5.4 Answers

5.4.1 Solution to 5.2 Example

$$\begin{aligned}
f(x) &= x\,e^x & &\Rightarrow f(0) = 0 \\
f'(x) &= x\,e^x + e^x & & \\
&= f(x) + e^x & &\Rightarrow f'(0) = 1 \\
f''(x) &= [\, f'(x) \,] + e^x & & \\
&= [\, f(x) + e^x \,] + e^x & & \\
&= f(x) + 2\,e^x & &\Rightarrow f''(0) = 2 \\
f'''(x) &= [\, f'(x) \,] + 2\,e^x & & \\
&= [\, f(x) + e^x \,] + 2\,e^x & & \\
&= f(x) + 3\,e^x & &\Rightarrow f'''(0) = 3 \\
&\;\;\ldots & & \\
f^{(n)}(x) &= f(x) + n\,e^x & &\Rightarrow f^{(n)}(0) = n
\end{aligned}$$

(All set for a proof by induction !)

The Maclaurin Series

A given function, $f(x)$, may be written as the polynomial,

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \ldots + \frac{f^{(r)}(0)}{r!}\,x^r + \ldots$$

provided that $f(0)$, $f'(0)$, $f''(0)$, ..., $f^{(r)}(0)$, ... all have finite values.

$$f(x) \approx x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120}$$

[8 marks]

5.4.2 Solutions to 5.3 Exercise

Answer 1

(i)

$$\begin{aligned}
f(x) &= e^{1-2x} \\
f'(x) &= e^{1-2x}(-2) \\
&= -2f(x) \\
f''(x) &= -2\,[\,f'(x)\,] \\
&= -2[-2f(x)\,] \\
&= 4f(x) \\
f'''(x) &= 4\,[\,f'(x)\,] \\
&= 4\,[-2f(x)\,] \\
&= -8f(x) \qquad \square
\end{aligned}$$

[3 marks]

(ii) $$f^{(n)}(x) = (-1)^n\, 2^n f(x)$$

[2 marks]

(iii) As $f(0) = e, \quad f^{(n)} = (-1)^n\, 2^n e$

[1 mark]

(iv)

$$\begin{aligned}
f(x) &= f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!} + \ldots \\
&= e - 2e\,x + \frac{4e}{2}x^2 - \frac{8e}{6}x^3 + \frac{16e}{24}x^4 - \frac{32e}{120}x^5 + \frac{64e}{720}x^6 - \ldots \\
&\approx e - 2e\,x + 2e\,x^2 - \frac{4e}{3}x^3 + \frac{2e}{3}x^4 - \frac{4}{15}x^5 + \frac{4e}{45}x^6
\end{aligned}$$

[3 marks]

(v) From $e^x = 1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^r}{r!} + ..$ valid for all x

with x replaced with $(1 - 2x)$ comes,

$$e^{1-2x} = 1 + (1 - 2x) + \frac{1}{2!}(1 - 2x)^2 + \frac{1}{3!}(1 - 2x)^3 + \ldots$$

[1 mark]

(vi) Every bracket when expanded gives a number, and so the first term of the Maclaurin series is obtained as an infinite series rather than a single number. It is,

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

The same issue arises in trying to determine the coefficient of x or, indeed, the coefficient of any power of x in the series.

[2 marks]

(vii)

$$f(x) = e^{1-2x}$$
$$= e^{1} e^{-2x}$$
$$= e\left(1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + \frac{(-2x)^5}{5!} + \frac{(-2x)^6}{6!} + \ldots\right)$$
$$= e\left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 + \frac{4}{45}x^6 + \ldots\right)$$
$$\approx e - 2e\,x + 2e\,x^2 - \frac{4e}{3}x^3 + \frac{2e}{3}x^4 - \frac{4e}{15}x^5 + \frac{4e}{45}x^6$$

Which is exactly the same sextic as was found in part (iv) □

[2 marks]

