Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

TELESCOPING SERIES

THE METHOD OF DIFFERENCES

TELESCOPING SERIES

THE METHOD OF DIFFERENCES

Lesson 1

Further A-Level Pure Mathematics, Core 2

Telescoping Series

1.1 The Classic Telescoping Series Example

Often in mathematics, a seemingly unremarkable question can lead to an ingenious solution with an underlying technique, then of use in solving many similar and seeming harder questions. This in turn leads to a focus on questions that one feels should be solvable in a likewise manner but which are not.

A excellent example of such an apparently unremarkable question is the sum,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \dots + \frac{1}{n(n+1)}$$

One approach is to look at the initial partial sums of the series,

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6}$$

A reasonable guess at this stage would be that,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

This guess could be proven correct with a proof by induction.

There is an unsatisfactory messiness with this approach for it does not give insightful about why the result is true.

For the "insight" an old tool is needed, that of partial fractions, along with a willingness to explore the resulting equivalent series.

Teaching Video : http://www.NumberWonder.co.uk/v9097/1a.mp4
http://www.NumberWonder.co.uk/v9097/1b.mp4

<= Part 1 Part 2 =>

After watching the video, write out the details of the Telescoping Series method of showing that,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

[6 marks]

1.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 34

Question 1

(a) Express $\dfrac{1}{(r+2)(r+3)}$ in partial fractions

[1 mark]

(b) Hence use the Telescoping Series method to find the sum of the series,

$$\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)}$$

[5 marks]

The Telescoping Series are the result of applying a technique known as
The Method of Differences

Question 2

Further A-Level Examination Question from June 2010, FP2, Q1

(a) Express $\dfrac{3}{(3r-1)(3r+2)}$ in partial fractions

[2 marks]

(b) Using your answer to part (a) and the method of differences, show that

$$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{3n}{2(3n+2)}$$

[3 marks]

(c) Evaluate $\displaystyle\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)}$ giving your answer to 3 significant figures

[2 marks]

Question 3

(a) Express $\dfrac{1}{r(r+2)}$ in partial fractions

[1 mark]

(b) Hence find the sum of the series $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+2)}$ using the method of differences

[5 marks]

Question 4

Further A-Level Examination Question from June 2018, IAL, F2, Q5

(a) Express $\dfrac{4r+2}{r(r+1)(r+2)}$ in partial fractions

[3 marks]

(b) Hence using the method of differences, prove that

$$\sum_{r=1}^{n} \frac{4r+2}{r(r+1)(r+2)} = \frac{n(an+b)}{2(n+1)(n+2)}$$

where a and b are constants to be found

[5 marks]

Question 5

Further A-Level Examination Question from June 2017, FP2, Q1

(a) Show that, for $r > 0$

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} \equiv \frac{2r+1}{r^2(r+1)^2}$$

[1 mark]

(b) Hence prove that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{n(n+2)}{(n+1)^2}$$

[3 marks]

(c) Show that, for $n \in \mathbb{N}$, $n > 1$

$$\sum_{r=n}^{3n} \frac{6r + 3}{r^2 (r + 1)^2} = \frac{a n^2 + bn + c}{n^2 (3n + 1)^2}$$

where a, b and c are constants to be found.

[3 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”
© 2025 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com

1.3 Answers

1.3.1 Solution to 1.1 Example (Teaching Video)

$$\frac{1}{r(r+1)} = \frac{A}{r} + \frac{B}{(r+1)}$$

$$1 = A(r+1) + Br$$

$$\text{Let } r = 0 \Rightarrow A = 1$$

$$\text{Let } r = -1 \Rightarrow B = -1$$

$$\therefore \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{(r+1)}$$

$$\begin{aligned}
\therefore \sum_{r=1}^{n} \frac{1}{r(r+1)} &= \sum_{r=1}^{n} \frac{1}{r} - \sum_{r=1}^{n} \frac{1}{r+1} \\
&= \frac{1}{1} - \frac{1}{2} \\
&+ \frac{1}{2} - \frac{1}{3} \\
&+ \frac{1}{3} - \frac{1}{4} \\
&+ \frac{1}{4} - \dots \\
&+ \dots - \frac{1}{n} \\
&+ \frac{1}{n} - \frac{1}{n+1}
\end{aligned}$$

All terms cancel except the first and the last to give,

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{r(r+1)} &= 1 - \frac{1}{n+1} \\
&= \frac{n+1-1}{n+1} \\
&= \frac{n}{n+1}
\end{aligned}$$

[6 marks]

1.3.2 Solutions to 1.2 Exercise

Answer 1

(a)

$$\frac{1}{(r+2)(r+3)} = \frac{A}{(r+2)} + \frac{B}{(r+3)}$$

$$1 = A(r+3) + B(r+2)$$

$$\text{Let } r = -2 \Rightarrow A = 1$$

$$\text{Let } r = -3 \Rightarrow B = -1$$

$$\therefore \frac{1}{(r+2)(r+3)} = \frac{1}{(r+2)} - \frac{1}{(r+3)}$$

[1 mark]

(b)

$$\therefore \sum_{r=1}^{n} \frac{1}{(r+2)(r+3)} = \sum_{r=1}^{n} \frac{1}{r+2} - \sum_{r=1}^{n} \frac{1}{r+3}$$

$$= \frac{1}{3} - \frac{1}{4}$$

$$+ \frac{1}{4} - \frac{1}{5}$$

$$+ \frac{1}{5} - \ldots$$

$$+ \ldots - \frac{1}{n+2}$$

$$+ \frac{1}{n+2} - \frac{1}{n+3}$$

All terms cancel except the first and the last to give,

$$\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)} = \frac{1}{3} - \frac{1}{n+3}$$

$$= \frac{n+3-3}{3(n+3)}$$

$$= \frac{n}{3(n+3)}$$

[5 marks]

In general,

$$\sum_{r=1}^{n} \frac{1}{(r+m)(r+m+1)} = \frac{n}{(m+1)(n+m+1)}$$

Answer 2

(a)

$$\frac{3}{(3r-1)(3r+2)} = \frac{A}{(3r-1)} + \frac{B}{(3r+2)}$$

$$3 = A(3r+2) + B(3r-1)$$

$$\text{Let } r = \frac{1}{3} \Rightarrow A = 1$$

$$\text{Let } r = -\frac{2}{3} \Rightarrow B = -1$$

$$\therefore \frac{3}{(3r-1)(3r+2)} = \frac{1}{(3r-1)} - \frac{1}{(3r+2)}$$

[2 marks]

