is the correct initial term

Assume that when $n = k, \; u_k = 5 \times 6^{k-1} + 1$ in which case...

$$\begin{aligned} u_{k+1} &= 6\,u_k - 5 \\ &= 6 \times \left(5 \times 6^{k-1} + 1\right) - 5 \\ &= 6 \times 5 \times 6^{k-1} + 6 - 5 \\ &= 5 \times 6^{k} + 1 \\ &= 5 \times 6^{(k+1)-1} + 1 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction ☐

[5 marks]

Answer 6

Statement

For all positive integers n, $f(n) = 5^n + 8n + 3$ is divisible by 4

Proof by Induction

When $n = 1$, $f(1) = 5^1 + 8 \times 1 + 3$

$= 16$ which is divisible by 4

Assume that when $n = k$, $f(k) = 5^k + 8k + 3$ is divisible by 4
in which case...

$$
\begin{aligned}
f(k+1) &= 5^{k+1} + 8(k+1) + 3 \\
&= 5 \times 5^k + 8k + 8 + 3 \\
&= (1 + 4) \times 5^k + 8k + 8 + 3 \\
&= 5^k + 8k + 3 + 4 \times 5^k + 8 \\
&= \left(5^k + 8k + 3\right) + 4\left(5^k + 2\right) \\
f(k+1) &= f(k) + 4\left(5^k + 2\right)
\end{aligned}
$$

$\therefore f(k+1)$ is divisible by 4

Therefore, if $f(n)$ is divisible by 4 when $n = k$,

then $f(n)$ is also divisible by 4 when $n = k + 1$

As $f(n)$ is divisible by 4 when $n = 1$, $f(n)$ is also divisible by 4 for all $n \in \mathbb{Z}^+$
by mathematical induction. $\square$

[7 marks]