2.4 Answers to 2.3 Exercise

Answer 1

(a) No mention of partial fractions but that's the solution method,

$$\frac{1}{(5r-2)(5r+3)} = \frac{A}{(5r-2)} + \frac{B}{(5r+3)}$$

$$1 = A(5r+3) + B(5r-2)$$

$$\text{Let } r = \frac{2}{5} \Rightarrow A = \frac{1}{5}$$

$$\text{Let } r = -\frac{3}{5} \Rightarrow B = -\frac{1}{5}$$

$$\therefore \frac{1}{(5r-2)(5r+3)} = \frac{1}{5}\left(\frac{1}{5r-2} - \frac{1}{5r+3}\right)$$

$$\text{Thus, } \sum_{r=1}^{n} \frac{1}{(5r-2)(5r+3)} = \frac{1}{5}\sum_{r=1}^{n}\left(\frac{1}{5r-2} - \frac{1}{5r+3}\right)$$

$$\begin{aligned} &= \frac{1}{5}\left(\frac{1}{3} - \frac{1}{8}\right. \\ &+ \quad \frac{1}{8} - \frac{1}{13} \\ &+ \quad \frac{1}{13} - \ldots \\ &+ \quad \ldots - \frac{1}{5n-2} \\ &+ \left.\frac{1}{5n-2} - \frac{1}{5n+3}\right) \end{aligned}$$

All terms cancel except the first and the last to give,

$$\begin{aligned} \sum_{r=1}^{n} \frac{1}{(5r-2)(5r+3)} &= \frac{1}{5}\left(\frac{1}{3} - \frac{1}{5n+3}\right) \\ &= \frac{1}{5}\left(\frac{5n+3-3}{3(5n+3)}\right) \\ &= \frac{n}{3(5n+3)} \end{aligned}$$

[5 marks]

(b)

$$\begin{aligned} \sum_{r=10}^{50} \frac{1}{(5r-2)(5r+3)} &= \frac{50}{3(5\times 50+3)} - \frac{9}{3(5\times 9+3)} \\ &= \frac{50}{759} - \frac{9}{144} \\ &= \frac{41}{12144} \end{aligned}$$

[2 marks]

Answer 2

Again, there is no mention of partial fractions although that is the solution method,

$$\frac{1}{(r+1)(r+2)(r+3)} = \frac{A}{(r+1)} + \frac{B}{(r+2)} + \frac{C}{(r+3)}$$

$$1 = A(r+2)(r+3) + B\,(r+1)(r+3) + C(r+1)(r+2)$$

$$\text{Let } r = -1 \Rightarrow 1 = A(1)(2) \quad \Rightarrow A = \frac{1}{2}$$

$$\text{Let } r = -2 \Rightarrow 1 = B(-1)(1) \quad \Rightarrow B = -1$$

$$\text{Let } r = -3 \Rightarrow 1 = C(-2)(-1) \Rightarrow C = \frac{1}{2}$$

$$\therefore \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2}\left(\frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}\right)$$

$$\begin{aligned}\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} &= \frac{1}{2}\sum_{r=0}^{n}\left(\frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}\right) \\ &= \frac{1}{2}\Big(\frac{1}{1} - \frac{2}{2} + \frac{1}{3} \\ &\quad + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \\ &\quad + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \\ &\quad + \frac{1}{4} - \dots + \dots \\ &\quad + \dots \quad - \frac{2}{n+1} + \frac{1}{n+2} \\ &\quad + \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}\Big)\end{aligned}$$

Most terms cancel in 'threes' in lines rising diagonally.

$$\begin{aligned}\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} &= \frac{1}{2}\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+2} - \frac{2}{n+2} + \frac{1}{n+3}\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+2} + \frac{1}{n+3}\right) \\ &= \frac{1}{2}\left(\frac{(n+2)(n+3) - 2(n+3) + 2(n+2)}{2(n+2)(n+3)}\right) \\ &= \frac{1}{2}\left(\frac{n^2 + 5n + 6 - 2n - 6 + 2n + 4}{2(n+2)(n+3)}\right) \\ &= \frac{n^2 + 5n + 4}{4(n+2)(n+3)} \\ &= \frac{(n+1)(n+4)}{4(n+2)(n+3)} \quad \therefore a = 1.\, b = 4,\ c = 4\end{aligned}$$

[5 marks]

Answer 3
(a)