(b)

$$\therefore \sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \sum_{r=1}^{n} \frac{1}{3r-1} - \sum_{r=1}^{n} \frac{1}{3r+2}$$

$$= \frac{1}{2} - \frac{1}{5}$$

$$+ \frac{1}{5} - \frac{1}{8}$$

$$+ \frac{1}{8} - \ldots$$

$$+ \ldots - \frac{1}{3n-1}$$

$$+ \frac{1}{3n-1} - \frac{1}{3n+2}$$

All terms cancel except the first and the last to give,

$$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{3n+2}$$

$$= \frac{3n+2-2}{2(3n+2)}$$

$$= \frac{3n}{2(3n+2)}$$

[3 marks]

(c)

$$\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)} = \sum_{r=1}^{1000} \frac{3}{(3r-1)(3r+2)} - \sum_{r=1}^{99} \frac{3}{(3r-1)(3r+2)}$$

$$= \frac{3000}{2 \times 3002} - \frac{297}{2 \times 299}$$

$$= 0.00301$$

[2 marks]

Answer 3

(a)

$$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{(r+2)}$$

$$1 = A(r+2) + Br$$

Let $r = 0 \Rightarrow A = \dfrac{1}{2}$ and let $r = -2 \Rightarrow B = -\dfrac{1}{2}$

$$\therefore \frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

[1 mark]

(b)

$$\therefore \sum_{r=1}^{n} \frac{1}{r(r+2)} = \sum_{r=1}^{n} \frac{1}{2r} - \sum_{r=1}^{n} \frac{1}{2(r+2)}$$

$$= \frac{1}{2} - \frac{1}{6}$$

$$+ \frac{1}{4} - \frac{1}{8}$$

$$+ \frac{1}{6} - \frac{1}{10}$$

$$+ \frac{1}{8} - \ldots$$

$$+ \ldots - \frac{1}{2n}$$

$$+ \frac{1}{2n-2} - \frac{1}{2n+2}$$

$$+ \frac{1}{2n} - \frac{1}{2n+4}$$

All terms cancel except the first two and the last two to give,

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2n+2} - \frac{1}{2n+4}$$

$$= \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$$

$$= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)}$$

$$= \frac{3n^2 + 5n}{4(n+1)(n+2)}$$

$$= \frac{n(3n+5)}{4(n+1)(n+2)}$$

[5 marks]

Answer 4

(a) $$\frac{4r+2}{r(r+1)(r+2)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$$

$$4r+2 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$$

$$\text{Let } r = 0 : \quad 2 = A(1)(2) \Rightarrow A = 1$$

$$\text{Let } r = -1 : -2 = B(-1)(1) \Rightarrow B = 2$$

$$\text{Let } r = -2 : -6 = C(-2)(-1) \Rightarrow C = -3$$

$$\therefore \frac{4r+2}{r(r+1)(r+2)} = \frac{1}{r} + \frac{2}{r+1} - \frac{3}{r+2}$$

[3 marks]

(b) $$\therefore \sum_{r=1}^{n} \frac{4r+2}{r(r+1)(r+2)} = \sum_{r=1}^{n} \frac{1}{r} + \sum_{r=1}^{n} \frac{2}{r+1} - \sum_{r=1}^{n} \frac{3}{r+2}$$

$$\begin{aligned}
&= \frac{1}{1} + \frac{2}{2} - \frac{3}{3} \\
&+ \frac{1}{2} + \frac{2}{3} - \frac{3}{4} \\
&+ \frac{1}{3} + \frac{2}{4} - \frac{3}{5} \\
&+ \frac{1}{4} + \ldots - \ldots \\
&+ \ldots + \ldots - \frac{3}{n} \\
&+ \frac{1}{n-1} + \frac{2}{n} - \frac{3}{n+1} \\
&+ \frac{1}{n} + \frac{2}{n+1} - \frac{3}{n+2}
\end{aligned}$$

All terms cancel except the top left and bottom right triangles of three.

$$\begin{aligned}
\sum_{r=1}^{n} \frac{4r+2}{r(r+1)(r+2)} &= \frac{1}{1} + \frac{2}{2} + \frac{1}{2} - \frac{3}{n+1} + \frac{2}{n+1} - \frac{3}{n+2} \\
&= \frac{5}{2} - \frac{1}{n+1} - \frac{3}{n+2} \\
&= \frac{5(n+1)(n+2) - 2(n+2) - 6(n+1)}{2(n+1)(n+2)} \\
&= \frac{5n^2 + 15n + 10 - 2n - 4 - 6n - 6}{2(n+1)(n+2)} \\
&= \frac{n(5n+7)}{2(n+1)(n+2)}
\end{aligned}$$

[5 marks]

Answer 5

(a)

$$\text{LHS} = \frac{1}{r^2} - \frac{1}{(r+1)^2}$$

$$= \frac{(r+1)^2 - r^2}{r^2(r+1)^2}$$

$$= \frac{r^2 + 2r + 1 - r^2}{r^2(r+1)^2}$$

$$= \frac{2r+1}{r^2(r+1)^2}$$

$$= \text{RHS} \qquad \square$$

[1 mark]

(b)

$$\text{LHS} = \sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$$

$$= \sum_{r=1}^{n} \frac{1}{r^2} - \sum_{r=1}^{n} \frac{1}{(r+1)^2}$$

$$= \frac{1}{1} - \frac{1}{4}$$

$$+ \frac{1}{4} - \frac{1}{9}$$

$$+ \frac{1}{9} - \ldots$$

$$+ \ldots - \frac{1}{n^2}$$

$$+ \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

All terms cancel except the first and the last to give,

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{1}{1} - \frac{1}{(n+1)^2}$$

$$= \frac{(n+1)^2 - 1}{(n+1)^2}$$

$$= \frac{n^2 + 2n + 1 - 1}{(n+1)^2}$$

$$= \frac{n(n+2)}{(n+1)^2}$$

$$= \text{RHS} \qquad \square$$

[3 marks]

(c)

$$\begin{aligned}
\text{LHS} &= \sum_{r=n}^{3n} \frac{6r + 3}{r^2 (r+1)^2} \\
&= 3\left(\sum_{r=1}^{3n} \frac{2r+1}{r^2(r+1)^2} - \sum_{r=1}^{n-1} \frac{2r+1}{r^2(r+1)^2} \right) \\
&= 3\left(\frac{3n(3n+2)}{(3n+1)^2} - \frac{(n-1)(n+1)}{n^2} \right) \\
&= 3\left(\frac{3n^3\left(3n+2\right) - \left(n^2-1\right)\left(3n+1\right)^2}{n^2(3n+1)^2} \right) \\
&= 3\left(\frac{9n^4 + 6n^3 - \left(n^2-1\right)\left(9n^2+6n+1\right)}{n^2(3n+1)^2} \right) \\
&= 3\left(\frac{9n^4 + 6n^3 - \left(9n^4 + 6n^3 + n^2 - 9n^2 - 6n - 1\right)}{n^2(3n+1)^2} \right) \\
&= 3\left(\frac{9n^4 + 6n^3 - 9n^4 - 6n^3 + 8n^2 + 6n + 1}{n^2(3n+1)^2} \right) \\
&= 3\left(\frac{8n^2 + 6n + 1}{n^2(3n+1)^2} \right) \\
&= \frac{24n^2 + 18n + 3}{n^2(3n+1)^2} \\
&= \text{RHS with } a = 24,\ b = 18 \text{ and } c = 3 \qquad \square
\end{aligned}$$

[3 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2025 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com

2.1 A Refusal To Telescope

The “Method of Differences” is a powerful tool in finding the sum to n terms of many series but, as the name implies, it relies on there being a subtraction part in any partial fraction expansion so that the resulting equivalent series telescopes.

