3.5.1 Solution to 3.3.Example (Teaching Video)

From the standard results that, $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots$

and $\quad sin\, x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots$

$$e^{\frac{x}{\pi}} sin\left(x^2\right) = \left(1 + \frac{x}{\pi} + \frac{x^2}{2\pi^2} + \frac{x^3}{6\pi^3} + \ldots\right)\left(x^2 - \frac{x^6}{6} + \ldots\right)$$

$$= x^2 + \frac{x^3}{\pi} + \frac{x^4}{2\pi^2} + \frac{x^5}{6\pi^3} + \ldots$$

So $f(x) \approx x^2 + \frac{x^3}{\pi} + \frac{x^4}{2\pi^2} + \frac{x^5}{6\pi^3}$

[6 marks]

3.5.2 Solutions to 3.4 Exercise

Answer 1

(i) From the standard result that,

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{valid for all } x$$

$$g(x) = \frac{1}{e^x}$$

$$= e^{-x}$$

$$= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \ldots$$

$$\approx 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$$

(ii) It's valid for all x, but increasingly inaccurate as one moves away from its centred value of $x = 0$

[3 marks]

Answer 2

(i) From the standard result that,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{valid for all } x$$

$$\begin{aligned} h(x) &= 2x \sin(3x) \\ &= 2x\left(3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} + \dots\right) \\ &\approx 6x^2 - 9x^4 + \frac{81}{20}x^6 \end{aligned}$$

(ii) It's valid for all x, but increasingly inaccurate as one moves away from its centred value of $x = 0$, as is obvious from the provided graph.

[3 marks]

Answer 3

(i) The standard result is,

- $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots + (-1)^r \dfrac{x^{2r+1}}{2r+1} + \dots \quad -1 < x \leqslant 1$

With $x = 1$ this yields,

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

Remembering to use radians,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\right)$$

[2 marks]

(ii) Using the first five terms of the series,

$$\begin{aligned} \pi &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\right) \\ &= \frac{1052}{315} \\ &= 3.3396825\dots \end{aligned}$$

[2 marks]

Answer 4

$$g(x) = \frac{1}{1 + x^2}$$
$$= \frac{1}{1 - \left(-x^2\right)}$$
$$= \left(1 - \left(-x^2\right)\right)^{-1}$$

Substituting $\left(-x^2\right)$ into

- $(1 - x)^{-1} = 1 + x + x^2 + \ldots + x^r + \ldots$ valid for $-1 < x < 1$

gives

$$g(x) = 1 + \left(-x^2\right) + \left(-x^2\right)^2 + \left(-x^2\right)^3 + \left(-x^2\right)^4 + \ldots$$
$$\approx 1 - x^2 + x^4 - x^6 + x^8$$

[3 marks]

Answer 5

(i) From the standard result that,

- $ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots + (-1)^{r+1}\dfrac{x^r}{r} + \ldots \quad -1 < x \leqslant 1$

and the substitution of $(-x)$ in place of x it is deduced that,

$$ln(1 - x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \ldots \qquad -1 \leqslant x < 1$$

$$ln\left(\frac{1 + x}{1 - x}\right) = ln(1 + x) - ln(1 - x)$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right)$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \ldots$$
$$\approx 2x + \frac{2x^3}{3} + \frac{2x^5}{5} \qquad \text{valid for } -1 < x < 1$$

[4 marks]

(ii)

$$\frac{1 + x}{1 - x} = 3$$
$$1 + x = 3(1 - x)$$
$$1 + x = 3 - 3x$$
$$4x = 2$$
$$x = \frac{1}{2}$$
$$\therefore\ ln\,3 \approx 2 \times \frac{1}{2} + \frac{2}{3} \times \left(\frac{1}{2}\right)^3 + \frac{2}{5} \times \left(\frac{1}{2}\right)^5$$
$$ln\,3 \approx 1 + \frac{1}{12} + \frac{1}{80}$$
$$ln\,3 \approx 1.096$$

