4.4 Answers

4.4.1 Solution to 4.1 Example #1 (Teaching Video)

A Proof that the derivative of $\arcsin x$ is $\dfrac{1}{\sqrt{1 - x^2}}$

$$y = \arcsin x$$

$$\therefore x = \sin y$$

$$\frac{dx}{dy} = \cos y$$

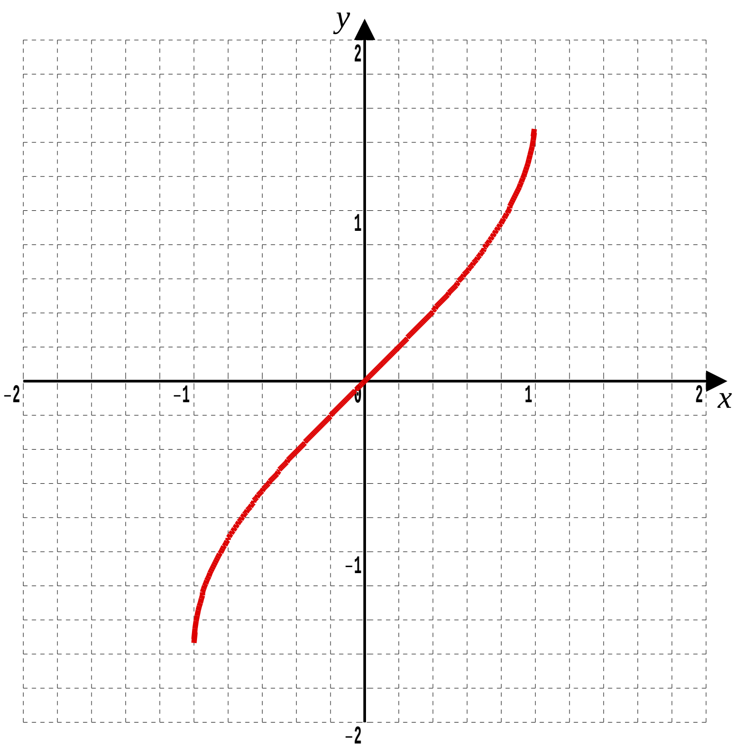
Now use the fact that $\cos^2 y + \sin^2 y = 1$ to get...

$$\frac{dx}{dy} = \pm\sqrt{1 - \sin^2 y}$$

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1 - x^2}}$$

From the graph of $y = \arcsin x$ we can see that the gradient is always positive

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \qquad \square$$



[5 marks]

4.4.2 Solution to 4.2 Example #2 (Teaching Video)

With the help of the Binomial Theorem, find the first four non-zero terms in the Maclaurin series for *arcsin x*

The Binomial Theorem begins;

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

and it's known that the derivative of $arcsin\, x$ is $\dfrac{1}{\sqrt{1-x^2}}$

$$\frac{1}{\sqrt{1-x^2}}$$

$$= \left(1-x^2\right)^{-\frac{1}{2}}$$

$$= \left(1+\left(-x^2\right)\right)^{-\frac{1}{2}}$$

$$= 1 + \left(-\frac{1}{2}\right)\left(-x^2\right) + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-x^2\right)^2 + \frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-x^2\right)^3 + \ldots$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \ldots \qquad \text{valid for } -1 < x < 1$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \ldots\,dx$$

$$arcsin\, x + c = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots$$

As $arcsin\, x = 0$ when $x = 0$, the constant of the integration, $c = 0$

$$\therefore\ arcsin\, x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots \qquad \text{valid for } -1 < x < 1$$

Note :

In fact the series is valid for $-1 \leqslant x \leqslant 1$ but proving that is outwith the course.

[8 marks]

4.4.3 Solutions to 4.3 Exercise

Answer 1

$$y = arccos\,x$$
$$\therefore x = cos\,y$$
$$\frac{dx}{dy} = -sin\,y$$

Now use the fact that $cos^2 y + sin^2 y = 1$ to get...

$$\frac{dx}{dy} = \pm\sqrt{1 - cos^2 y}$$
$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1 - x^2}}$$

From the graph of $y = arccos\,x$ it can be seen that the gradient is always negative

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \qquad \square$$

[5 marks]

Answer 2

(i)

$$\frac{1}{\sqrt{1 - x^2}} + \left(-\frac{1}{\sqrt{1 - x^2}}\right) = 0$$
$$\int \frac{1}{\sqrt{1 - x^2}}\,dx + \int\left(-\frac{1}{\sqrt{1 - x^2}}\right)dx = \int 0\,dx$$
$$arcsin\,x + arccos\,x = c$$

When $x = 0$, $arcsin\,x = 0$ and $arccos\,x = \frac{\pi}{2}$

$$\Rightarrow \quad c = \frac{\pi}{2}$$
$$\therefore arcsin\,x + arccos\,x = \frac{\pi}{2}$$

[3 marks]

(ii) Given that, from Example #2,

$$arcsin\,x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots \qquad \text{valid for } -1 < x < 1$$

the part (i) relationship gives,

$$arccos\,x = \frac{\pi}{2} - arcsin\,x$$
$$= \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots\right) \text{ valid for } -1 < x < 1$$

Note :

In fact the series is valid for $-1 \leqslant x \leqslant 1$ but proving that is outwith the course.

