7.4 Answers

Answer 1

$$\cos\left(x - \frac{\pi}{4}\right) = \cos x \cos\left(\frac{\pi}{4}\right) + \sin x \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2}\ (\cos x + \sin x)$$

$$= \frac{\sqrt{2}}{2}\left(\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\right)$$

$$= \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right)$$

$$= \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right) \qquad \square$$

[3 marks]

Answer 2

(a)

$$f(x) = \ln(\sec x + \tan x) \qquad \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{\sec x + \tan x} \times \left(\sec x \tan x + \sec^2 x\right)$$

$$= \frac{1}{\sec x + \tan x} \times \sec x\ (\tan x + \sec x)$$

$$= \sec x \qquad \square \qquad \Rightarrow f'(0) = 1$$

(b)

$$f''(x) = \sec x \tan x \qquad \Rightarrow f''(0) = 0$$

$$f'''(x) = \sec x \sec^2 x + \sec x \tan x \tan x$$

$$= \sec^3 x + \sec x \tan^2 x$$

$$= \sec^3 x + \sec x\left(\sec^2 x - 1\right)$$

$$= 2\sec^3 x - \sec x \qquad \Rightarrow f'''(0) = 1$$

(c)

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$$

$$f(x) \approx x + \frac{x^3}{6}$$

(d) With $x = \frac{\pi}{4}$,

$$\ln\left(\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right) \approx \frac{\pi}{4} + \frac{\pi^3}{384}$$

$$\ln\left(\sqrt{2} + 1\right) \approx \frac{\pi}{4}\left(1 + \frac{\pi^2}{96}\right)$$

[2, 2, 2, 2 marks]

Answer 3

(i) From the standard result that $\frac{d}{dx}(arcsin\, x) = \frac{1}{\sqrt{1 - x^2}}$

$$f(x) = arcsin(4x)$$

$$f'(x) = \frac{1}{\sqrt{1 - (4x)^2}} \times 4$$

$$= \frac{4}{\sqrt{1 - 16x^2}}$$

[2 marks]

(ii)

$$\frac{4}{\sqrt{1 - 16x^2}} = 4\left(1 + \left(-16x^2\right)\right)^{-\frac{1}{2}}$$

$$= 4\left(1 - \frac{1}{2}\left(-16x^2\right) + \frac{3}{8}\left(-16x^2\right)^2 - ...\right)$$

$$= 4\left(1 + 8x^2 + 96x^4 + ...\right)$$

$$f'(x) = 4 + 32x^2 + 384x^4 - ...$$

Integrate both sides...

$$f(x) + c = 4x + \frac{32}{3}x^3 + \frac{384}{5}x^5 + ...$$

$$\text{As } f(0) = arcsin(0)$$

$$= 0 \Rightarrow c = 0$$

$$\therefore f(x) = 4x + \frac{32}{3}x^3 + \frac{384}{5}x^5 + ...$$

[4 marks]

Answer 4

$$f(x) = \arctan(1 + x)$$

$$\text{when } x = 0,\ f(0) = \arctan(1) \Rightarrow f(0) = \frac{\pi}{4}$$

$$\begin{aligned}
f'(x) &= \frac{1}{1 + (1 + x)^2} \\
&= \frac{1}{1 + 1 + 2x + x^2} \\
&= \frac{1}{2 + 2x + x^2} \qquad \Rightarrow f'(0) = \frac{1}{2} \\
&= \left(2 + 2x + x^2\right)^{-1}
\end{aligned}$$

$$\begin{aligned}
f''(x) &= -\left(2 + 2x + x^2\right)^{-2}(2 + 2x) \\
&= -\frac{2(1 + x)}{\left(2 + 2x + x^2\right)^2} \qquad \Rightarrow f''(0) = -\frac{2}{4} \\
&\qquad\qquad\qquad\qquad\qquad\qquad = -\frac{1}{2}
\end{aligned}$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

$$f(x) \approx \frac{\pi}{4} + \frac{x}{2} - \frac{x^2}{4}$$

[5 marks]

Answer 5

$$\sin x \times \frac{1}{\sin x} = 1$$

$$\sin x \times \csc x = 1$$

$$\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\left(\frac{1}{x} + ax + bx^3\right) \approx 1$$

$$1 + ax^2 + bx^4 - \frac{1}{6}x^2 - \frac{1}{6}ax^4 + \frac{1}{120}x^4 + O\left(x^6\right) \approx 1$$

For coefficients of x^2,

$$a - \frac{1}{6} = 0$$

$$a = \frac{1}{6}$$

For coefficients of x^4,

$$b - \frac{1}{6}a + \frac{1}{120} = 0$$

$$b = \frac{1}{6} \times \frac{1}{6} - \frac{1}{120}$$

$$= \frac{7}{360}$$

Thus $\sec x \approx \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^2$

[5 marks]

Answer 6

(a)

$$f(x) = e^{2x} \sin x \qquad \Rightarrow \; f(0) = 0$$

$$f'(x) = e^{2x} \cos x + 2\, e^{2x} \sin x$$

$$= e^{2x} \cos x + 2 f(x) \qquad \Rightarrow \; f'(0) = 1$$

$$f''(x) = e^{2x} (-\sin x) + 2e^{2x} \cos x + 2 f'(x)$$

$$= -f(x) + 2\, e^{2x} \cos x + 2 f'(x)$$

From, $f'(x) = e^{2x} \cos x + 2 f(x)$

$$e^{2x} \cos x = f'(x) - 2 f(x)$$

$$\therefore \; f''(x) = -f(x) + 2\big(f'(x) - 2 f(x)\big) + 2 f'(x)$$

$$= -f(x) + 2 f'(x) - 4 f(x) + 2 f'(x)$$

$$= -5 f(x) + 4 f'(x) \qquad \Rightarrow \; f''(0) = 4$$

That is, $p = -5$ and $q = 4$

[5 marks]

(b)

$$f'''(x) = -5 f'(x) + 4 f''(x)$$

$$= -5 f'(x) + 4 \big(-5 f(x) + 4 f'(x)\big)$$

$$= -20 f(x) + 11 f'(x) \qquad \Rightarrow \; f'''(0) = 11$$

$$f^{(4)}(x) = -20 f'(x) + 11 f''(x)$$

$$= -20 f'(x) + 11 \big(-5 f(x) + 4 f'(x)\big)$$

$$= -55 f(x) + 24 f'(x) \qquad \Rightarrow \; f^{(4)}(0) = 24$$

$$f^{(5)}(x) = -55 f'(x) + 24 f''(x)$$

$$= -55 f'(x) + 24 \big(-5 f(x) + 4 f'(x)\big)$$

$$= -120 f(x) + 41 f'(x) \qquad \Rightarrow \; f^{(5)}(0) = 41$$

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(r)}(0)}{r!} x^r + \ldots$$

$$f(x) \approx x + 2 x^2 + \frac{11}{6} x^3 + x^4 + \frac{41}{120} x^5$$

[3 marks]