Answer 1

$$\begin{aligned} sin\left(x + \frac{\pi}{4}\right) &= sin\, x\, cos\left(\frac{\pi}{4}\right) + cos\, x\, sin\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}\,(\, sin\, x \;+\; cos\, x\,) \\ &= \frac{\sqrt{2}}{2}\left(\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)\right) \\ &= \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right) \\ &= \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right) \qquad \square \end{aligned}$$

[4 marks]

Answer 2

(a)

$$f\,(x) = ln\, sec\, x \qquad \Rightarrow f\,(0) = 0$$

$$f'\,(x) = \frac{1}{sec\, x} \times sec\, x\, tan\, x$$

$$= tan\, x \qquad \square \qquad \Rightarrow f'\,(0) = 0$$

(b)

$$f''\,(x) = sec^2\, x$$

$$= 1 + tan^2\, x \qquad \Rightarrow f''\,(0) = 1$$

$$f'''\,(x) = 2\, tan\, x\, sec^2\, x \qquad \Rightarrow f'''\,(0) = 0$$

$$= 2\, tan\, x\left(1 + tan^2\, x\right)$$

$$= 2\, tan\, x + 2\, tan^3\, x$$

$$f''''\,(x) = 2\, sec^2\, x + 6\, tan^2\, x\, sec^2\, x$$

$$= 2\left(1 + tan^2\, x\right) + 6\, tan^2\, x\left(1 + tan^2\, x\right)$$

$$= 2 + 8\, tan^2\, x + 6\, tan^4\, x \qquad \Rightarrow f''''\,(x) = 2$$

(c)

$$f\,(x) = f\,(0) + f'\,(0)\, x + \frac{f''\,(0)}{2!}\, x^2 + \ldots + \frac{f^{(r)}\,(0)}{r!}\, x^r + \ldots$$

$$f\,(x) \approx \frac{x^2}{2} + \frac{x^4}{12}$$

(d) With $x = \frac{\pi}{4}$, $ln\, sec\left(\frac{\pi}{4}\right) \approx \frac{\pi^2}{32} + \frac{\pi^4}{3072}$

$$ln\,\sqrt{2} \approx \frac{\pi^2}{32}\left(1 + \frac{\pi^2}{96}\right) \Rightarrow ln\, 2 \approx \frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$$

[2, 2, 2, 2 marks]

Answer 3

(i) From the standard result that $\frac{d}{dx}(arccos\, x) = -\frac{1}{\sqrt{1 - x^2}}$

$$f(x) = arccos\,(2x)$$

$$f'(x) = -\frac{1}{\sqrt{1 - (2x)^2}} \times 2$$

$$= -\frac{2}{\sqrt{1 - 4x^2}}$$

[2 marks]

(ii)

$$-\frac{2}{\sqrt{1 - 4x^2}} = (-2)\left(1 + \left(-4x^2\right)\right)^{-\frac{1}{2}}$$

$$= (-2)\left(1 - \frac{1}{2}\left(-4x^2\right) + \frac{3}{8}\left(-4x^2\right)^2 - \ldots\right)$$

$$= (-2)\left(1 + 2x^2 + 6x^4 + \ldots\right)$$

$$f'(x) = -2 - 4x^2 - 12x^4 - \ldots$$

Integrate both sides...

$$f(x) + c = -2x - \frac{4}{3}x^3 - \frac{12}{5}x^5 - \ldots$$

$$\text{As } f(0) = arccos\,(0)$$

$$= \frac{\pi}{2} \Rightarrow c = -\frac{\pi}{2}$$

$$\therefore\ f(x) = \frac{\pi}{2} - 2x - \frac{4}{3}x^2 - \frac{12}{5}x^5 - \ldots$$

[4 marks]

