

4.2 Answers

Further A-Level Pure Mathematics, Core 2 Telescoping Series

Answer 1

(i)

$$\begin{aligned}\text{LHS} &= \frac{1}{r^2} - \frac{1}{(r+1)^2} \\&= \frac{1}{r^2} \times \frac{(r+1)^2}{(r+1)^2} - \frac{1}{(r+1)^2} \times \frac{r^2}{r^2} \\&= \frac{r^2 + 2r + 1 - r^2}{r^2(r+1)^2} \\&= \frac{2r + 1}{r^2(r+1)^2} \\&= \text{RHS} \quad \square\end{aligned}$$

[1 mark]

(ii)

$$\begin{aligned}\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2} &= \sum_{r=1}^n \left(\frac{1}{r^2} - \frac{1}{(r+1)^2} \right) \\&= \frac{1}{1^2} - \frac{1}{2^2} \\&\quad + \frac{1}{2^2} - \frac{1}{3^2} \\&\quad + \frac{1}{3^2} - \frac{1}{4^2} \\&\quad + \dots \\&\quad + \frac{1}{n^2} - \frac{1}{(n+1)^2}\end{aligned}$$

All terms cancel except the first and last

$$\therefore \sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

[5 marks]

(iii) Convergent

$$\text{As } n \rightarrow \infty, \frac{1}{(n+1)^2} \rightarrow 0$$

$$\therefore \sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2} = 1$$

[2 marks]

Answer 2

(a)

$$\frac{2}{(r+2)(r+4)} = \frac{A}{(r+2)} + \frac{B}{(r+4)}$$

Multiply through by $[(r+2)(r+4)]$

$$2 = A(r+4) + B(r+2)$$

$$\text{Let } r = -2 \Rightarrow 2 = 2A \therefore A = 1$$

$$\text{Let } r = -4 \Rightarrow 2 = -2B \therefore B = -1$$

$$\text{Thus, } \frac{2}{(r+2)(r+4)} = \frac{1}{(r+2)} - \frac{1}{(r+4)}$$

[1 mark]

(b)

$$\text{LHS} = \sum_{r=1}^n \frac{2}{(r+2)(r+4)}$$

$$= \sum_{r=1}^n \left(\frac{1}{(r+2)} - \frac{1}{(r+4)} \right)$$

$$= \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6}$$

$$+ \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8}$$

$$+ \frac{1}{7} - \frac{1}{9} + \frac{1}{8} - \frac{1}{10}$$

+ ...

$$+ \frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+2} - \frac{1}{n+4}$$

Most terms cancel, leaving first, third, third last and last

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{n+3} - \frac{1}{n+4}$$

$$= \frac{7(n+3)(n+4) - 12(n+4) - 12(n+3)}{12(n+3)(n+4)}$$

$$= \frac{7n^2 + 49n + 84 - 12n - 48 - 12n - 36}{12(n+3)(n+4)}$$

$$= \frac{7n^2 + 25n}{12(n+3)(n+4)}$$

$$= \frac{n(7n+25)}{12(n+3)(n+4)}$$

$$= \text{RHS} \quad \square$$

[5 marks]

Answer 3**(i)**

$$\begin{aligned}
\sum_{r=1}^n \left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}} \right) &= 1024^{\frac{1}{1}} - 1024^{\frac{1}{2}} \\
&+ 1024^{\frac{1}{2}} - 1024^{\frac{1}{3}} \\
&+ 1024^{\frac{1}{3}} - 1024^{\frac{1}{4}} \\
&+ \dots \\
&+ 1024^{\frac{1}{n}} - 1024^{\frac{1}{n+1}}
\end{aligned}$$

All terms cancel except the first and last

$$\begin{aligned}
&= 1024^{\frac{1}{1}} - 1024^{\frac{1}{n+1}} \\
&= 1024 - 1024^{\frac{1}{n+1}}
\end{aligned}$$

[3 marks]**(ii)**

$$\begin{aligned}
\sum_{r=5}^9 \left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}} \right) &= \sum_{r=1}^9 \left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}} \right) - \sum_{r=1}^4 \left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}} \right) \\
&= 1024 - 1024^{\frac{1}{10}} - 1024 + 1024^{\frac{1}{5}} \\
&= -2 + 4 \\
&= 2
\end{aligned}$$

[3 marks]

Answer 4**(a)**

$$\begin{aligned}
\text{LHS} &= r - 3 + \frac{1}{r+1} - \frac{1}{r+2} \\
&= \frac{(r-3)(r+1)(r+2) + (r+2) - (r+1)}{(r+1)(r+2)} \\
&= \frac{(r-3)(r^2+3r+2) + r+2 - r-1}{(r+1)(r+2)} \\
&= \frac{r^3 + 3r^2 + 2r - 3r^2 - 9r - 6 + 1}{(r+1)(r+2)} \\
&= \frac{r^3 - 7r - 5}{(r+1)(r+2)} \\
&= \text{RHS} \qquad \qquad \qquad \square
\end{aligned}$$