The sum, $\displaystyle\sum_{r=0}^{n} \frac{2r+3}{(r+1)(r+2)}$ looks similar to those considered previously.

Yet applying the partial fraction technique does not give any subtraction part.

$$
\begin{aligned}
\sum_{r=0}^{n} \frac{2r+3}{(r+1)(r+2)} &= \sum_{r=0}^{n} \frac{1}{r+1} + \sum_{r=0}^{n} \frac{1}{r+2} \\
&= \frac{1}{1} + \frac{1}{2} \\
&+ \frac{1}{2} + \frac{1}{3} \\
&+ \frac{1}{3} + \frac{1}{4} \\
&+ \frac{1}{4} + \ldots \\
&+ \ldots + \frac{1}{n+1} \\
&+ \frac{1}{n+1} + \frac{1}{n+2} \\
&= 1 + 2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n+1}\right) + \frac{1}{n+2}
\end{aligned}
$$

The problem identified by the failure of the method of differences to generate a telescoping series is more serious than it might at first seem for the series at the heart of the problem is The Harmonic Series,

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

This is divergent and the partial sums have no simple formula in terms of n.

Teaching Video : http://www.NumberWonder.co.uk/v9097/2.mp4

The Teaching video will talk through the above example

2.2 In The Exam

In the Further A-Level examination, the questions asked will avoid problems such as that presented in the Teaching Video. Unless a question states otherwise, the default technique, ***even when not explicitly stated***, is to apply the partial fractions method.

2.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 34

Question 1

Further A-Level Examination Question from 2018, Mock Paper, Core 1, Q4

(a) Prove that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{1}{(5r-2)(5r+3)} \equiv \frac{n}{a(bn+c)}$$

where a, b and c are integers to be determined

[5 marks]

(b) Hence, showing your working, find the exact value of

$$\sum_{r=10}^{50} \frac{1}{(5r-2)(5r+3)}$$

[2 marks]

Question 2

Further A-Level Examination Question from June 2019, Core 1, Q4

Prove that, for $n \in \mathbb{Z}, n \geqslant 0$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

where a, b and c are integers to be found

[5 marks]

Question 3

(a) By multiplying the numerator and the denominator by the denominator's conjugate, turn the following into a telescoping series and hence obtain an expression for the sum to *n* terms,

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r} + \sqrt{r+1}}$$

[4 marks]

(b) Hence evaluate,

$$\frac{1}{\sqrt{4} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{6}} + \dots + \frac{1}{\sqrt{80} + \sqrt{81}}$$

[2 marks]

Question 4

(a) Show that $\dfrac{r}{(r+1)!} = \dfrac{1}{r!} - \dfrac{1}{(r+1)!}$

[2 marks]

(b) Hence find $\displaystyle\sum_{r=1}^{n} \frac{r}{(r+1)!}$

[5 marks]

Question 5

Determine the exact value of,

$$\sum_{r=1}^{30}\left(\frac{1}{ln(r+1)}-\frac{1}{ln(r+2)}\right)$$

[5 marks]

Question 6

Show that the partial fractions method fails when trying to determine the following sum via a telescoping series,

$$\sum_{r=1}^{n} \frac{1}{(3r - 2)(3r + 2)}$$

Interestingly, this #FAIL example does have a subtraction component. Thus it is a step on from the #FAIL example at the start of this lesson which didn't work because there were no subtractions at all. However, this example still cannot be made to work, essentially because $3p - 2 \neq 3q + 2$ for any integers p and q.

[4 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2025 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com

2.4 Answers to 2.3 Exercise

Answer 1

(a) No mention of partial fractions but that's the solution method,

$$\frac{1}{(5r-2)(5r+3)} = \frac{A}{(5r-2)} + \frac{B}{(5r+3)}$$

$$1 = A(5r+3) + B(5r-2)$$

$$\text{Let } r = \frac{2}{5} \Rightarrow A = \frac{1}{5}$$

$$\text{Let } r = -\frac{3}{5} \Rightarrow B = -\frac{1}{5}$$

$$\therefore \frac{1}{(5r-2)(5r+3)} = \frac{1}{5}\left(\frac{1}{5r-2} - \frac{1}{5r+3}\right)$$

Thus, $$\sum_{r=1}^{n} \frac{1}{(5r-2)(5r+3)} = \frac{1}{5}\sum_{r=1}^{n}\left(\frac{1}{5r-2} - \frac{1}{5r+3}\right)$$

$$\begin{aligned} &= \frac{1}{5}\left(\frac{1}{3} - \frac{1}{8}\right. \\ &+ \quad \frac{1}{8} - \frac{1}{13} \\ &+ \quad \frac{1}{13} - \ldots \\ &+ \quad \ldots - \frac{1}{5n-2} \\ &+ \left.\frac{1}{5n-2} - \frac{1}{5n+3}\right) \end{aligned}$$

All terms cancel except the first and the last to give,

$$\begin{aligned} \sum_{r=1}^{n} \frac{1}{(5r-2)(5r+3)} &= \frac{1}{5}\left(\frac{1}{3} - \frac{1}{5n+3}\right) \\ &= \frac{1}{5}\left(\frac{5n+3-3}{3(5n+3)}\right) \\ &= \frac{n}{3(5n+3)} \end{aligned}$$

[5 marks]

(b) $$\begin{aligned} \sum_{r=10}^{50} \frac{1}{(5r-2)(5r+3)} &= \frac{50}{3(5\times 50+3)} - \frac{9}{3(5\times 9+3)} \\ &= \frac{50}{759} - \frac{9}{144} \\ &= \frac{41}{12144} \end{aligned}$$

[2 marks]