[2 marks]

Answer 3

$$y = arctan\,x$$

$$\therefore\ x = tan\,y$$

$$\frac{dx}{dy} = sec^2\,y$$

Now use the fact that $1 + tan^2\,y = sec^2\,y$ to get...

$$\frac{dx}{dy} = 1 + tan^2\,y$$

$$\frac{dx}{dy} = 1 + x^2$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2} \qquad \square$$

[5 marks]

Answer 4
With the help of the Binomial Theorem, find the first four non-zero terms in the Maclaurin series for *arcsin x*

The Binomial Theorem begins;

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ ...$$

and it's known that the derivative of $arctan\,x$ is $\dfrac{1}{1 + x^2}$

$$\frac{1}{1 + x^2} = \left(1 + x^2\right)^{-1}$$

$$= 1 + (-1)\left(x^2\right) + \frac{1}{2!}(-1)(-2)\left(x^2\right)^2 + \frac{1}{3!}(-1)(-2)(-3)\left(x^2\right)^3 + \ ...$$

$$= 1 - x^2 + x^4 - x^6 + \ ... \qquad \text{valid for } -1 < x < 1$$

$$\int \frac{1}{1 + x^2}\,dx = \int 1 - x^2 + x^4 - x^6 + \ ...\ dx$$

$$arctan\,x + c = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ ...$$

As $arctan\,x = 0$ when $x = 0$, the constant of the integration, $c = 0$

$$\therefore\ arctan\,x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ ... \qquad \text{valid for } -1 < x < 1$$

Note :

In fact the series is valid for $-1 \leqslant x \leqslant 1$ but proving that is outwith the course.

[8 marks]

Answer 5

$$y = arccot\, x$$
$$\therefore x = cot\, y$$
$$\frac{dx}{dy} = -csc^2 y$$

Now use the fact that $cot^2 y + 1 = csc^2 y$ to get...

$$\frac{dx}{dy} = (-1)\left(cot^2 y + 1\right)$$
$$\frac{dx}{dy} = (-1)\left(x^2 + 1\right)$$
$$\frac{dy}{dx} = -\frac{1}{1 + x^2} \qquad \square$$

[5 marks]

Answer 6

(i)
$$\frac{1}{1 + x^2} + \left(-\frac{1}{1 + x^2}\right) = 0$$
$$\int \frac{1}{1 + x^2}\, dx + \int \left(-\frac{1}{1 + x^2}\right) dx = \int 0\, dx$$
$$arctan\, x + arccot\, x = c$$

When $x = 0$, $arctan\, x = 0$ and $arccot\, x = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{2}$

$$\therefore arctan\, x + arccot\, x = \frac{\pi}{2}$$

Note :

$$arccot\,(0) = X \Rightarrow cot\,(X) = 0 \Rightarrow cos\,(X) = 0$$
$$\Rightarrow arccos\,(0) = X \Rightarrow X = \frac{\pi}{2}$$

[3 marks]

(ii) Given that, from Question 4,

$arctan\, x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ valid for $-1 < x < 1$

the part (i) relationship gives,

$$arccot\, x = \frac{\pi}{2} - arctan\, x$$
$$= \frac{\pi}{2} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right) \text{ valid for } -1 < x < 1$$

Note :

In fact the series is valid for $-1 \leqslant x \leqslant 1$
but proving that is beyond the Further A-Level course.

[2 marks]

Answer 7

$$y = arcsec\,x$$

$$\therefore x = sec\,y$$

$$\frac{dx}{dy} = sec\,y\ \ tan\,y$$

Now use the fact that $1 + tan^2 y = sec^2 y$ to get...

$$\frac{dx}{dy} = \pm sec\,y \sqrt{sec^2 y - 1}$$

$$\frac{dx}{dy} = \pm x \sqrt{x^2 - 1}$$

$$\frac{dy}{dx} = \pm \frac{1}{x \sqrt{x^2 - 1}}$$

From the graph of $y = arcsec\,x$ it can be seen that the gradient is always positive

(Each of its two parts are an increasing function)

$$\frac{dy}{dx} = \frac{1}{x \sqrt{x^2 - 1}} \qquad \square$$

[5 marks]

Answer 8

The quickest reason to give to explain why no Maclaurin series can exist for *arcsec x* is simply to observe that the function does not have a crossing point of the *y*-axis, the graph having been provided in question 7.

Another reason that is easily given is to observe that for a Maclaurin series to exist, all of $f(0)$, $f'(0)$, $f''(0)$, ..., $f^{(r)}(0)$, ... must have finite values, a fact clearly stated when the theorem "Maclaurin Series" was given.
So, simply observe that *arcsec*(0) does not exist or, using the derivative from question 7, that $f'(0)$ is undefined due to a division by zero.

[2 marks]

Graph Plotting

To draw the graphs of *arcsec x*, *arccsc x* and *arccot x*, or calculate their value at a particular point, use the relationships,

$$arcsec\,x = arccos\left(\frac{1}{x}\right)$$

$$arccsc\,x = arcsin\left(\frac{1}{x}\right)$$

$$arccot\,x = arctan\left(\frac{1}{x}\right)$$