2.4 Answers

2.4.1 Solution to 2.2 Example (Teaching Video)

$$f(x) = e^{\sin x} \qquad \Rightarrow f(0) = 1$$

$$f'(x) = e^{\sin x} \cos x \qquad \Rightarrow f'(0) = 1$$

$$f''(x) = e^{\sin x}(-\sin x) + e^{\sin x}\cos x \cos x$$

$$= e^{\sin x}(\cos^2 x - \sin x) \qquad \Rightarrow f''(0) = 1$$

By Maclaurin,

$$f(x) \approx f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2$$

$$f(x) \approx 1 + x + \frac{x^2}{2}$$

[6 marks]

2.4.2 Solutions to 2.3 Exercise

Answer 1

$$f(x) = e^{\cos x} \qquad \Rightarrow f(0) = e$$

$$f'(x) = e^{\cos x}(-\sin x) \qquad \Rightarrow f'(0) = 0$$

$$f''(x) = e^{\cos x}(-\cos x) + e^{\cos x}(-\sin x)(-\sin x)$$

$$= e^{\cos x}\left(\sin^2 x - \cos x\right) \qquad \Rightarrow f''(0) = -e$$

By Maclaurin,

$$f(x) \approx f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2$$

$$f(x) \approx e - \frac{e\,x^2}{2}$$

[6 marks]

Answer 2

$$f(x) = \cos\left(\pi e^x\right) \qquad \Rightarrow f(0) = -1$$

$$f'(x) = -\sin\left(\pi e^x\right)\pi e^x \qquad \Rightarrow f'(0) = 0$$

$$f''(x) = -\sin\left(\pi e^x\right)\pi e^x - \cos\left(\pi e^x\right)\pi e^x\,\pi e^x$$

$$= \pi e^x\left(-\sin\left(\pi e^x\right) - \pi e^x \cos\left(\pi e^x\right)\right) \qquad \Rightarrow f''(0) = \pi^2$$

By Maclaurin,

$$f(x) \approx f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2$$

$$f(x) \approx -1 + \frac{\pi^2}{2}x^2$$

[6 marks]

Answer 3

$$f(x) = \sin(\pi \cos x) \qquad \Rightarrow f(0) = 0$$

$$f'(x) = \cos(\pi \cos x)\ \pi\ (-\sin x) \qquad \Rightarrow f'(0) = 0$$

$$f''(x) = \cos(\pi \cos x)\ \pi\ (-\cos x) + (-\sin(\pi \cos x)\ \pi\ (-\sin x)\ \pi\ (-\sin x))$$

$$= -\pi \cos x \cos(\pi \cos x) - \pi^2 \sin^2 x \sin(\pi \cos x) \qquad \Rightarrow f''(0) = \pi$$

By Maclaurin, $\quad f(x) \approx f(0) + f'(0)\ x + \dfrac{f''(0)}{2!} x^2$

$$f(x) \approx \frac{\pi}{2} x^2$$

[6 marks]

Answer 4

$$f(x) = e^{1-x^2} \qquad \Rightarrow f(0) = e$$

$$f'(x) = e^{1-x^2}\ (-2x) \qquad \Rightarrow f'(0) = 0$$

$$f''(x) = e^{1-x^2}(-2) + e^{1-x^2}(-2x)(-2x)$$

$$= -2e^{1-x^2} + 4x^2 e^{1-x^2} \qquad \Rightarrow f''(0) = -2e$$

By Maclaurin, $\quad f(x) \approx f(0) + f'(0)\ x + \dfrac{f''(0)}{2!} x^2$

$$f(x) \approx e - e x^2$$

[6 mark]

Answer 5

(i)

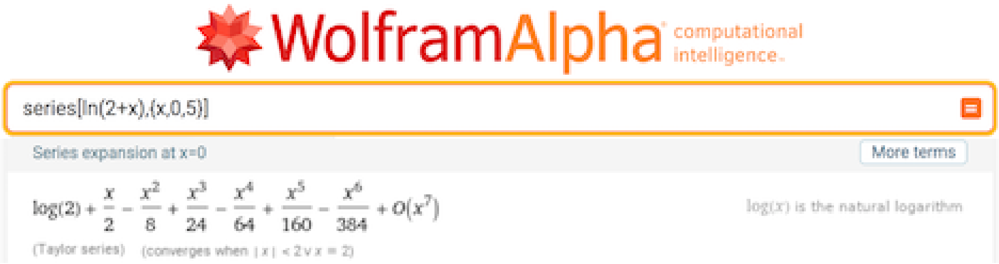
$f(x) = \ln(2 + x)$	when $x = 0$,	$f(0) = \ln 2$
$f'(x) = \dfrac{1}{2 + x}$	when $x = 0$,	$f'(0) = \dfrac{1}{2}$
$f''(x) = -\dfrac{1}{(2 + x)^2}$	when $x = 0$,	$f''(0) = -\dfrac{1}{4}$
$f'''(x) = \dfrac{2}{(2 + x)^3}$	when $x = 0$,	$f'''(0) = \dfrac{1}{4}$
$f^{(4)}(x) = -\dfrac{6}{(2 + x)^4}$	when $x = 0$,	$f^{(4)}(0) = -\dfrac{3}{8}$
$f^{(5)}(x) = \dfrac{24}{(2 + x)^5}$	when $x = 0$,	$f^{(5)}(0) = \dfrac{3}{4}$

(ii) $$f(x) \approx \ln 2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} - \frac{x^4}{64} + \frac{x^5}{160}$$

[8 marks]

Wolfram Alpha can do Maclaurin series.
To check your answer to question 5,

$$series[\ ln(2 + x),\ \{x,\ 0,\ 5\}\ ]$$



Answer 6

$f(x) = \ln(1 + \sin x)$	when $x = 0$,	$f(0) = 0$
$f'(x) = \dfrac{\cos x}{1 + \sin x}$	when $x = 0$,	$f'(0) = 1$
$f''(x) = -\dfrac{1}{1 + \sin x}$	when $x = 0$,	$f''(0) = -1$
$f'''(x) = -\dfrac{\cos x}{(1 + \sin x)^2}$	when $x = 0$,	$f'''(0) = -1$
$f^{(4)}(x) = \dfrac{\sin x - \sin^2 x + 2\cos^2 x}{(1 + \sin x)^3}$	when $x = 0$,	$f^{(4)}(0) = 2$

$$f(x) \approx x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12}$$

[10 marks]