Answer 4

$$f(x) = arccot(1 + x)$$

when $x = 0$, $arccot(1) = y \Rightarrow cot\, y = 1 \Rightarrow tan\, y = 1 \Rightarrow y = \frac{\pi}{4}$

$$\Rightarrow f(0) = \frac{\pi}{4}$$

$$\begin{aligned} f'(x) &= -\frac{1}{1 + (1 + x)^2} \\ &= -\frac{1}{1 + 1 + 2x + x^2} \\ &= -\frac{1}{2 + 2x + x^2} \qquad \Rightarrow f'(0) = -\frac{1}{2} \\ &= -\left(2 + 2x + x^2\right)^{-1} \end{aligned}$$

$$\begin{aligned} f''(x) &= \left(2 + 2x + x^2\right)^{-2}(2 + 2x) \\ &= \frac{2(1 + x)}{(2 + 2x + x^2)^2} \qquad \Rightarrow f''(0) = \frac{2}{4} \\ &\qquad\qquad\qquad\qquad = \frac{1}{2} \end{aligned}$$

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(r)}(0)}{r!} x^r + \ldots$$

$$f(x) \approx \frac{\pi}{4} - \frac{x}{2} + \frac{x^2}{4}$$

[5 marks]

Answer 5

$$\cos x \times \frac{1}{\cos x} = 1$$

$$\cos x \times \sec x = 1$$

$$\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)\left(1 + ax^2 + bx^4\right) \approx 1$$

$$1 + ax^2 + bx^4 - \frac{1}{2}x^2 - \frac{1}{2}ax^4 + \frac{1}{24}x^4 + O\left(x^6\right) \approx 1$$

For coefficients of x^2,

$$a - \frac{1}{2} = 0$$

$$a = \frac{1}{2}$$

For coefficients of x^4,

$$b - \frac{1}{2}a + \frac{1}{24} = 0$$

$$b = \frac{1}{2} \times \frac{1}{2} - \frac{1}{24}$$

$$= \frac{5}{24}$$

Thus $\sec x \approx 1 + \frac{1}{2}x + \frac{5}{24}x^2$

[5 marks]

Answer 6

(i)

$$f(x) = ln\left(x + \sqrt{1 + x^2}\right)$$

$$f'(x) = \frac{1}{x + \sqrt{1 + x^2}} \times \left(1 + \frac{1}{2}\left(1 + x^2\right)^{-\frac{1}{2}} \times 2x\right)$$

$$= \frac{1}{x + \sqrt{1 + x^2}} \times \left(1 + \frac{x}{\sqrt{1 + x^2}}\right)$$

$$= \frac{1}{x + \sqrt{1 + x^2}} \times \left(\frac{\sqrt{1 + x^2}}{\sqrt{1 + x^2}} + \frac{x}{\sqrt{1 + x^2}}\right)$$

$$= \frac{1}{x + \sqrt{1 + x^2}} \times \frac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}}$$

$$= \frac{1}{\sqrt{1 + x^2}} \qquad \therefore \sqrt{1 + x^2}\, f'(x) = 1 \qquad \square$$

[2 marks]

(ii)

$$\left(1 + x^2\right)^{\frac{1}{2}} f'(x) = 1$$

$$\left(1 + x^2\right)^{\frac{1}{2}} f''(x) + \frac{1}{2}\left(1 + x^2\right)^{-\frac{1}{2}} \times 2x\, f'(x) = 0$$

$$\sqrt{1 + x^2}\, f''(x) + \frac{x}{\sqrt{1 + x^2}} f'(x) = 0$$

multiply through by $\sqrt{1 + x^2}$

$$\left(1 + x^2\right) f''(x) + x\, f'(x) = 0 \qquad \square$$

[2 marks]

(iii)

$$\left(1 + x^2\right) f'''(x) + 2x\, f''(x) + x\, f''(x) + f'(x) = 0$$

$$\left(1 + x^2\right) f'''(x) + 3x\, f''(x) + f'(x) = 0$$

[2 marks]

(iv)

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1$$

[2 marks]

(v)

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(r)}(0)}{r!} x^r + \ldots$$

$$f(x) \approx x - \frac{x^3}{6}$$

[2 marks]