3.4 Answers

3.4.1 Solution to 3.2 Example (Teaching Video)

$$\begin{aligned}
\text{RHS} &= \frac{1}{4}\big(r(r+1)(r+2)(r+3) - (r-1)r(r+1)(r+2)\big) \\
&= \frac{1}{4}r(r+1)(r+2)\big((r+3)-(r-1)\big) \\
&= \frac{1}{4}r(r+1)(r+2)(4) \\
&= r(r+1)(r+2) \\
&= \text{LHS} \qquad \square
\end{aligned}$$

With $f(r) = (r-1)r(r+1)(r+2)$

$$\begin{aligned}
\sum_{r=1}^{n} r(r+1)(r+2) &= \frac{1}{4}\sum_{r=1}^{n}\big(f(r+1)-f(r)\big) \\
&= \frac{1}{4}\big(f(n+1)-f(1)\big) \\
&= \frac{1}{4}\big(n(n+1)(n+2)(n+3)-0\big) \\
&= \frac{1}{4}n(n+1)(n+2)(n+3)
\end{aligned}$$

3.4.2 Solutions to 3.3 Exercise

Answer 1

(a)

$$\begin{aligned} \text{LHS} &= r(r+1) - (r-1)r \\ &= r^2 + r - r^2 + r \\ &= 2r \\ &= \text{RHS} \qquad \square \end{aligned}$$

[2 marks]

(b) Version 1

$$\begin{aligned} \sum_{r=1}^{n} r &= \frac{1}{2} \sum_{r=1}^{n} \big(r(r+1) - (r-1)r \big) \\ &= \frac{1}{2} \big(2 - 0 \\ &\quad + 6 - 2 \\ &\quad + 12 - 6 \\ &\quad + \ \ldots \\ &\quad + n(n+1) - (n-1)n \ \big) \end{aligned}$$

All terms cancel except the second last

$$\therefore \quad \sum_{r=1}^{n} r = \frac{1}{2} n(n+1) \qquad \square$$

[4 marks]

(b) Version 2

$$r = \frac{1}{2} r(r+1) - \frac{1}{2}(r-1)r$$

$$f(r) = \frac{1}{2} r(r+1) \text{ and } f(r+1) = \frac{1}{2}(r+1)r$$

$$\begin{aligned} \sum_{r=1}^{n} r &= \sum_{r=1}^{n} \left(\frac{1}{2} r(r+1) - \frac{1}{2}(r-1)r \right) \\ &= \sum_{r=1}^{n} \big(f(r+1) - f(r) \big) \\ &= f(n+1) - f(1) \\ &= \frac{1}{2}(n+1)n - 0 \\ &= \frac{1}{2} n(n+1) \qquad \square \end{aligned}$$

[4 marks]

Answer 2

Version 1 for first part

$$(r+1)^3 - r^3 = 3r^2 + 3r + 1$$

$$\therefore \quad \sum_{r=1}^{n}\left((r+1)^3 - r^3\right) = \sum_{r=1}^{n}\left(3r^2 + 3r + 1\right)$$

Considering the LHS,

$$\begin{aligned}\sum_{r=1}^{n}\left((r+1)^3 - r^3\right) = {} & 8 - 1 \\ & + 27 - 8 \\ & + 64 - 27 \\ & + \ldots \\ & + (n+1)^3 - n^3\end{aligned}$$

All terms cancel except second and second last

$$\begin{aligned}\therefore \quad \sum_{r=1}^{n}\left((r+1)^3 - r^3\right) &= (n+1)^3 - 1 \\ &= n^3 + 3n^2 + 3n + 1 - 1 \\ &= n^3 + 3n^2 + 3n\end{aligned}$$

