

3.5 Answers

Further A-Level Pure Mathematics, Core 2 Maclaurin Series

3.5.1 Solution to 3.3.Example (Teaching Video)

From the standard results that, $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots$

$$\text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots$$

$$\begin{aligned} e^{\frac{x}{\pi}} \sin(x^2) &= \left(1 + \frac{x}{\pi} + \frac{x^2}{2\pi^2} + \frac{x^3}{6\pi^3} + \dots \right) \left(x^2 - \frac{x^6}{6} + \dots \right) \\ &= x^2 + \frac{x^3}{\pi} + \frac{x^4}{2\pi^2} + \frac{x^5}{6\pi^3} + \dots \end{aligned}$$

$$\text{So } f(x) \approx x^2 + \frac{x^3}{\pi} + \frac{x^4}{2\pi^2} + \frac{x^5}{6\pi^3}$$

[6 marks]

3.5.2 Solutions to 3.4 Exercise

Answer 1

(i) From the standard result that,

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots \quad \text{valid for all } x$$

$$g(x) = \frac{1}{e^x}$$

$$= e^{-x}$$

$$= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \dots$$

$$\approx 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$$

(ii) It's valid for all x , but increasingly inaccurate as one moves away from its centred value of $x = 0$

[3 marks]

Answer 2

- (i) From the standard result that,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{valid for all } x$$

$$h(x) = 2x \sin(3x)$$

$$= 2x \left(3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \dots \right)$$

$$\approx 6x^2 - 9x^4 + \frac{81}{20}x^6$$

- (ii) It's valid for all x , but increasingly inaccurate as one moves away from its centred value of $x = 0$, as is obvious from the provided graph.

[3 marks]

Answer 3

- (i) The standard result is,

$$\bullet \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^r \frac{x^{2r+1}}{2r+1} + \dots \quad -1 < x \leq 1$$

With $x = 1$ this yields,

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

Remembering to use radians,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

$$\pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \right)$$

[2 marks]

- (ii) Using the first five terms of the series,

$$\pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \right)$$

$$= \frac{1052}{315}$$

$$= 3.3396825\dots$$

[2 marks]

Answer 4

$$\begin{aligned}
 g(x) &= \frac{1}{1+x^2} \\
 &= \frac{1}{1-(-x^2)} \\
 &= (1-(-x^2))^{-1}
 \end{aligned}$$

Substituting $(-x^2)$ into

• $(1-x)^{-1} = 1 + x + x^2 + \dots + x^r + \dots$ valid for $-1 < x < 1$
gives

$$\begin{aligned}
 g(x) &= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \dots \\
 &\approx 1 - x^2 + x^4 - x^6 + x^8
 \end{aligned}$$

[3 marks]

Answer 5

(i) From the standard result that,

$$\bullet \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1} \frac{x^r}{r} + \dots \quad -1 < x \leq 1$$

and the substitution of $(-x)$ in place of x it is deduced that,

$$\ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \dots \quad -1 \leq x < 1$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$\begin{aligned}
 &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\right) \\
 &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \dots \\
 &\approx 2x + \frac{2x^3}{3} + \frac{2x^5}{5} \quad \text{valid for } -1 < x < 1
 \end{aligned}$$

[4 marks]

(ii)

$$\frac{1+x}{1-x} = 3$$

$$1+x = 3(1-x)$$

$$1+x = 3-3x$$

$$4x = 2$$

$$x = \frac{1}{2}$$

$$\therefore \ln 3 \approx 2 \times \frac{1}{2} + \frac{2}{3} \times \left(\frac{1}{2}\right)^3 + \frac{2}{5} \times \left(\frac{1}{2}\right)^5$$

$$\ln 3 \approx 1 + \frac{1}{12} + \frac{1}{80}$$

$$\ln 3 \approx 1.096$$

[4 marks]

Answer 6

(i) From the standard result that,

$$\bullet \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1} \frac{x^r}{r} + \dots \quad -1 < x \leq 1$$

and the substitution of $(2x)$ in place of x it is deduced that,

$$\ln(1+2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \dots \quad -\frac{1}{2} < x \leq \frac{1}{2}$$

$$f(x) = (1-3x) \ln(1+2x)$$

$$f(x) \approx (1-3x) \left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 \right)$$

$$f(x) \approx 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 - 6x^2 + 6x^3 - 8x^4$$

$$f(x) \approx 2x - 8x^2 + \frac{26}{3}x^3 - 12x^4$$

(ii) It's valid for $-\frac{1}{2} < x \leq \frac{1}{2}$

[4 marks]

Answer 7**(i)** From the standard result that,

$$\bullet \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1} \frac{x^r}{r} + \dots \quad -1 < x \leq 1$$

and the substitution of $\left(\frac{3x}{2}\right)$ in place of x it is deduced that,

$$\begin{aligned} \ln\left(1 + \frac{3x}{2}\right) &= \left(\frac{3x}{2}\right) - \frac{1}{2}\left(\frac{3x}{2}\right)^2 + \frac{1}{3}\left(\frac{3x}{2}\right)^3 - \frac{1}{4}\left(\frac{3x}{2}\right)^4 + \dots \\ &= \frac{3x}{2} - \frac{9x^2}{8} + \frac{9x^3}{8} - \frac{81x^4}{64} + \dots \quad -\frac{2}{3} < x \leq \frac{2}{3} \end{aligned}$$

$$g(x) = 64x \ln(2 + 3x)$$

$$= 64x \ln\left(2\left(1 + \frac{3x}{2}\right)\right)$$

$$= 64x \ln(2) + 64x \ln\left(1 + \frac{3x}{2}\right)$$

$$g(x) \approx 64 \ln(2)x + 64x \left(\frac{3x}{2} - \frac{9x^2}{8} + \frac{9x^3}{8} - \frac{81x^4}{64}\right)$$

$$g(x) \approx 64 \ln(2)x + 96x^2 - 72x^3 + 72x^4 - 81x^5$$

(ii) It's valid for $-\frac{2}{3} < x \leq \frac{2}{3}$ **[4 marks]**

Answer 8

(i) From the standard result that, $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots$

$$\begin{aligned} e^{-\frac{x^2}{2}} &= 1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2!}\left(-\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x^2}{2}\right)^3 + \frac{1}{4!}\left(-\frac{x^2}{2}\right)^4 + \dots \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384} + \dots \end{aligned}$$

[3 marks]**(ii)**

$$\begin{aligned} \int_{-1}^1 e^{-\frac{x^2}{2}} dx &\approx \int_{-1}^1 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384}\right) dx \\ &= \left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456} \right]_{-1}^1 \\ &= \left[1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} \right] - \left[-1 + \frac{1}{6} - \frac{1}{40} + \frac{1}{336} - \frac{1}{3456} \right] \\ &= 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} + 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} \\ &= 2 \times \left[1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} \right] \\ &= 1.711 \end{aligned}$$

[3 marks]**Answer 9**

$$\begin{aligned} e^t &\approx 1 + t + \frac{t^2}{2!} \\ \frac{t}{e^t - 1} &\approx \frac{t}{1 + t + \frac{t^2}{2!} - 1} = \frac{t}{t + \frac{t^2}{2!}} \\ &= \frac{1}{1 + \frac{t}{2}} \\ &= \left(1 + \frac{t}{2}\right)^{-1} \end{aligned}$$

By the binomial theorem, $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$

$$\therefore \left(1 + \frac{t}{2}\right)^{-1} \approx 1 + (-1)\left(\frac{t}{2}\right) = 1 - \frac{1}{2}t \quad \square$$

[5 marks]

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