(viii) From part (v),

$$e^{1-2x} = 1 + (1 - 2x) + \frac{1}{2!}(1 - 2x)^2 + \frac{1}{3!}(1 - 2x)^3 + \ldots$$

with $x = 0$

$$e^{1} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots\ldots.$$

[2 marks]

Answer 2

(i)

$$f(x) = e^{3x} - e^{-3x}$$
$$f'(x) = 3e^{3x} + 3e^{-3x}$$
$$= 3\left(e^{3x} + e^{-3x}\right)$$
$$f''(x) = 3\left(3e^{3x} - 3e^{-3x}\right)$$
$$= 3^2\left(e^{3x} - e^{-3x}\right)$$
$$= 3^2 f(x) \quad □$$

[2 marks]

(ii)

$$f'''(x) = 3^2 f'(x)$$
$$f^{(4)}(x) = 3^2\,[\,f''(x)\,]$$
$$= 3^2[\,3^2 f(x)\,]$$
$$= 3^4 f(x) \quad □$$

[2 marks]

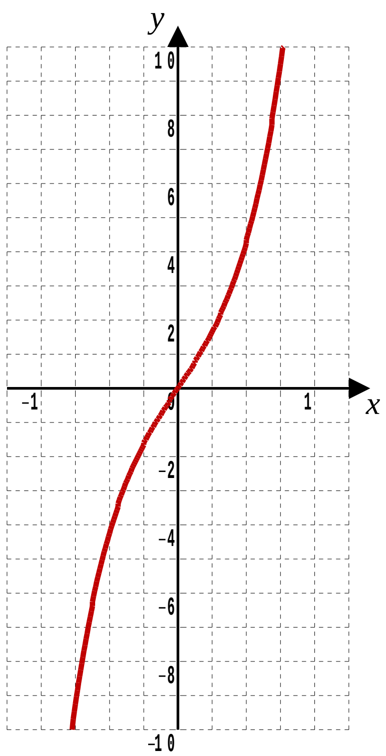
(iii) A reasonable deduction is that, $f^{(2n)}(x) = 3^{2n} f(x)$

As $f(x) = 0$ it follows that $f^{(2n)}(0) = 0$

In the Maclaurin series for $f(x)$ the coefficients of all the even powers of x are thus zero. That is, the series will not contain even powers of x ☐

An alternative argument can be made involving odd functions.

It is simply that, as $f(x)$ is an odd function it can contain only odd powers of x. That is, the series will not contain even powers of x ☐



Let $f(x) = e^{3x} - e^{-3x}$ in which case;

$$\begin{aligned} f(-x) &= e^{3(-x)} - e^{-3(-x)} \\ &= e^{-3x} - e^{3x} \\ &= -e^{3x} + e^{-3x} \\ &= -\left(e^{3x} - e^{-3x}\right) \\ &= -f(x) \end{aligned}$$

$\therefore\ f(x)$ is an odd function

The graph of the odd function $f(x) = e^{3x} - e^{-3x}$

[2 marks]

(iv)

$$f'(x) = 3\left(e^{3x} + e^{-3x}\right) \Rightarrow f'(0) = 3 \times 2 \qquad 1^{\text{st}} \text{ term}$$

$$f'''(x) = 3^2 f'(x) \Rightarrow f'''(0) = 3^3 \times 2 \qquad 2^{\text{nd}} \text{ term}$$

$$f^{(5)}(x) = 3^4 f'(x) \Rightarrow f^{(5)}(0) = 3^5 \times 2 \qquad 3^{\text{rd}} \text{ term}$$

$$\therefore\ f(x) \approx 6x + 9x^3 + \frac{81}{20}x^5$$

[3 marks]

(v) n^{th} term will be $3^{2n-1} \times 2 \times \dfrac{1}{(2n-1)!} x^{2n-1}$

[2 marks]