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r} + \sqrt{r+1}} = \sum_{r=1}^{n} \frac{1}{\sqrt{r} + \sqrt{r+1}} \times \frac{\sqrt{r} - \sqrt{r+1}}{\sqrt{r} - \sqrt{r+1}}$$

$$= \sum_{r=1}^{n} \frac{\sqrt{r} - \sqrt{r+1}}{r - (r+1)}$$

$$= \sum_{r=1}^{n} \sqrt{r+1} - \sqrt{r}$$

$$= \sqrt{2} - \sqrt{1}$$
$$+ \sqrt{3} - \sqrt{2}$$
$$+ \sqrt{4} - \sqrt{3}$$
$$+ \; ...$$
$$+ \sqrt{n} - \sqrt{n-1}$$
$$+ \sqrt{n+1} - \sqrt{n}$$

All terms cancel except the second and the second last

$$\therefore \sum_{r=1}^{n} \frac{1}{\sqrt{r} + \sqrt{r+1}} = \sqrt{n+1} - 1$$

[4 marks]

(b)

$$\frac{1}{\sqrt{4} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{6}} + ... + \frac{1}{\sqrt{80} + \sqrt{81}}$$

$$= \sum_{r=1}^{80} \frac{1}{\sqrt{r} + \sqrt{r+1}} - \sum_{r=1}^{3} \frac{1}{\sqrt{r} + \sqrt{r+1}}$$

$$= \sqrt{81} - 1 - \left(\sqrt{4} - 1\right)$$

$$= 9 - 1 - 2 + 1$$

$$= 7$$

[2 marks]

Answer 4

(a)

$$\begin{aligned} \text{RHS} &= \frac{1}{r!} - \frac{1}{(r+1)!} \\ &= \frac{1}{r!} \times \frac{(r+1)}{(r+1)} - \frac{1}{(r+1)!} \\ &= \frac{r+1-1}{(r+1)!} \\ &= \frac{r}{(r+1)!} \\ &= \text{LHS} \qquad \square \end{aligned}$$

[2 marks]

(b)

$$\begin{aligned} \sum_{r=1}^{n} \frac{r}{(r+1)!} &= \sum_{r=1}^{n} \left(\frac{1}{r!} - \frac{1}{(r+1)!} \right) \\ &= \frac{1}{1!} - \frac{1}{2!} \\ &\quad + \frac{1}{2!} - \frac{1}{3!} \\ &\quad + \frac{1}{3!} - \ \dots \\ &\quad + \ \dots - \ \dots \\ &\quad + \ \dots - \frac{1}{n!} \\ &\quad + \frac{1}{n!} - \frac{1}{(n+1)!} \end{aligned}$$

All terms cancel except the first and last

$$\therefore \sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$$

[5 marks]

Answer 5

$$\sum_{r=1}^{30}\left(\frac{1}{ln(r+1)}-\frac{1}{ln(r+2)}\right)=\frac{1}{ln\,2}-\frac{1}{ln\,3}$$

$$+\frac{1}{ln\,3}-\frac{1}{ln\,4}$$

$$+\frac{1}{ln\,4}-\ ...$$

$$+\ ...$$

$$+\ ...\ -\frac{1}{ln\,31}$$

$$+\frac{1}{ln\,31}-\frac{1}{ln\,32}$$

$$=\frac{1}{ln\,2}-\frac{1}{ln\,32}$$

$$=\frac{1}{ln\,2}-\frac{1}{ln\,2^5}$$

$$=\frac{1}{ln\,2}-\frac{1}{5\,ln\,2}$$

$$=\frac{5-1}{5\,ln\,2}$$

$$=\frac{4}{5\,ln\,2}$$

[5 marks]

Answer 6

$$\frac{1}{(3r-2)(3r+2)} = \frac{A}{(3r-2)} + \frac{B}{(3r+2)}$$

$$1 = A(3r+2) + B(3r-2)$$

$$\text{Let } r = \frac{2}{3} \Rightarrow A = \frac{1}{4}$$

$$\text{Let } r = -\frac{2}{5} \Rightarrow B = -\frac{1}{4}$$

$$\therefore \frac{1}{(3r-2)(3r+2)} = \frac{1}{4}\left(\frac{1}{3r-2} - \frac{1}{3r+2}\right)$$

$$\begin{aligned}
\text{Thus, } \sum_{r=1}^{n} \frac{1}{(3r-2)(3r+2)} &= \frac{1}{4}\sum_{r=1}^{n}\left(\frac{1}{3r-2} - \frac{1}{3r+2}\right) \\
&= \frac{1}{4}\left(\frac{1}{1} - \frac{1}{5}\right. \\
&+ \quad \frac{1}{4} - \frac{1}{8} \\
&+ \quad \frac{1}{7} - \frac{1}{11} \\
&+ \quad \frac{1}{10} - \frac{1}{14} \\
&+ \quad \ldots - \frac{1}{3n-1} \\
&+ \left.\frac{1}{3n-2} - \frac{1}{3n+2}\right)
\end{aligned}$$

And there is no cancelling to be had !

[4 marks]