Answer 2

Again, there is no mention of partial fractions although that is the solution method,

$$\frac{1}{(r+1)(r+2)(r+3)} = \frac{A}{(r+1)} + \frac{B}{(r+2)} + \frac{C}{(r+3)}$$

$$1 = A(r+2)(r+3) + B\,(r+1)(r+3) + C(r+1)(r+2)$$

$$\text{Let } r = -1 \Rightarrow 1 = A(1)(2) \quad \Rightarrow A = \frac{1}{2}$$

$$\text{Let } r = -2 \Rightarrow 1 = B(-1)(1) \quad \Rightarrow B = -1$$

$$\text{Let } r = -3 \Rightarrow 1 = C(-2)(-1) \Rightarrow C = \frac{1}{2}$$

$$\therefore \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2}\left(\frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}\right)$$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2}\sum_{r=0}^{n}\left(\frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}\right)$$

$$\begin{aligned} &= \frac{1}{2}\Big(\frac{1}{1} - \frac{2}{2} + \frac{1}{3} \\ &\quad + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \\ &\quad + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \\ &\quad + \frac{1}{4} - \ldots + \ldots \\ &\quad + \ldots - \frac{2}{n+1} + \frac{1}{n+2} \\ &\quad + \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}\Big) \end{aligned}$$

Most terms cancel in 'threes' in lines rising diagonally.

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2}\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+2} - \frac{2}{n+2} + \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+2} + \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\left(\frac{(n+2)(n+3) - 2(n+3) + 2(n+2)}{2(n+2)(n+3)}\right)$$

$$= \frac{1}{2}\left(\frac{n^2 + 5n + 6 - 2n - 6 + 2n + 4}{2(n+2)(n+3)}\right)$$

$$= \frac{n^2 + 5n + 4}{4(n+2)(n+3)}$$

$$= \frac{(n+1)(n+4)}{4(n+2)(n+3)} \quad \therefore a = 1.\ b = 4,\ c = 4$$

[5 marks]

Answer 3

(a)

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r} + \sqrt{r+1}} = \sum_{r=1}^{n} \frac{1}{\sqrt{r} + \sqrt{r+1}} \times \frac{\sqrt{r} - \sqrt{r+1}}{\sqrt{r} - \sqrt{r+1}}$$

$$= \sum_{r=1}^{n} \frac{\sqrt{r} - \sqrt{r+1}}{r - (r+1)}$$

$$= \sum_{r=1}^{n} \sqrt{r+1} - \sqrt{r}$$

$$\begin{aligned} &= \sqrt{2} - \sqrt{1} \\ &+ \sqrt{3} - \sqrt{2} \\ &+ \sqrt{4} - \sqrt{3} \\ &+ \ \ldots \\ &+ \sqrt{n} - \sqrt{n-1} \\ &+ \sqrt{n+1} - \sqrt{n} \end{aligned}$$

All terms cancel except the second and the second last

$$\therefore \sum_{r=1}^{n} \frac{1}{\sqrt{r} + \sqrt{r+1}} = \sqrt{n+1} - 1$$

[4 marks]

(b)

$$\frac{1}{\sqrt{4} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{6}} + \ldots + \frac{1}{\sqrt{80} + \sqrt{81}}$$

$$= \sum_{r=1}^{80} \frac{1}{\sqrt{r} + \sqrt{r+1}} - \sum_{r=1}^{3} \frac{1}{\sqrt{r} + \sqrt{r+1}}$$

$$= \sqrt{81} - 1 - \left(\sqrt{4} - 1\right)$$

$$= 9 - 1 - 2 + 1$$

$$= 7$$

[2 marks]

Answer 4

(a)

$$\begin{aligned}
\text{RHS} &= \frac{1}{r!} - \frac{1}{(r+1)!} \\
&= \frac{1}{r!} \times \frac{(r+1)}{(r+1)} - \frac{1}{(r+1)!} \\
&= \frac{r+1-1}{(r+1)!} \\
&= \frac{r}{(r+1)!} \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[2 marks]

(b)

$$\begin{aligned}
\sum_{r=1}^{n} \frac{r}{(r+1)!} &= \sum_{r=1}^{n} \left(\frac{1}{r!} - \frac{1}{(r+1)!} \right) \\
&= \frac{1}{1!} - \frac{1}{2!} \\
&+ \frac{1}{2!} - \frac{1}{3!} \\
&+ \frac{1}{3!} - \ldots \\
&+ \ldots - \ldots \\
&+ \ldots - \frac{1}{n!} \\
&+ \frac{1}{n!} - \frac{1}{(n+1)!}
\end{aligned}$$

All terms cancel except the first and last

$$\therefore \sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$$

[5 marks]

Answer 5

$$\sum_{r=1}^{30}\left(\frac{1}{ln(r+1)}-\frac{1}{ln(r+2)}\right)=\frac{1}{ln\,2}-\frac{1}{ln\,3}$$

$$+\frac{1}{ln\,3}-\frac{1}{ln\,4}$$

$$+\frac{1}{ln\,4}-\ \dots$$

$$+\ \dots$$

$$+\ \dots\ -\frac{1}{ln\,31}$$

$$+\frac{1}{ln\,31}-\frac{1}{ln\,32}$$

$$=\frac{1}{ln\,2}-\frac{1}{ln\,32}$$

$$=\frac{1}{ln\,2}-\frac{1}{ln\,2^5}$$

$$=\frac{1}{ln\,2}-\frac{1}{5\,ln\,2}$$

$$=\frac{5-1}{5\,ln\,2}$$

$$=\frac{4}{5\,ln\,2}$$

[5 marks]

Answer 6

$$\frac{1}{(3r-2)(3r+2)} = \frac{A}{(3r-2)} + \frac{B}{(3r+2)}$$

$$1 = A(3r+2) + B(3r-2)$$

$$\text{Let } r = \frac{2}{3} \Rightarrow A = \frac{1}{4}$$

$$\text{Let } r = -\frac{2}{5} \Rightarrow B = -\frac{1}{4}$$

$$\therefore \frac{1}{(3r-2)(3r+2)} = \frac{1}{4}\left(\frac{1}{3r-2} - \frac{1}{3r+2}\right)$$

$$\text{Thus, } \sum_{r=1}^{n} \frac{1}{(3r-2)(3r+2)} = \frac{1}{4}\sum_{r=1}^{n}\left(\frac{1}{3r-2} - \frac{1}{3r+2}\right)$$

$$\begin{aligned} &= \frac{1}{4}\Bigg(\frac{1}{1} - \frac{1}{5} \\ &+ \frac{1}{4} - \frac{1}{8} \\ &+ \frac{1}{7} - \frac{1}{11} \\ &+ \frac{1}{10} - \frac{1}{14} \\ &+ \ldots - \frac{1}{3n-1} \\ &+ \frac{1}{3n-2} - \frac{1}{3n+2}\Bigg) \end{aligned}$$

And there is no cancelling to be had !