Answer 3

(a)

$$f(x) = e^{2x} \cos x \qquad \Rightarrow f(0) = 1$$

$$f'(x) = e^{2x}(-\sin x) + 2e^{2x}\cos x$$

$$= -e^{2x}\sin x + 2f(x) \qquad \Rightarrow f'(0) = 2$$

$$f''(x) = -e^{2x}\cos x + \left(-2e^{2x}\right)\sin x + 2f'(x)$$

$$= -f(x) - 2e^{2x}\sin x + 2f'(x)$$

From, $f'(x) = -e^{2x}\sin x + 2f(x)$

$$e^{2x}\sin x = 2f(x) - f'(x)$$

$$\therefore \; f''(x) = -f(x) - 2\left(2f(x) - f'(x)\right) + 2f'(x)$$

$$= -f(x) - 4f(x) + 2f'(x) + 2f'(x)$$

$$= -5f(x) + 4f'(x) \qquad \Rightarrow f''(0) = 3$$

That is, $p = -5$ and $q = 4$

[5 marks]

(b)

$$f'''(x) = -5f'(x) + 4f''(x)$$

$$= -5f'(x) + 4\left(-5f(x) + 4f'(x)\right)$$

$$= -20f(x) + 11f'(x) \qquad \Rightarrow f'''(0) = 2$$

$$f^{(4)}(x) = -20f'(x) + 11f''(x)$$

$$= -20f'(x) + 11\left(-5f(x) + 4f'(x)\right)$$

$$= -55f(x) + 24f'(x) \qquad \Rightarrow f^{(4)}(0) = -7$$

$$f^{(5)}(x) = -55f'(x) + 24f''(x)$$

$$= -55f'(x) + 24\left(-5f(x) + 4f'(x)\right)$$

$$= -120f(x) + 41f'(x) \qquad \Rightarrow f^{(5)}(0) = -38$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$$

$$f(x) \approx 1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3 - \frac{7}{24}x^4 - \frac{19}{60}x^5$$

[3 marks]