Version 2 for first part

Let $f(r) = r^3$ in which case $f(r+1) = (r+1)^3$

$$\begin{aligned}\text{Then, } \sum_{r=1}^{n}(r+1)^3 - r^3 &= \sum_{r=1}^{n} f(r+1) - f(r) \\ &= f(n+1) - f(1) \\ &= ((n+1)^3 - 1^3 \\ &= n^3 + 3n^2 + 3n + 1 - 1 \\ &= n^3 + 3n^2 + 3n\end{aligned}$$

$$\text{RHS} = \sum_{r=1}^{n}\left(3r^2 + 3r + 1\right) = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= 3\sum_{r=1}^{n} r^2 + 3 \times \frac{1}{2} n(n+1) + n$$

$$3\sum_{r=1}^{n} r^2 + 3 \times \frac{1}{2} n(n+1) + n = n^3 + 3n^2 + 3n$$

$$6\sum_{r=1}^{n} r^2 + 3n(n+1) + 2n = 2n^3 + 6n^2 + 6n$$

$$6\sum_{r=1}^{n} r^2 = 2n^3 + 6n^2 + 6n - 3n(n+1) - 2n$$

$$= 2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n$$

$$= 2n^3 + 3n^2 + n$$

$$= n\left(2n^2 + 3n + 1\right)$$

$$= n(n+1)(2n+1)$$

$$\therefore \sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$$

[6 marks]

Answer 3

(a)

$$
\begin{aligned}
\text{LHS} &= r^2(r+1)^2 - (r-1)^2 r^2 \\
&= r^2\left(r^2+2r+1\right) - \left(r^2-2r+1\right)r^2 \\
&= r^4 + 2r^3 + r^2 - r^4 + 2r^3 - r^2 \\
&= 4r^3 \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[3 marks]

(b) To prove that,

$$\left(1^3+2^3+3^3+\ldots+n^3\right) = \left(1+2+3+\ldots+n\right)^2$$

$$
\begin{aligned}
\text{LHS} = \left(1^3+2^3+3^3+\ldots+n^3\right) &= \sum_{r=1}^{n} r^3 \\
&= \frac{1}{4}\sum_{r=1}^{n}\left(r^2(r+1)^2 - (r-1)^2 r^2\right) \\
&= \frac{1}{4}\big(4 - 0 \\
&\quad + 36 - 4 \\
&\quad + 144 - 36 \\
&\quad + \ldots \\
&\quad + n^2(n+1)^2 - (n-1)n^2 \;\big)
\end{aligned}
$$

All terms cancel except the second (which is zero) and the second last

$$\therefore \left(1^3+2^3+3^3+\ldots+n^3\right) = \frac{1}{4}n^2(n+1)^2$$

$$
\begin{aligned}
\text{RHS} = (1+2+3+\ldots+n)^2 &= \left(\sum_{r=1}^{n} r\right)^2 \\
&= \left(\frac{1}{2}n(n+1)\right)^2 \\
&= \frac{1}{4}n^2(n+1)^2 \\
&= \text{LHS}
\end{aligned}
$$

$\therefore$ The claimed result is true $\qquad \square$

[4 marks]

Answer 4

(a)

$$\begin{aligned} \text{LHS} &= \big((r+1)! - 1\big) - \big(r! - 1\big) \\ &= (r+1)! - 1 - r! + 1 \\ &= (r+1)\,r! - r! \\ &= r \times r! + r! - r! \\ &= r \times r! \\ &= \text{RHS} \qquad\qquad \square \end{aligned}$$

[2 marks]

(b)

$$\begin{aligned} \sum_{r=1}^{n} (r \times r!) &= \sum_{r=1}^{n} \Big(\big((r+1)! - 1\big) - \big(r! - 1\big)\Big) \\ &= 1 - 0 \\ &\quad + 5 - 1 \\ &\quad + 23 - 5 \\ &\quad + \ \ldots \\ &\quad + \big((n+1)! - 1\big) - \big(n! - 1\big) \end{aligned}$$

