1.3 Answers

Further A-Level Pure Mathematics, Core 2
Telescoping Series

1.3.1 Solution to 1.1 Example (Teaching Video)

$$\frac{1}{r(r+1)} = \frac{A}{r} + \frac{B}{(r+1)}$$

$$1 = A(r+1) + Br$$

$$\text{Let } r = 0 \Rightarrow A = 1$$

$$\text{Let } r = -1 \Rightarrow B = -1$$

$$\therefore \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{(r+1)}$$

$$\begin{aligned}
\therefore \sum_{r=1}^{n} \frac{1}{r(r+1)} &= \sum_{r=1}^{n} \frac{1}{r} - \sum_{r=1}^{n} \frac{1}{r+1} \\
&= \frac{1}{1} - \frac{1}{2} \\
&+ \frac{1}{2} - \frac{1}{3} \\
&+ \frac{1}{3} - \frac{1}{4} \\
&+ \frac{1}{4} - \dots \\
&+ \dots - \frac{1}{n} \\
&+ \frac{1}{n} - \frac{1}{n+1}
\end{aligned}$$

All terms cancel except the first and the last to give,

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{r(r+1)} &= 1 - \frac{1}{n+1} \\
&= \frac{n+1-1}{n+1} \\
&= \frac{n}{n+1}
\end{aligned}$$

[6 marks]

1.3.2 Solutions to 1.2 Exercise

Answer 1

(a)

$$\frac{1}{(r+2)(r+3)} = \frac{A}{(r+2)} + \frac{B}{(r+3)}$$

$$1 = A(r+3) + B(r+2)$$

$$\text{Let } r = -2 \Rightarrow A = 1$$

$$\text{Let } r = -3 \Rightarrow B = -1$$

$$\therefore \frac{1}{(r+2)(r+3)} = \frac{1}{(r+2)} - \frac{1}{(r+3)}$$

[1 mark]

(b)

$$\therefore \sum_{r=1}^{n} \frac{1}{(r+2)(r+3)} = \sum_{r=1}^{n} \frac{1}{r+2} - \sum_{r=1}^{n} \frac{1}{r+3}$$

$$\begin{aligned} &= \frac{1}{3} - \frac{1}{4} \\ &+ \frac{1}{4} - \frac{1}{5} \\ &+ \frac{1}{5} - \ldots \\ &+ \ldots - \frac{1}{n+2} \\ &+ \frac{1}{n+2} - \frac{1}{n+3} \end{aligned}$$

All terms cancel except the first and the last to give,

$$\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)} = \frac{1}{3} - \frac{1}{n+3}$$

$$= \frac{n+3-3}{3(n+3)}$$

$$= \frac{n}{3(n+3)}$$

[5 marks]

In general,

$$\sum_{r=1}^{n} \frac{1}{(r+m)(r+m+1)} = \frac{n}{(m+1)(n+m+1)}$$

Answer 2

(a)

$$\frac{3}{(3r-1)(3r+2)} = \frac{A}{(3r-1)} + \frac{B}{(3r+2)}$$

$$3 = A(3r+2) + B(3r-1)$$

$$\text{Let } r = \frac{1}{3} \Rightarrow A = 1$$

$$\text{Let } r = -\frac{2}{3} \Rightarrow B = -1$$

$$\therefore \frac{3}{(3r-1)(3r+2)} = \frac{1}{(3r-1)} - \frac{1}{(3r+2)}$$

[2 marks]

(b)

$$\therefore \sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \sum_{r=1}^{n} \frac{1}{3r-1} - \sum_{r=1}^{n} \frac{1}{3r+2}$$

$$= \frac{1}{2} - \frac{1}{5}$$

$$+ \frac{1}{5} - \frac{1}{8}$$

$$+ \frac{1}{8} - \ldots$$

$$+ \ldots - \frac{1}{3n-1}$$

$$+ \frac{1}{3n-1} - \frac{1}{3n+2}$$

All terms cancel except the first and the last to give,

$$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{3n+2}$$

$$= \frac{3n+2-2}{2(3n+2)}$$

$$= \frac{3n}{2(3n+2)}$$

[3 marks]

(c)

$$\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)} = \sum_{r=1}^{1000} \frac{3}{(3r-1)(3r+2)} - \sum_{r=1}^{99} \frac{3}{(3r-1)(3r+2)}$$

$$= \frac{3000}{2 \times 3002} - \frac{297}{2 \times 299}$$

$$= 0.00301$$

[2 marks]

Answer 3

(a)

$$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{(r+2)}$$

$$1 = A(r+2) + Br$$

Let $r = 0 \Rightarrow A = \frac{1}{2}$ and let $r = -2 \Rightarrow B = -\frac{1}{2}$

$$\therefore \frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

[1 mark]

(b)

$$\therefore \sum_{r=1}^{n} \frac{1}{r(r+2)} = \sum_{r=1}^{n} \frac{1}{2r} - \sum_{r=1}^{n} \frac{1}{2(r+2)}$$

$$= \frac{1}{2} - \frac{1}{6}$$

$$+ \frac{1}{4} - \frac{1}{8}$$

$$+ \frac{1}{6} - \frac{1}{10}$$

$$+ \frac{1}{8} - \ldots$$

$$+ \ldots - \frac{1}{2n}$$

$$+ \frac{1}{2n-2} - \frac{1}{2n+2}$$

$$+ \frac{1}{2n} - \frac{1}{2n+4}$$

All terms cancel except the first two and the last two to give,

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2n+2} - \frac{1}{2n+4}$$

$$= \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$$

$$= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)}$$

$$= \frac{3n^2 + 5n}{4(n+1)(n+2)}$$

$$= \frac{n(3n+5)}{4(n+1)(n+2)}$$

[5 marks]

