4.2 Answers

Answer 1

(i)

$$\begin{aligned}
\text{LHS} &= \frac{1}{r^2} - \frac{1}{(r+1)^2} \\
&= \frac{1}{r^2} \times \frac{(r+1)^2}{(r+1)^2} - \frac{1}{(r+1)^2} \times \frac{r^2}{r^2} \\
&= \frac{r^2 + 2r + 1 - r^2}{r^2(r+1)^2} \\
&= \frac{2r+1}{r^2(r+1)^2} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[1 mark]

(ii)

$$\begin{aligned}
\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} &= \sum_{r=1}^{n} \left(\frac{1}{r^2} - \frac{1}{(r+1)^2} \right) \\
&= \frac{1}{1^2} - \frac{1}{2^2} \\
&+ \frac{1}{2^2} - \frac{1}{3^2} \\
&+ \frac{1}{3^2} - \frac{1}{4^2} \\
&+ \ldots \\
&+ \frac{1}{n^2} - \frac{1}{(n+1)^2}
\end{aligned}$$

All terms cancel except the first and last

$$\therefore \quad \sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

[5 marks]

(iii) Convergent

$$\text{As } n \to \infty, \quad \frac{1}{(n+1)^2} \to 0$$

$$\therefore \quad \sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1$$

[2 marks]

Answer 2

(a)

$$\frac{2}{(r+2)(r+4)} = \frac{A}{(r+2)} + \frac{B}{(r+4)}$$

Multiply through by $\left[\ (r+2)(r+4)\ \right]$

$$2 = A(r+4) + B(r+2)$$

Let $r = -2 \Rightarrow 2 = 2A \ \therefore\ A = 1$

Let $r = -4 \Rightarrow 2 = -2B \ \therefore\ B = -1$

Thus, $$\frac{2}{(r+2)(r+4)} = \frac{1}{(r+2)} - \frac{1}{(r+4)}$$

[1 mark]

(b)

$$\text{LHS} = \sum_{r=1}^{n} \frac{2}{(r+2)(r+4)}$$

$$= \sum_{r=1}^{n} \left(\frac{1}{(r+2)} - \frac{1}{(r+4)} \right)$$

$$= \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6}$$

$$+ \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8}$$

$$+ \frac{1}{7} - \frac{1}{9} + \frac{1}{8} - \frac{1}{10}$$

$$+ \ldots$$

$$+ \frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+2} - \frac{1}{n+4}$$

Most terms cancel, leaving first, third, third last and last

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{n+3} - \frac{1}{n+4}$$

$$= \frac{7(n+3)(n+4) - 12(n+4) - 12(n+3)}{12(n+3)(n+4)}$$

$$= \frac{7n^2 + 49n + 84 - 12n - 48 - 12n - 36}{12(n+3)(n+4)}$$

$$= \frac{7n^2 + 25n}{12(n+3)(n+4)}$$

$$= \frac{n(7n+25)}{12(n+3)(n+4)}$$

$$= \text{RHS} \qquad \square$$

[5 marks]

Answer 3

(i)

$$\sum_{r=1}^{n}\left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}}\right) = 1024^{\frac{1}{1}} - 1024^{\frac{1}{2}}$$

$$+ 1024^{\frac{1}{2}} - 1024^{\frac{1}{3}}$$

$$+ 1024^{\frac{1}{3}} - 1024^{\frac{1}{4}}$$

$$+ \ ...$$

$$+ 1024^{\frac{1}{n}} - 1024^{\frac{1}{n+1}}$$

All terms cancel except the first and last

$$= 1024^{\frac{1}{1}} - 1024^{\frac{1}{n+1}}$$

$$= 1024 - 1024^{\frac{1}{n+1}}$$

[3 marks]

(ii)

$$\sum_{r=5}^{9}\left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}}\right) = \sum_{r=1}^{9}\left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}}\right) - \sum_{r=1}^{4}\left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}}\right)$$

$$= 1024 - 1024^{\frac{1}{10}} - 1024 + 1024^{\frac{1}{5}}$$

$$= -2 + 4$$

$$= 2$$

[3 marks]

Answer 4
(a)

$$
\begin{aligned}
\text{LHS} &= r - 3 + \frac{1}{r+1} - \frac{1}{r+2} \\
&= \frac{(r-3)(r+1)(r+2) + (r+2) - (r+1)}{(r+1)(r+2)} \\
&= \frac{(r-3)\left(r^2 + 3r + 2\right) + r + 2 - r - 1}{(r+1)(r+2)} \\
&= \frac{r^3 + 3r^2 + 2r - 3r^2 - 9r - 6 + 1}{(r+1)(r+2)} \\
&= \frac{r^3 - 7r - 5}{(r+1)(r+2)} \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[2 marks]