Answer 4

$$g(x) = e^x \sin x \qquad\qquad \Rightarrow \; g(0) = 0$$

$$\therefore \; g'(x) = e^x \cos x + e^x \sin x$$

$$= h(x) + g(x) \qquad \text{(Note also } g'(x) = f(x)\text{)}$$

$$h(x) = e^x \cos x \qquad\qquad \Rightarrow \; h(0) = 1$$

$$h'(x) = e^x(-\sin x) + e^x \cos x$$

$$= h(x) - g(x)$$

$$f'(x) = g'(x) + h'(x)$$

$$= h(x) + g(x) + h(x) - g(x)$$

$$= 2\,h(x) \qquad\qquad \Rightarrow \; f'(0) = 2$$

$$f''(x) = 2\,h'(x)$$

$$= 2\,h(x) - 2\,g(x) \qquad\qquad \Rightarrow \; f''(0) = 2$$

$$f'''(x) = 2\,h'(x) - 2\,g'(x)$$

$$= 2\,h(x) - 2\,g(x) - 2\,h(x) - 2\,g(x)$$

$$= -4\,g(x) \qquad\qquad \Rightarrow \; f'''(0) = 0$$

$$f^{(4)}(x) = -4\,g'(x)$$

$$= -4\,h(x) - 4\,g(x) \qquad\qquad \Rightarrow \; f^{(4)}(0) = -4$$

$$f^{(5)}(x) = -4\,h'(x) - 4\,g'(x)$$

$$= -4\,h(x) + 4\,g(x) - 4\,h(x) - 4\,g(x)$$

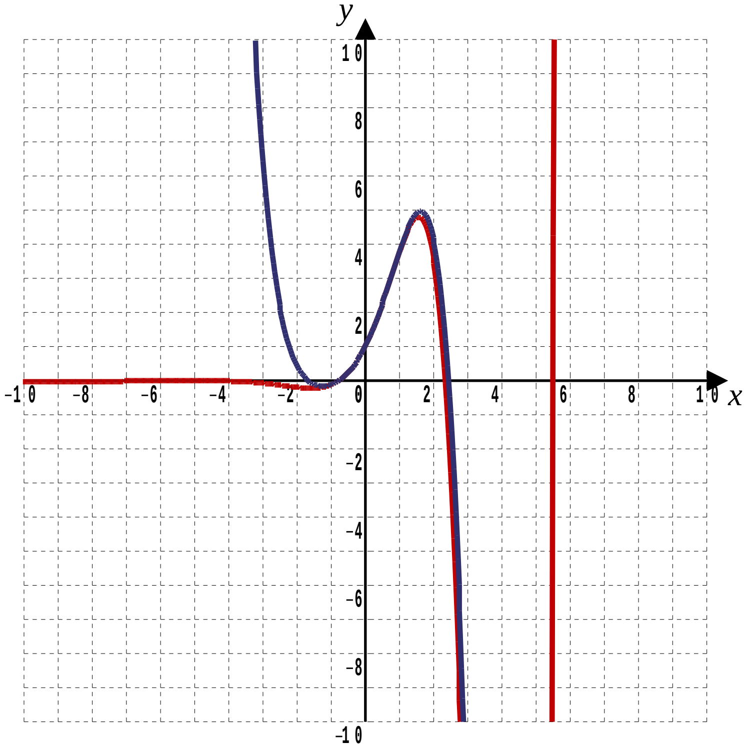
$$= -8\,h(x) \qquad\qquad \Rightarrow \; f^{(5)}(0) = -8$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

$$f(x) \approx 1 + 2x + x^2 - \frac{1}{6}x^4 - \frac{1}{15}x^5$$

[8 marks]

Answer 4 : Graph



Answer 4 : Checking Coefficients using Wolfram Alpha

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,0}]

$$1$$

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,1}]

$$2$$

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,2}]

$$1$$

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,3}]

$$0$$

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,4}]

$$-\frac{1}{6}$$

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,5}]

$$-\frac{1}{15}$$

Answer 5

(i) Mr Hansen's Method

$$f(x) = ln(1 + cos\,2x)$$

$$f'(x) = \frac{1}{1 + cos\,2x} \times (-sin\,2x) \times 2$$

$$= -2 \times \frac{sin\,2x}{1 + cos\,2x}$$

$$= -2 \times \frac{2\,sin\,x\,cos\,x}{cos^2 x + sin^2 x + cos^2 x - sin^2 x}$$

$$= -2 \times \frac{2\,sin\,x\,cos\,x}{2\,cos^2 x}$$

$$= -2 \times \frac{sin\,x}{cos\,x}$$

$$= -2\,tan\,x \qquad \square$$

Sheldon's Method

$$f(x) = ln(1 + cos\,2x)$$

$$= ln\left(cos^2 x + sin^2 x + cos^2 x - sin^2 x\right)$$

$$= ln\left(2\,cos^2 x\right)$$

$$= ln\,2 + ln\left(cos^2 x\right)$$

$$= ln\,2 + 2\,ln(cos\,x)$$

$$f'(x) = 0 + \frac{2}{cos\,x} \times (-sin\,x)$$

$$= -2\,tan\,x \qquad \square$$

[2 marks]

(ii)

$$f''(x) = -2\,sec^2 x$$

$$f'''(x) = -2 \times 2\,sec\,x\,sec\,x\,tan\,x$$

$$= -1 \times \left(-2\,sec^2 x\right)\left(-2\,tan\,x\right)$$

$$= -1\,f''(x)\,f'(x)$$

$$f''''(x) = -f''(x)\,f''(x) - f'''(x)\,f'(x)$$

$$= -\left(f'''(x)\,f'(x) + \left(f''(x)^2\right)\right)$$

[5 marks]

(iii)

$$f(0) = \ln 2$$
$$f'(0) = 0$$
$$f''(0) = -2$$
$$f'''(0) = 0$$
$$f''''(0) = -4$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

$$f(x) \approx \ln 2 - x^2 - \frac{1}{6}x^4$$

[8 marks]

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