[4 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2025 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com

3.1 The Method of Differences

Thus far, when the Method of Differences has been employed, the differences have occurred somewhat accidentally as a result of applying partial fractions to problems. The concept can be more purposefully set up and deployed to powerful effect.

First, however, a formal statement of exactly what the method of differences is:

The Method of Differences

If the general term, u_r, of a series can be expressed in the form

$$f(r+1) - f(r)$$

then $$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \big(f(r+1) - f(r) \big)$$

In consequence,
$$u_1 = f(2) - f(1)$$
$$u_2 = f(3) - f(2)$$
$$u_3 = f(4) - f(3)$$
$$\ldots$$
$$u_n = f(n+1) - f(n)$$
$$\therefore \quad \sum_{r=1}^{n} u_r = f(n+1) - f(1)$$

3.2 Example

Use the method of differences to find a formula for,

$$\sum_{r=1}^{n} r(r+1)(r+2)$$

Begin by showing that,

$$r(r+1)(r+2) = \frac{1}{4}\big(r(r+1)(r+2)(r+3) - (r-1)r(r+1)(r+2)\big)$$

Teaching Video : http://www.NumberWonder.co.uk/v9097/3.mp4

[6 marks]

3.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

(a) Prove that $r(r + 1) - (r - 1)r = 2r$

[2 mark]

(b) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n + 1)$$

[4 marks]

Question 2

Use the facts that

- $(r+1)^3 - r^3 = 3r^2 + 3r + 1$
- $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ (Proven in Question 1)
- $\sum_{r=1}^{n} 1 = n$ (Obvious !)

to construct a proof, using the method of differences, that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

[6 marks]

Question 3

Further A-Level Examination Question from June 2016, FP2, Q2

(a) Show that,

$$r^2 (r + 1)^2 - (r - 1)^2 r^2 \equiv 4 r^3$$

[3 marks]

Given that $\sum_{r=1}^{n} r = \frac{1}{2} n (n + 1)$

(b) use the identity in (a) and the method of differences to show that

$$\left(1^3 + 2^3 + 3^3 + ... + n^3 \right) = \left(1 + 2 + 3 + ... + n \right)^2$$

[4 marks]

Question 4

This question is about finding a formula for the series,

$$\sum_{r=1}^{n} (r \times r!) = 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n!$$

(a) Prove that $\left((r + 1)! - 1 \right) - \left(r! - 1 \right) = r \times r!$

[2 marks]

(b) Use the part (a) answer and the method of differences to find a formula for $\displaystyle\sum_{r=1}^{n} (r \times r!)$

[4 marks]

(c) Evaluate $\displaystyle\sum_{r=1}^{6} (r \times r!)$

[1 mark]

Question 5

This question is about finding a formula for the series,

$$\sum_{r=1}^{n}\left(r^2+1\right)r! = 2\times 1! + 5\times 2! + 10\times 3! + \dots + \left(n^2+1\right)n!$$

(a) Prove that $\left((r+2)! - (r+1)!\right) - 2\left((r+1)! - r!\right) = \left(r^2+1\right)r!$

[4 marks]

(b) Use the part (a) answer and the method of differences to find a formula for $\sum_{r=1}^{n}\left(r^2+1\right)r!$

[6 marks]

Question 6

Further A-Level Examination Question from June 2011, FP2, Q4

Given that,

$$(2r + 1)^3 = Ar^3 + Br^2 + Cr + 1$$

(a) find the values of the constants A, B and C

[2 marks]

(b) Show that,

$$(2r + 1)^3 - (2r - 1)^3 = 24r^2 + 2$$

[2 marks]

(c) Using the result in part (b) and the method of differences, show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n + 1)(2n + 1)$$

[5 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2025 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com

3.4 Answers

3.4.1 Solution to 3.2 Example (Teaching Video)

$$\begin{aligned}
\text{RHS} &= \frac{1}{4}\big(r(r+1)(r+2)(r+3) - (r-1)r(r+1)(r+2)\big) \\
&= \frac{1}{4}r(r+1)(r+2)\big((r+3)-(r-1)\big) \\
&= \frac{1}{4}r(r+1)(r+2)(4) \\
&= r(r+1)(r+2) \\
&= \text{LHS} \qquad \square
\end{aligned}$$

With $f(r) = (r-1)r(r+1)(r+2)$

$$\begin{aligned}
\sum_{r=1}^{n} r(r+1)(r+2) &= \frac{1}{4}\sum_{r=1}^{n}\big(f(r+1) - f(r)\big) \\
&= \frac{1}{4}\big(f(n+1) - f(1)\big) \\
&= \frac{1}{4}\big(n(n+1)(n+2)(n+3) - 0\big) \\
&= \frac{1}{4}n(n+1)(n+2)(n+3)
\end{aligned}$$

3.4.2 Solutions to 3.3 Exercise

Answer 1

(a)

$$\begin{aligned} \text{LHS} &= r(r+1) - (r-1)r \\ &= r^2 + r - r^2 + r \\ &= 2r \\ &= \text{RHS} \qquad \square \end{aligned}$$

[2 marks]

(b) Version 1

$$\begin{aligned} \sum_{r=1}^{n} r &= \frac{1}{2} \sum_{r=1}^{n} \big(r(r+1) - (r-1)r \big) \\ &= \frac{1}{2} \big(2 - 0 \\ &\quad + 6 - 2 \\ &\quad + 12 - 6 \\ &\quad + \ldots \\ &\quad + n(n+1) - (n-1)n \big) \end{aligned}$$

All terms cancel except the second last

$$\therefore \quad \sum_{r=1}^{n} r = \frac{1}{2} n(n+1) \qquad \square$$

[4 marks]

(b) Version 2

$$r = \frac{1}{2} r(r+1) - \frac{1}{2}(r-1)r$$

$$f(r) = \frac{1}{2} r(r+1) \text{ and } f(r+1) = \frac{1}{2}(r+1)r$$

$$\begin{aligned} \sum_{r=1}^{n} r &= \sum_{r=1}^{n} \left(\frac{1}{2} r(r+1) - \frac{1}{2}(r-1)r \right) \\ &= \sum_{r=1}^{n} \big(f(r+1) - f(r) \big) \\ &= f(n+1) - f(1) \\ &= \frac{1}{2}(n+1)n - 0 \\ &= \frac{1}{2} n(n+1) \qquad \square \end{aligned}$$

[4 marks]