(b) First of all,

$$
\begin{aligned}
\sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right) &= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n+1} - \frac{1}{n+2} \\
&= \frac{1}{2} - \frac{1}{n+2}
\end{aligned}
$$

In consequence,

$$
\begin{aligned}
\sum_{r=1}^{n} \frac{r^3 - 7r - 5}{(r+1)(r+2)} &= \sum_{r=1}^{n}\left(r - 3 + \frac{1}{r+1} - \frac{1}{r+2}\right) \\
&= \sum_{r=1}^{n} r - \sum_{r=1}^{n} 3 + \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right) \\
&= \frac{1}{2} n(n+1) - 3n + \frac{1}{2} - \frac{1}{n+2} \\
&= \frac{n(n+1)(n+2) - 6n(n+2) + (n+2) - 2}{2(n+2)} \\
&= \frac{n^3 + 3n^2 + 2n - 6n^2 - 12n + n + 2 - 2}{2(n+2)} \\
&= \frac{n^3 - 3n^2 - 9n}{2(n+2)} \\
&= \frac{n\left(n^2 - 3n - 9\right)}{2(n+2)}
\end{aligned}
$$

$$i.e.\ a = -3, \quad b = -9$$

[5 marks]

Answer 5
(i)

$$\begin{aligned}
\text{LHS} &= \frac{2^k}{k+1} - \frac{2^{k-1}}{k} \\
&= \frac{2 \times 2^{k-1}}{k+1} - \frac{2^{k-1}}{k} \\
&= \frac{2 \times 2^{k-1}}{k+1} \times \frac{k}{k} - \frac{2^{k-1}}{k} \times \frac{(k+1)}{(k+1)} \\
&= \frac{2^{k-1}(2k - k - 1)}{k(k+1)} \\
&= \frac{(k-1)\,2^{k-1}}{k(k+1)} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

(ii)

$$\begin{aligned}
\sum_{r=1}^{n}\left(\frac{(k-1)\,2^{k-1}}{k(k+1)}\right) &= \sum_{r=1}^{n}\left(\frac{2^k}{k+1} - \frac{2^{k-1}}{k}\right) \\
&= \frac{2^1}{2} - \frac{2^0}{1} \\
&+ \frac{2^2}{3} - \frac{2^1}{2} \\
&+ \frac{2^3}{4} - \frac{2^2}{3} \\
&+ \ \ldots \\
&+ \frac{2^n}{n+1} - \frac{2^{n-1}}{n}
\end{aligned}$$

All terms cancel except the second and second last

$$= \frac{2^n}{n+1} - 1$$

[4 marks]

Answer 6

The series needs manipulating into an expression that's a difference.

$$
\begin{aligned}
\frac{r}{(r-2)! + (r-1)! + r!} &= \frac{r}{(r-2)! + (r-1)(r-2)! + r(r-1)(r-2)!} \\
&= \frac{r}{(r-2)!\ \left[\, 1 + (r-1) + r(r-1)\, \right]} \\
&= \frac{r}{(r-2)!\ \left[\, 1 + r - 1 + r^2 - r\, \right]} \\
&= \frac{r}{(r-2)!\ \left[\, r^2\, \right]} \\
&= \frac{1}{r(r-2)!} \\
&= \frac{(r-1)}{r(r-1)(r-2)!} \\
&= \frac{r-1}{r!} \\
&= \frac{r}{r!} - \frac{1}{r!} \\
&= \frac{r}{r(r-1)!} - \frac{1}{r!} \\
&= \frac{1}{(r-1)!} - \frac{1}{r!}
\end{aligned}
$$

Thus,
$$
\begin{aligned}
\sum_{r=2}^{n} \frac{r}{(r-2)! + (r-1)! + r!} &= \sum_{r=2}^{n} \left(\frac{1}{(r-1)!} - \frac{1}{r!} \right) \\
&= \frac{1}{1!} - \frac{1}{2!} \\
&\quad + \frac{1}{2!} - \frac{1}{3!} \\
&\quad + \frac{1}{3!} - \frac{1}{4!} \\
&\quad + \ \dots \\
&\quad + \frac{1}{(n-1)!} - \frac{1}{n!}
\end{aligned}
$$

All terms cancel except the first and the last

$$
\therefore \quad \sum_{r=2}^{n} \frac{r}{(r-2)! + (r-1)! + r!} = 1 - \frac{1}{n!} \qquad \square
$$

[6 marks]