[2 marks]**(b)** First of all,

$$\begin{aligned}
\sum_{r=1}^n \left(\frac{1}{r+1} - \frac{1}{r+2} \right) &= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n+1} - \frac{1}{n+2} \\
&= \frac{1}{2} - \frac{1}{n+2}
\end{aligned}$$

In consequence,

$$\begin{aligned}
\sum_{r=1}^n \frac{r^3 - 7r - 5}{(r+1)(r+2)} &= \sum_{r=1}^n \left(r - 3 + \frac{1}{r+1} - \frac{1}{r+2} \right) \\
&= \sum_{r=1}^n r - \sum_{r=1}^n 3 + \sum_{r=1}^n \left(\frac{1}{r+1} - \frac{1}{r+2} \right) \\
&= \frac{1}{2} n(n+1) - 3n + \frac{1}{2} - \frac{1}{n+2} \\
&= \frac{n(n+1)(n+2) - 6n(n+2) + (n+2) - 2}{2(n+2)} \\
&= \frac{n^3 + 3n^2 + 2n - 6n^2 - 12n + n + 2 - 2}{2(n+2)} \\
&= \frac{n^3 - 3n^2 - 9n}{2(n+2)} \\
&= \frac{n(n^2 - 3n - 9)}{2(n+2)}
\end{aligned}$$

$$i.e. \ a = -3, \ b = -9$$

[5 marks]

Answer 5**(i)**

$$\begin{aligned}
\text{LHS} &= \frac{2^k}{k+1} - \frac{2^{k-1}}{k} \\
&= \frac{2 \times 2^{k-1}}{k+1} - \frac{2^{k-1}}{k} \\
&= \frac{2 \times 2^{k-1}}{k+1} \times \frac{k}{k} - \frac{2^{k-1}}{k} \times \frac{(k+1)}{(k+1)} \\
&= \frac{2^{k-1}(2k - k - 1)}{k(k+1)} \\
&= \frac{(k-1)2^{k-1}}{k(k+1)} \\
&= \text{RHS} \quad \square
\end{aligned}$$

[3 marks]**(ii)**

$$\begin{aligned}
\sum_{r=1}^n \left(\frac{(k-1)2^{k-1}}{k(k+1)} \right) &= \sum_{r=1}^n \left(\frac{2^k}{k+1} - \frac{2^{k-1}}{k} \right) \\
&= \frac{2^1}{2} - \frac{2^0}{1} \\
&\quad + \frac{2^2}{3} - \frac{2^1}{2} \\
&\quad + \frac{2^3}{4} - \frac{2^2}{3} \\
&\quad + \dots \\
&\quad + \frac{2^n}{n+1} - \frac{2^{n-1}}{n}
\end{aligned}$$

All terms cancel except the second and second last

$$= \frac{2^n}{n+1} - 1$$

[4 marks]

Answer 6

The series needs manipulating into an expression that's a difference.

$$\begin{aligned}
 \frac{r}{(r-2)! + (r-1)! + r!} &= \frac{r}{(r-2)! + (r-1)(r-2)! + r(r-1)(r-2)!} \\
 &= \frac{r}{(r-2)! [1 + (r-1) + r(r-1)]} \\
 &= \frac{r}{(r-2)! [1 + r - 1 + r^2 - r]} \\
 &= \frac{r}{(r-2)! [r^2]} \\
 &= \frac{1}{r(r-2)!} \\
 &= \frac{(r-1)}{r(r-1)(r-2)!} \\
 &= \frac{r-1}{r!} \\
 &= \frac{r}{r!} - \frac{1}{r!} \\
 &= \frac{r}{r(r-1)!} - \frac{1}{r!} \\
 &= \frac{1}{(r-1)!} - \frac{1}{r!}
 \end{aligned}$$

$$\begin{aligned}
 \text{Thus, } \sum_{r=2}^n \frac{r}{(r-2)! + (r-1)! + r!} &= \sum_{r=2}^n \left(\frac{1}{(r-1)!} - \frac{1}{r!} \right) \\
 &= \frac{1}{1!} - \frac{1}{2!} \\
 &\quad + \frac{1}{2!} - \frac{1}{3!} \\
 &\quad + \frac{1}{3!} - \frac{1}{4!} \\
 &\quad + \dots \\
 &\quad + \frac{1}{(n-1)!} - \frac{1}{n!}
 \end{aligned}$$

All terms cancel except the first and the last

$$\therefore \sum_{r=2}^n \frac{r}{(r-2)! + (r-1)! + r!} = 1 - \frac{1}{n!} \quad \square$$

[6 marks]

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