Answer 2

Version 1 for first part

$$(r+1)^3 - r^3 = 3r^2 + 3r + 1$$

$$\therefore \quad \sum_{r=1}^{n}\left((r+1)^3 - r^3\right) = \sum_{r=1}^{n}\left(3r^2 + 3r + 1\right)$$

Considering the LHS,

$$\begin{aligned}\sum_{r=1}^{n}\left((r+1)^3 - r^3\right) &= 8 - 1 \\ &+ 27 - 8 \\ &+ 64 - 27 \\ &+ \ldots \\ &+ (n+1)^3 - n^3\end{aligned}$$

All terms cancel except second and second last

$$\begin{aligned}\therefore \quad \sum_{r=1}^{n}\left((r+1)^3 - r^3\right) &= (n+1)^3 - 1 \\ &= n^3 + 3n^2 + 3n + 1 - 1 \\ &= n^3 + 3n^2 + 3n\end{aligned}$$

Version 2 for first part

Let $f(r) = r^3$ in which case $f(r+1) = (r+1)^3$

$$\begin{aligned}\text{Then,} \quad \sum_{r=1}^{n}(r+1)^3 - r^3 &= \sum_{r=1}^{n} f(r+1) - f(r) \\ &= f(n+1) - f(1) \\ &= ((n+1)^3 - 1^3 \\ &= n^3 + 3n^2 + 3n + 1 - 1 \\ &= n^3 + 3n^2 + 3n\end{aligned}$$

$$\text{RHS} = \sum_{r=1}^{n}\left(3r^2 + 3r + 1\right) = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= 3\sum_{r=1}^{n} r^2 + 3 \times \frac{1}{2} n(n+1) + n$$

$$3\sum_{r=1}^{n} r^2 + 3 \times \frac{1}{2} n(n+1) + n = n^3 + 3n^2 + 3n$$

$$6\sum_{r=1}^{n} r^2 + 3n(n+1) + 2n = 2n^3 + 6n^2 + 6n$$

$$6\sum_{r=1}^{n} r^2 = 2n^3 + 6n^2 + 6n - 3n(n+1) - 2n$$

$$= 2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n$$

$$= 2n^3 + 3n^2 + n$$

$$= n\left(2n^2 + 3n + 1\right)$$

$$= n(n+1)(2n+1)$$

$$\therefore \sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$$

[6 marks]

Answer 3
(a)

$$\begin{aligned} \text{LHS} &= r^2(r+1)^2 - (r-1)^2 r^2 \\ &= r^2\left(r^2+2r+1\right) - \left(r^2-2r+1\right)r^2 \\ &= r^4 + 2r^3 + r^2 - r^4 + 2r^3 - r^2 \\ &= 4r^3 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

(b) To prove that,

$$\left(1^3 + 2^3 + 3^3 + \ldots + n^3\right) = \left(1 + 2 + 3 + \ldots + n\right)^2$$

$$\begin{aligned} \text{LHS} = \left(1^3 + 2^3 + 3^3 + \ldots + n^3\right) &= \sum_{r=1}^{n} r^3 \\ &= \frac{1}{4}\sum_{r=1}^{n}\left(r^2(r+1)^2 - (r-1)^2 r^2\right) \\ &= \frac{1}{4}\left(4 - 0\right. \\ &\quad + 36 - 4 \\ &\quad + 144 - 36 \\ &\quad + \ldots \\ &\quad \left. + n^2(n+1)^2 - (n-1)n^2 \right) \end{aligned}$$

All terms cancel except the second (which is zero) and the second last

$$\therefore \left(1^3 + 2^3 + 3^3 + \ldots + n^3\right) = \frac{1}{4}n^2(n+1)^2$$

$$\begin{aligned} \text{RHS} = (1 + 2 + 3 + \ldots + n)^2 &= \left(\sum_{r=1}^{n} r\right)^2 \\ &= \left(\frac{1}{2}n(n+1)\right)^2 \\ &= \frac{1}{4}n^2(n+1)^2 \\ &= \text{LHS} \end{aligned}$$

$\therefore$ The claimed result is true $\square$

[4 marks]

Answer 4

(a)

$$\begin{aligned} \text{LHS} &= \big((r+1)! - 1\big) - \big(r! - 1\big) \\ &= (r+1)! - 1 - r! + 1 \\ &= (r+1)\, r! - r! \\ &= r \times r! + r! - r! \\ &= r \times r! \\ &= \text{RHS} \qquad \square \end{aligned}$$

[2 marks]

(b)

$$\begin{aligned} \sum_{r=1}^{n} (r \times r!) &= \sum_{r=1}^{n} \Big(\big((r+1)! - 1\big) - \big(r! - 1\big)\Big) \\ &= 1 - 0 \\ &\quad + 5 - 1 \\ &\quad + 23 - 5 \\ &\quad + \ \ldots \\ &\quad + \big((n+1)! - 1\big) - \big(n! - 1\big) \end{aligned}$$

All terms cancel except the second, which is zero, and the second last

$$\therefore \quad \sum_{r=1}^{n} (r \times r!) = (n+1)! - 1$$

[4 marks]

(c)

$$\begin{aligned} \sum_{r=1}^{6} (r \times r!) &= 7! - 1 \\ &= 5039 \end{aligned}$$

[1 mark]

Answer 5

(a)

$$\begin{aligned} \text{LHS} &= \big((r+2)! - (r+1)!\big) - 2\big((r+1)! - r!\big) \\ &= (r+2)(r+1)r! - (r+1)r! - 2(r+1)r! + 2r! \\ &= r!\big((r+2)(r+1) - 3(r+1) + 2\big) \\ &= r!\big(r^2 + 3r + 2 - 3r - 3 + 2\big) \\ &= r!\big(r^2 + 1\big) \\ &= \text{RHS} \qquad \square \end{aligned}$$

[4 marks]

(b)

$$\sum_{r=1}^{n}\left(r^2+1\right)r! = \sum_{r=1}^{n}\left((r+2)!-(r+1)!\right) - 2\sum_{r=1}^{n}\left((r+1)!-r!\right)$$

$$\begin{aligned}\sum_{r=1}^{n}\left((r+2)!-(r+1)!\right) &= 6-2\\ &+24-6\\ &+120-24\\ &+\ldots\\ &+(n+2)!-(n+1)!\end{aligned}$$

All terms cancel except the second and second last

$$\therefore \quad \sum_{r=1}^{n}\left((r+2)!-(r+1)!\right) = (n+2)!-2$$

$$\begin{aligned}\sum_{r=1}^{n}\left((r+1)!-r!\right) &= 2-1\\ &+6-2\\ &+24-6\\ &+\ldots\\ &+(n+1)!-n!\end{aligned}$$