All terms cancel except the second, which is zero, and the second last

$$\therefore \ \sum_{r=1}^{n} (r \times r!) = (n+1)! - 1$$

[4 marks]

(c)

$$\begin{aligned} \sum_{r=1}^{6} (r \times r!) &= 7! - 1 \\ &= 5039 \end{aligned}$$

[1 mark]

Answer 5

(a)

$$\begin{aligned} \text{LHS} &= \big((r+2)! - (r+1)!\big) - 2\big((r+1)! - r!\big) \\ &= (r+2)(r+1)r! - (r+1)r! - 2(r+1)r! + 2r! \\ &= r!\big((r+2)(r+1) - 3(r+1) + 2\big) \\ &= r!\big(r^2 + 3r + 2 - 3r - 3 + 2\big) \\ &= r!\big(r^2 + 1\big) \\ &= \text{RHS} \qquad\qquad \square \end{aligned}$$

[4 marks]

(b)

$$\sum_{r=1}^{n}\left(r^2+1\right)r! = \sum_{r=1}^{n}\left((r+2)! - (r+1)!\right) - 2\sum_{r=1}^{n}\left((r+1)! - r!\right)$$

$$\begin{aligned}\sum_{r=1}^{n}\left((r+2)! - (r+1)!\right) &= 6 - 2 \\ &+ 24 - 6 \\ &+ 120 - 24 \\ &+ \ldots \\ &+ (n+2)! - (n+1)!\end{aligned}$$

All terms cancel except the second and second last

$$\therefore \quad \sum_{r=1}^{n}\left((r+2)! - (r+1)!\right) = (n+2)! - 2$$

$$\begin{aligned}\sum_{r=1}^{n}\left((r+1)! - r!\right) &= 2 - 1 \\ &+ 6 - 2 \\ &+ 24 - 6 \\ &+ \ldots \\ &+ (n+1)! - n!\end{aligned}$$

All terms cancel except the second and second last

$$\therefore \quad \sum_{r=1}^{n}\left((r+1)! - r!\right) = (n+1)! - 1$$

$$\begin{aligned}\sum_{r=1}^{n}\left(r^2+1\right)r! &= (n+2)! - 2 - 2\left((n+1)! - 1\right) \\ &= (n+2)! - 2 - 2(n+1)! + 2 \\ &= (n+2)(n+1)! - 2(n+1)! \\ &= (n+1)!\left(n+2-2\right) \\ &= n(n+1)!\end{aligned}$$

[6 marks]

Answer 6

(a)

$$(2r+1)^3 = 8r^3 + 3\times 4r^2 + 3\times 2r + 1$$
$$= 8r^3 + 12r^2 + 6r + 1$$
$$\therefore A = 8,\quad B = 12 \text{ and } C = 6$$

[2 marks]

(b)

$$\text{LHS} = (2r+1)^3 - (2r-1)^3$$
$$= 8r^3 + 12r^2 + 6r + 1 - \left(8r^3 - 12r^2 + 6r - 1\right)$$
$$= 24r^2 + 2$$
$$= \text{RHS} \qquad \square$$

[2 marks]

(c) From part (b), $(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$

$$\therefore \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right) = 24\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} 1$$

$$\begin{aligned}\text{LHS} = \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right) &= 27 - 1\\ &+ 125 - 27\\ &+ 343 - 125\\ &+ \ldots\\ &+ (2n+1)^3 - (2n-1)^3\end{aligned}$$

$$\therefore \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right) = (2n+1)^3 - 1$$

Thus, $24\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} 1 = (2n+1)^3 - 1$

$$\begin{aligned}24\sum_{r=1}^{n} r^2 &= 8n^3 + 12n^2 + 6n + 1 - 1 - 2n\\ &= 8n^3 + 12n^2 + 4n\\ &= 4n\left(2n^2 + 3n + 1\right)\\ &= 4n(n+1)(2n+1)\end{aligned}$$

$$\therefore \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \square$$

[5 marks]