

6.4 Answers

Further A-Level Pure Mathematics, Core 2

Maclaurin Series

Answer 1

$$\begin{aligned}
 \sin\left(x + \frac{\pi}{4}\right) &= \sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right) \\
 &= \frac{\sqrt{2}}{2} (\sin x + \cos x) \\
 &= \frac{\sqrt{2}}{2} \left(\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) \right) \\
 &= \frac{\sqrt{2}}{2} \left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \right) \\
 &= \frac{\sqrt{2}}{2} \left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \dots \right) \quad \square
 \end{aligned}$$

[4 marks]

Answer 2

$$(a) \quad f(x) = \ln \sec x \quad \Rightarrow f(0) = 0$$

$$\begin{aligned}
 f'(x) &= \frac{1}{\sec x} \times \sec x \tan x \\
 &= \tan x \quad \square \quad \Rightarrow f'(0) = 0
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad f''(x) &= \sec^2 x \\
 &= 1 + \tan^2 x \quad \Rightarrow f''(0) = 1
 \end{aligned}$$

$$\begin{aligned}
 f'''(x) &= 2 \tan x \sec^2 x \quad \Rightarrow f'''(0) = 0 \\
 &= 2 \tan x (1 + \tan^2 x) \\
 &= 2 \tan x + 2 \tan^3 x
 \end{aligned}$$

$$\begin{aligned}
 f''''(x) &= 2 \sec^2 x + 6 \tan^2 x \sec^2 x \\
 &= 2(1 + \tan^2 x) + 6 \tan^2 x (1 + \tan^2 x) \\
 &= 2 + 8 \tan^2 x + 6 \tan^4 x \quad \Rightarrow f''''(x) = 2
 \end{aligned}$$

$$(c) \quad f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

$$f(x) \approx \frac{x^2}{2} + \frac{x^4}{12}$$

$$(d) \quad \text{With } x = \frac{\pi}{4}, \ln \sec\left(\frac{\pi}{4}\right) \approx \frac{\pi^2}{32} + \frac{\pi^4}{3072}$$

$$\ln \sqrt{2} \approx \frac{\pi^2}{32} \left(1 + \frac{\pi^2}{96} \right) \Rightarrow \ln 2 \approx \frac{\pi^2}{16} \left(1 + \frac{\pi^2}{96} \right)$$

[2, 2, 2, 2 marks]

Answer 3

(i) From the standard result that $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$

$$f(x) = \arccos(2x)$$

$$\begin{aligned} f'(x) &= -\frac{1}{\sqrt{1-(2x)^2}} \times 2 \\ &= -\frac{2}{\sqrt{1-4x^2}} \end{aligned}$$

[2 marks]

$$\begin{aligned} \text{(ii)} \quad -\frac{2}{\sqrt{1-4x^2}} &= (-2) (1 + (-4x^2))^{-\frac{1}{2}} \\ &= (-2) \left(1 - \frac{1}{2}(-4x^2) + \frac{3}{8}(-4x^2)^2 - \dots \right) \\ &= (-2) (1 + 2x^2 + 6x^4 + \dots) \\ f'(x) &= -2 - 4x^2 - 12x^4 - \dots \end{aligned}$$

Integrate both sides...

$$f(x) + c = -2x - \frac{4}{3}x^3 - \frac{12}{5}x^5 - \dots$$

$$\text{As } f(0) = \arccos(0)$$

$$= \frac{\pi}{2} \Rightarrow c = -\frac{\pi}{2}$$

$$\therefore f(x) = \frac{\pi}{2} - 2x - \frac{4}{3}x^3 - \frac{12}{5}x^5 - \dots$$

[4 marks]

Answer 4

$$f(x) = \operatorname{arccot}(1+x)$$

$$\text{when } x = 0, \operatorname{arccot}(1) = y \Rightarrow \cot y = 1 \Rightarrow \tan y = 1 \Rightarrow y = \frac{\pi}{4}$$

$$\Rightarrow f(0) = \frac{\pi}{4}$$

$$\begin{aligned} f'(x) &= -\frac{1}{1+(1+x)^2} \\ &= -\frac{1}{1+1+2x+x^2} \\ &= -\frac{1}{2+2x+x^2} \quad \Rightarrow f'(0) = -\frac{1}{2} \\ &= -(2+2x+x^2)^{-1} \end{aligned}$$

$$\begin{aligned} f''(x) &= (2+2x+x^2)^{-2} (2+2x) \\ &= \frac{2(1+x)}{(2+2x+x^2)^2} \quad \Rightarrow f''(0) = \frac{2}{4} \\ &= \frac{1}{2} \end{aligned}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

$$f(x) \approx \frac{\pi}{4} - \frac{x}{2} + \frac{x^2}{4}$$

[5 marks]

Answer 5

$$\cos x \times \frac{1}{\cos x} = 1$$

$$\cos x \times \sec x = 1$$

$$\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)\left(1 + ax^2 + bx^4\right) \approx 1$$

$$1 + ax^2 + bx^4 - \frac{1}{2}x^2 - \frac{1}{2}ax^4 + \frac{1}{24}x^4 + O(x^6) \approx 1$$

For coefficients of x^2 ,

$$a - \frac{1}{2} = 0$$

$$a = \frac{1}{2}$$

For coefficients of x^4 ,

$$b - \frac{1}{2}a + \frac{1}{24} = 0$$

$$b = \frac{1}{2} \times \frac{1}{2} - \frac{1}{24}$$

$$= \frac{5}{24}$$

$$\text{Thus } \sec x \approx 1 + \frac{1}{2}x + \frac{5}{24}x^2$$

[5 marks]

Answer 6

(i) $f(x) = \ln(x + \sqrt{1+x^2})$

$$f'(x) = \frac{1}{x + \sqrt{1+x^2}} \times \left(1 + \frac{1}{2} (1+x^2)^{-\frac{1}{2}} \times 2x \right)$$

$$= \frac{1}{x + \sqrt{1+x^2}} \times \left(1 + \frac{x}{\sqrt{1+x^2}} \right)$$

$$= \frac{1}{x + \sqrt{1+x^2}} \times \left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} \right)$$

$$= \frac{1}{x + \sqrt{1+x^2}} \times \frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}} \quad \therefore \sqrt{1+x^2} f'(x) = 1 \quad \square$$

[2 marks]

(ii) $(1+x^2)^{\frac{1}{2}} f'(x) = 1$

$$(1+x^2)^{\frac{1}{2}} f''(x) + \frac{1}{2} (1+x^2)^{-\frac{1}{2}} \times 2x f'(x) = 0$$

$$\sqrt{1+x^2} f''(x) + \frac{x}{\sqrt{1+x^2}} f'(x) = 0$$

multiply through by $\sqrt{1+x^2}$

$$(1+x^2) f''(x) + x f'(x) = 0 \quad \square$$

[2 marks]

(iii)

$$(1+x^2) f'''(x) + 2x f''(x) + x f''(x) + f'(x) = 0$$

$$(1+x^2) f'''(x) + 3x f''(x) + f'(x) = 0$$

[2 marks]

(iv)

$$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1$$

[2 marks]

(v)

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

$$f(x) \approx x - \frac{x^3}{6}$$

[2 marks]

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