All terms cancel except the second and second last

$$\therefore \quad \sum_{r=1}^{n}\left((r+1)!-r!\right) = (n+1)!-1$$

$$\begin{aligned}\sum_{r=1}^{n}\left(r^2+1\right)r! &= (n+2)!-2-2\left((n+1)!-1\right)\\ &= (n+2)!-2-2(n+1)!+2\\ &= (n+2)(n+1)!-2(n+1)!\\ &= (n+1)!\,\left(n+2-2\right)\\ &= n(n+1)!\end{aligned}$$

[6 marks]

Answer 6

(a)

$$(2r+1)^3 = 8r^3 + 3\times 4r^2 + 3\times 2r + 1$$
$$= 8r^3 + 12r^2 + 6r + 1$$
$$\therefore A = 8,\ \ B = 12 \text{ and } C = 6$$

[2 marks]

(b)

$$\text{LHS} = (2r+1)^3 - (2r-1)^3$$
$$= 8r^3 + 12r^2 + 6r + 1 - \left(8r^3 - 12r^2 + 6r - 1\right)$$
$$= 24r^2 + 2$$
$$= \text{RHS} \qquad \square$$

[2 marks]

(c) From part (b), $(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$

$$\therefore \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right) = 24\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} 1$$

$$\begin{aligned}\text{LHS} = \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right) &= 27 - 1\\ &+ 125 - 27\\ &+ 343 - 125\\ &+ \ldots\\ &+ (2n+1)^3 - (2n-1)^3\end{aligned}$$

$$\therefore \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right) = (2n+1)^3 - 1$$

Thus, $24\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} 1 = (2n+1)^3 - 1$

$$\begin{aligned}24\sum_{r=1}^{n} r^2 &= 8n^3 + 12n^2 + 6n + 1 - 1 - 2n\\ &= 8n^3 + 12n^2 + 4n\\ &= 4n\left(2n^2 + 3n + 1\right)\\ &= 4n(n+1)(2n+1)\end{aligned}$$

$$\therefore \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \square$$

[5 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2025 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com

4.1 Test

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

(i) Prove that, $\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$

[1 marks]

(ii) Hence, using the method of differences, find a formula for,

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$$

[5 marks]

(iii) State if the sum to infinity, $\sum_{r=1}^{\infty} \frac{2r+1}{r^2(r+1)^2}$, is convergent or divergent.

Give a reason for your answer.

[2 marks]

Question 2

Further A-Level Examination Question from June 2014, FP2, Q1

(a) Express $\dfrac{2}{(r+2)(r+4)}$ in partial fractions.

[1 mark]

(b) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \frac{n(7n+25)}{12(n+3)(n+4)}$

[5 marks]

Question 3

(i) Use the method of differences to obtain a formula for,

$$\sum_{r=1}^{n}\left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}}\right)$$

[3 marks]

(ii) Hence, or otherwise, determine the exact value of,

$$\sum_{r=5}^{9}\left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}}\right)$$

[3 marks]

Question 4

Further A-Level Examination Question from June 2016, FP2, Q2

(a) Show that, for $r > 0$

$$r - 3 + \frac{1}{r+1} - \frac{1}{r+2} = \frac{r^3 - 7r - 5}{(r+1)(r+2)}$$

[2 marks]

(b) Hence prove, using the method of differences, that

$$\sum_{r=1}^{n} \frac{r^3 - 7r - 5}{(r+1)(r+2)} = \frac{n\left(n^2 + an + b\right)}{2(n+2)}$$

where a and b are constants to be found

[5 marks]

Question 5

(i) Prove that $\dfrac{2^k}{k+1} - \dfrac{2^{k-1}}{k} = \dfrac{(k-1)\,2^{k-1}}{k(k+1)}$

[3 marks]

(ii) Use the method of differences to obtain a formula for,

$$\sum_{r=1}^{n}\left(\frac{(k-1)\,2^{k-1}}{k(k+1)}\right)$$

[4 marks]

Question 6

Prove that,

$$\frac{2}{0! + 1! + 2!} + \frac{3}{1! + 2! + 3!} + ... + \frac{n}{(n - 2)! + (n - 1)! + n!} = 1 - \frac{1}{n!}$$

[6 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”
© 2025 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com

4.2 Answers

Answer 1

(i)

$$\begin{aligned}\text{LHS} &= \frac{1}{r^2} - \frac{1}{(r+1)^2} \\ &= \frac{1}{r^2} \times \frac{(r+1)^2}{(r+1)^2} - \frac{1}{(r+1)^2} \times \frac{r^2}{r^2} \\ &= \frac{r^2 + 2r + 1 - r^2}{r^2(r+1)^2} \\ &= \frac{2r+1}{r^2(r+1)^2} \\ &= \text{RHS} \qquad \square\end{aligned}$$

[1 mark]

(ii)

$$\begin{aligned}\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} &= \sum_{r=1}^{n} \left(\frac{1}{r^2} - \frac{1}{(r+1)^2} \right) \\ &= \frac{1}{1^2} - \frac{1}{2^2} \\ &+ \frac{1}{2^2} - \frac{1}{3^2} \\ &+ \frac{1}{3^2} - \frac{1}{4^2} \\ &+ \ldots \\ &+ \frac{1}{n^2} - \frac{1}{(n+1)^2}\end{aligned}$$

All terms cancel except the first and last

$$\therefore \sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

[5 marks]

(iii) Convergent

$$\text{As } n \to \infty, \quad \frac{1}{(n+1)^2} \to 0$$

$$\therefore \sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1$$

[2 marks]

Answer 2

(a)

$$\frac{2}{(r+2)(r+4)} = \frac{A}{(r+2)} + \frac{B}{(r+4)}$$

Multiply through by $\left[\ (r+2)(r+4)\ \right]$

$$2 = A(r+4) + B(r+2)$$

Let $r = -2 \Rightarrow 2 = 2A \ \therefore\ A = 1$

Let $r = -4 \Rightarrow 2 = -2B \ \therefore\ B = -1$

Thus, $$\frac{2}{(r+2)(r+4)} = \frac{1}{(r+2)} - \frac{1}{(r+4)}$$

[1 mark]

(b)

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} \\
&= \sum_{r=1}^{n} \left(\frac{1}{(r+2)} - \frac{1}{(r+4)} \right) \\
&= \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} \\
&\quad + \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8} \\
&\quad + \frac{1}{7} - \frac{1}{9} + \frac{1}{8} - \frac{1}{10} \\
&\quad + \ldots \\
&\quad + \frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+2} - \frac{1}{n+4}
\end{aligned}$$