Answer 4

(a)
$$\frac{4r+2}{r(r+1)(r+2)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$$

$$4r+2 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$$

$$\text{Let } r = 0 : \quad 2 = A(1)(2) \Rightarrow A = 1$$

$$\text{Let } r = -1 : -2 = B(-1)(1) \Rightarrow B = 2$$

$$\text{Let } r = -2 : -6 = C(-2)(-1) \Rightarrow C = -3$$

$$\therefore \frac{4r+2}{r(r+1)(r+2)} = \frac{1}{r} + \frac{2}{r+1} - \frac{3}{r+2}$$

[3 marks]

(b)
$$\therefore \sum_{r=1}^{n} \frac{4r+2}{r(r+1)(r+2)} = \sum_{r=1}^{n} \frac{1}{r} + \sum_{r=1}^{n} \frac{2}{r+1} - \sum_{r=1}^{n} \frac{3}{r+2}$$

$$\begin{aligned} &= \frac{1}{1} + \frac{2}{2} - \frac{3}{3} \\ &+ \frac{1}{2} + \frac{2}{3} - \frac{3}{4} \\ &+ \frac{1}{3} + \frac{2}{4} - \frac{3}{5} \\ &+ \frac{1}{4} + \ldots - \ldots \\ &+ \ldots + \ldots - \frac{3}{n} \\ &+ \frac{1}{n-1} + \frac{2}{n} - \frac{3}{n+1} \\ &+ \frac{1}{n} + \frac{2}{n+1} - \frac{3}{n+2} \end{aligned}$$

All terms cancel except the top left and bottom right triangles of three.

$$\begin{aligned} \sum_{r=1}^{n} \frac{4r+2}{r(r+1)(r+2)} &= \frac{1}{1} + \frac{2}{2} + \frac{1}{2} - \frac{3}{n+1} + \frac{2}{n+1} - \frac{3}{n+2} \\ &= \frac{5}{2} - \frac{1}{n+1} - \frac{3}{n+2} \\ &= \frac{5(n+1)(n+2) - 2(n+2) - 6(n+1)}{2(n+1)(n+2)} \\ &= \frac{5n^2 + 15n + 10 - 2n - 4 - 6n - 6}{2(n+1)(n+2)} \\ &= \frac{n(5n+7)}{2(n+1)(n+2)} \end{aligned}$$

[5 marks]

Answer 5

(a)

$$\begin{aligned} \text{LHS} &= \frac{1}{r^2} - \frac{1}{(r+1)^2} \\ &= \frac{(r+1)^2 - r^2}{r^2(r+1)^2} \\ &= \frac{r^2 + 2r + 1 - r^2}{r^2(r+1)^2} \\ &= \frac{2r+1}{r^2(r+1)^2} \\ &= \text{RHS} \qquad \square \end{aligned}$$

[1 mark]

(b)

$$\begin{aligned} \text{LHS} &= \sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} \\ &= \sum_{r=1}^{n} \frac{1}{r^2} - \sum_{r=1}^{n} \frac{1}{(r+1)^2} \\ &= \frac{1}{1} - \frac{1}{4} \\ &+ \frac{1}{4} - \frac{1}{9} \\ &+ \frac{1}{9} - \ldots \\ &+ \ldots - \frac{1}{n^2} \\ &+ \frac{1}{n^2} - \frac{1}{(n+1)^2} \end{aligned}$$

All terms cancel except the first and the last to give,

$$\begin{aligned} \sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} &= \frac{1}{1} - \frac{1}{(n+1)^2} \\ &= \frac{(n+1)^2 - 1}{(n+1)^2} \\ &= \frac{n^2 + 2n + 1 - 1}{(n+1)^2} \\ &= \frac{n(n+2)}{(n+1)^2} \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

(c)

$$\text{LHS} = \sum_{r=n}^{3n} \frac{6r + 3}{r^2 (r + 1)^2}$$

$$= 3\left(\sum_{r=1}^{3n} \frac{2r + 1}{r^2 (r + 1)^2} - \sum_{r=1}^{n-1} \frac{2r + 1}{r^2 \; (r + 1)^2} \right)$$

$$= 3\left(\frac{3n (3n + 2)}{(3n + 1)^2} - \frac{(n - 1)(n + 1)}{n^2} \right)$$

$$= 3\left(\frac{3 n^3 \left(3n + 2 \right) - \left(n^2 - 1 \right)\left(3n + 1 \right)^2}{n^2 (3n + 1)^2} \right)$$

$$= 3\left(\frac{9 n^4 + 6 n^3 - \left(n^2 - 1 \right)\left(9 n^2 + 6n + 1 \right)}{n^2 (3n + 1)^2} \right)$$

$$= 3\left(\frac{9 n^4 + 6 n^3 - \left(9 n^4 + 6 n^3 + n^2 - 9 n^2 - 6n - 1 \right)}{n^2 (3n + 1)^2} \right)$$

$$= 3\left(\frac{9 n^4 + 6 n^3 - 9 n^4 - 6 n^3 + 8 n^2 + 6n + 1}{n^2 (3n + 1)^2} \right)$$

$$= 3\left(\frac{8 n^2 + 6n + 1}{n^2 (3n + 1)^2} \right)$$

$$= \frac{24 n^2 + 18 n + 3}{n^2 \; (3n + 1)^2}$$

$= \text{RHS}$ with $a = 24$, $b = 18$ and $c = 3$ □

[3 marks]