[4 marks]

Answer 6

(i) From the standard result that,

- $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{r+1}\frac{x^r}{r} + ... \qquad -1 < x \leqslant 1$

and the substitution of ($2x$) in place of x it is deduced that,

$$ln(1 + 2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + ... \qquad -\frac{1}{2} < x \leqslant \frac{1}{2}$$
$$f(x) = (1 - 3x)\ ln(1 + 2x)$$
$$f(x) \approx (1 - 3x)\left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4\right)$$
$$f(x) \approx 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 - 6x^2 + 6x^3 - 8x^4$$
$$f(x) \approx 2x - 8x^2 + \frac{26}{3}x^3 - 12x^4$$

(ii) It's valid for $-\frac{1}{2} < x \leqslant \frac{1}{2}$

[4 marks]

Answer 7

(i) From the standard result that,

- $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{r+1}\frac{x^r}{r} + ... \quad -1 < x \leqslant 1$

and the substitution of $\left(\frac{3x}{2}\right)$ in place of x it is deduced that,

$$ln\left(1 + \frac{3x}{2}\right) = \left(\frac{3x}{2}\right) - \frac{1}{2}\left(\frac{3x}{2}\right)^2 + \frac{1}{3}\left(\frac{3x}{2}\right)^3 - \frac{1}{4}\left(\frac{3x}{2}\right)^4 + ...$$

$$= \frac{3x}{2} - \frac{9x^2}{8} + \frac{9x^3}{8} - \frac{81x^4}{64} + ... \qquad -\frac{2}{3} < x \leqslant \frac{2}{3}$$

$$g(x) = 64x\, ln(2 + 3x)$$

$$= 64x\, ln\left(2\left(1 + \frac{3x}{2}\right)\right)$$

$$= 64x\, ln(2) + 64x\, ln\left(1 + \frac{3x}{2}\right)$$

$$g(x) \approx 64\, ln(2)x + 64x\left(\frac{3x}{2} - \frac{9x^2}{8} + \frac{9x^3}{8} - \frac{81x^4}{64}\right)$$

$$g(x) \approx 64\, ln(2)x + 96x^2 - 72x^3 + 72x^4 - 81x^5$$

(ii) It's valid for $-\frac{2}{3} < x \leqslant \frac{2}{3}$

[4 marks]

Answer 8

(i) From the standard result that, $e^{x} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots$

$$e^{-\frac{x^2}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2!}\left(-\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x^2}{2}\right)^3 + \frac{1}{4!}\left(-\frac{x^2}{2}\right)^4 + \ldots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384} + \ldots$$

[3 marks]

(ii)

$$\int_{-1}^{1} e^{-\frac{x^2}{2}}\,dx \approx \int_{-1}^{1}\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384}\right)dx$$

$$= \left[\, x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456}\,\right]_{-1}^{1}$$

$$= \left[1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456}\right] - \left[-1 + \frac{1}{6} - \frac{1}{40} + \frac{1}{336} - \frac{1}{3456}\right]$$

$$= 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} + 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456}$$

$$= 2 \times \left[1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456}\right]$$

$$= 1.711$$

[3 marks]

Answer 9

$$e^{t} \approx 1 + t + \frac{t^2}{2!}$$

$$\frac{t}{e^{t} - 1} \approx \frac{t}{1 + t + \frac{t^2}{2!} - 1} = \frac{t}{t + \frac{t^2}{2!}}$$

$$= \frac{1}{1 + \frac{t}{2}}$$

$$= \left(1 + \frac{t}{2}\right)^{-1}$$

By the binomial theorem, $(\,1 + x\,)^{n} = 1 + nx + \frac{n\,(n - 1)}{2!}\,x^2 + \ldots$

$$\therefore \left(1 + \frac{t}{2}\right)^{-1} \approx 1 + (-1)\left(\frac{t}{2}\right) = 1 - \frac{1}{2}\,t \qquad \square$$

[5 marks]