Most terms cancel, leaving first, third, third last and last

$$\begin{aligned}
&= \frac{1}{3} + \frac{1}{4} - \frac{1}{n+3} - \frac{1}{n+4} \\
&= \frac{7(n+3)(n+4) - 12(n+4) - 12(n+3)}{12(n+3)(n+4)} \\
&= \frac{7n^2 + 49n + 84 - 12n - 48 - 12n - 36}{12(n+3)(n+4)} \\
&= \frac{7n^2 + 25n}{12(n+3)(n+4)} \\
&= \frac{n(7n+25)}{12(n+3)(n+4)} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

Answer 3

(i)

$$\sum_{r=1}^{n}\left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}}\right) = 1024^{\frac{1}{1}} - 1024^{\frac{1}{2}}$$

$$+\ 1024^{\frac{1}{2}} - 1024^{\frac{1}{3}}$$

$$+\ 1024^{\frac{1}{3}} - 1024^{\frac{1}{4}}$$

$$+\ \ldots$$

$$+\ 1024^{\frac{1}{n}} - 1024^{\frac{1}{n+1}}$$

All terms cancel except the first and last

$$= 1024^{\frac{1}{1}} - 1024^{\frac{1}{n+1}}$$

$$= 1024 - 1024^{\frac{1}{n+1}}$$

[3 marks]

(ii)

$$\sum_{r=5}^{9}\left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}}\right) = \sum_{r=1}^{9}\left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}}\right) - \sum_{r=1}^{4}\left(1024^{\frac{1}{r}} - 1024^{\frac{1}{r+1}}\right)$$

$$= 1024 - 1024^{\frac{1}{10}} - 1024 + 1024^{\frac{1}{5}}$$

$$= -2 + 4$$

$$= 2$$

[3 marks]

Answer 4

(a)

$$\begin{aligned}\text{LHS} &= r - 3 + \frac{1}{r+1} - \frac{1}{r+2} \\ &= \frac{(r-3)(r+1)(r+2) + (r+2) - (r+1)}{(r+1)(r+2)} \\ &= \frac{(r-3)\left(r^2+3r+2\right) + r + 2 - r - 1}{(r+1)(r+2)} \\ &= \frac{r^3 + 3r^2 + 2r - 3r^2 - 9r - 6 + 1}{(r+1)(r+2)} \\ &= \frac{r^3 - 7r - 5}{(r+1)(r+2)} \\ &= \text{RHS} \qquad \square\end{aligned}$$

[2 marks]

(b) First of all,

$$\begin{aligned}\sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right) &= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n+1} - \frac{1}{n+2} \\ &= \frac{1}{2} - \frac{1}{n+2}\end{aligned}$$

In consequence,

$$\begin{aligned}\sum_{r=1}^{n}\frac{r^3 - 7r - 5}{(r+1)(r+2)} &= \sum_{r=1}^{n}\left(r - 3 + \frac{1}{r+1} - \frac{1}{r+2}\right) \\ &= \sum_{r=1}^{n} r - \sum_{r=1}^{n} 3 + \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right) \\ &= \frac{1}{2}n(n+1) - 3n + \frac{1}{2} - \frac{1}{n+2} \\ &= \frac{n(n+1)(n+2) - 6n(n+2) + (n+2) - 2}{2(n+2)} \\ &= \frac{n^3 + 3n^2 + 2n - 6n^2 - 12n + n + 2 - 2}{2(n+2)} \\ &= \frac{n^3 - 3n^2 - 9n}{2(n+2)} \\ &= \frac{n\left(n^2 - 3n - 9\right)}{2(n+2)}\end{aligned}$$

$$i.e.\ a = -3, \quad b = -9$$

[5 marks]

Answer 5

(i)

$$
\begin{aligned}
\text{LHS} &= \frac{2^k}{k+1} - \frac{2^{k-1}}{k} \\
&= \frac{2 \times 2^{k-1}}{k+1} - \frac{2^{k-1}}{k} \\
&= \frac{2 \times 2^{k-1}}{k+1} \times \frac{k}{k} - \frac{2^{k-1}}{k} \times \frac{(k+1)}{(k+1)} \\
&= \frac{2^{k-1}(2k - k - 1)}{k(k+1)} \\
&= \frac{(k-1)\,2^{k-1}}{k(k+1)} \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[3 marks]

(ii)

$$
\begin{aligned}
\sum_{r=1}^{n} \left(\frac{(k-1)\,2^{k-1}}{k(k+1)} \right) = \sum_{r=1}^{n} \left(\frac{2^k}{k+1} - \frac{2^{k-1}}{k} \right) &= \frac{2^1}{2} - \frac{2^0}{1} \\
&+ \frac{2^2}{3} - \frac{2^1}{2} \\
&+ \frac{2^3}{4} - \frac{2^2}{3} \\
&+ \ldots \\
&+ \frac{2^n}{n+1} - \frac{2^{n-1}}{n}
\end{aligned}
$$

All terms cancel except the second and second last

$$
= \frac{2^n}{n+1} - 1
$$

[4 marks]

Answer 6

The series needs manipulating into an expression that's a difference.

$$\frac{r}{(r-2)!+(r-1)!+r!} = \frac{r}{(r-2)!+(r-1)(r-2)!+r(r-1)(r-2)!}$$

$$= \frac{r}{(r-2)!\,\left[1+(r-1)+r(r-1)\right]}$$

$$= \frac{r}{(r-2)!\,\left[1+r-1+r^2-r\right]}$$

$$= \frac{r}{(r-2)!\,\left[r^2\right]}$$

$$= \frac{1}{r(r-2)!}$$

$$= \frac{(r-1)}{r(r-1)(r-2)!}$$

$$= \frac{r-1}{r!}$$

$$= \frac{r}{r!} - \frac{1}{r!}$$

$$= \frac{r}{r(r-1)!} - \frac{1}{r!}$$

$$= \frac{1}{(r-1)!} - \frac{1}{r!}$$

Thus, $$\sum_{r=2}^{n} \frac{r}{(r-2)!+(r-1)!+r!} = \sum_{r=2}^{n}\left(\frac{1}{(r-1)!} - \frac{1}{r!}\right)$$

$$= \frac{1}{1!} - \frac{1}{2!}$$

$$+ \frac{1}{2!} - \frac{1}{3!}$$

$$+ \frac{1}{3!} - \frac{1}{4!}$$

$$+ \ \ldots$$

$$+ \frac{1}{(n-1)!} - \frac{1}{n!}$$

All terms cancel except the first and the last

$$\therefore \sum_{r=2}^{n} \frac{r}{(r-2)!+(r-1)!+r!} = 1 - \frac{1}{n!} \qquad \square$$

[6 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”
© 2025 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com