

7.4 Answers

Further A-Level Pure Mathematics, Core 2 Maclaurin Series

Answer 1

$$\begin{aligned}
 \cos\left(x - \frac{\pi}{4}\right) &= \cos x \cos\left(\frac{\pi}{4}\right) + \sin x \sin\left(\frac{\pi}{4}\right) \\
 &= \frac{\sqrt{2}}{2} (\cos x + \sin x) \\
 &= \frac{\sqrt{2}}{2} \left(\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \right) \\
 &= \frac{\sqrt{2}}{2} \left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right) \\
 &= \frac{\sqrt{2}}{2} \left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \dots\right) \quad \square
 \end{aligned}$$

[3 marks]

Answer 2

(a) $f(x) = \ln(\sec x + \tan x) \Rightarrow f(0) = 0$

$$\begin{aligned}
 f'(x) &= \frac{1}{\sec x + \tan x} \times (\sec x \tan x + \sec^2 x) \\
 &= \frac{1}{\sec x + \tan x} \times \sec x (\tan x + \sec x) \\
 &= \sec x \quad \square \quad \Rightarrow f'(0) = 1
 \end{aligned}$$

(b) $f''(x) = \sec x \tan x \Rightarrow f''(0) = 0$

$$\begin{aligned}
 f'''(x) &= \sec x \sec^2 x + \sec x \tan x \tan x \\
 &= \sec^3 x + \sec x \tan^2 x \\
 &= \sec^3 x + \sec x (\sec^2 x - 1) \\
 &= 2 \sec^3 x - \sec x \quad \Rightarrow f'''(0) = 1
 \end{aligned}$$

(c) $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$

$$f(x) \approx x + \frac{x^3}{6}$$

(d) With $x = \frac{\pi}{4}$,

$$\begin{aligned}
 \ln\left(\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right) &\approx \frac{\pi}{4} + \frac{\pi^3}{384} \\
 \ln(\sqrt{2} + 1) &\approx \frac{\pi}{4} \left(1 + \frac{\pi^2}{96}\right)
 \end{aligned}$$

[2, 2, 2, 2 marks]

Answer 3

(i) From the standard result that $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

$$f(x) = \arcsin(4x)$$

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-(4x)^2}} \times 4 \\ &= \frac{4}{\sqrt{1-16x^2}} \end{aligned}$$

[2 marks]

$$\begin{aligned} \text{(ii)} \quad \frac{4}{\sqrt{1-16x^2}} &= 4(1 + (-16x^2))^{-\frac{1}{2}} \\ &= 4\left(1 - \frac{1}{2}(-16x^2) + \frac{3}{8}(-16x^2)^2 - \dots\right) \\ &= 4(1 + 8x^2 + 96x^4 + \dots) \\ f'(x) &= 4 + 32x^2 + 384x^4 - \dots \end{aligned}$$

Integrate both sides...

$$f(x) + c = 4x + \frac{32}{3}x^3 + \frac{384}{5}x^5 + \dots$$

$$\text{As } f(0) = \arcsin(0)$$

$$= 0 \Rightarrow c = 0$$

$$\therefore f(x) = 4x + \frac{32}{3}x^3 + \frac{384}{5}x^5 + \dots$$

[4 marks]

Answer 4

$$f(x) = \arctan(1 + x)$$

$$\text{when } x = 0, f(0) = \arctan(1) \Rightarrow f(0) = \frac{\pi}{4}$$

$$\begin{aligned} f'(x) &= \frac{1}{1 + (1 + x)^2} \\ &= \frac{1}{1 + 1 + 2x + x^2} \\ &= \frac{1}{2 + 2x + x^2} \quad \Rightarrow f'(0) = \frac{1}{2} \\ &= (2 + 2x + x^2)^{-1} \end{aligned}$$

$$\begin{aligned} f''(x) &= -(2 + 2x + x^2)^{-2} (2 + 2x) \\ &= -\frac{2(1 + x)}{(2 + 2x + x^2)^2} \quad \Rightarrow f''(0) = -\frac{2}{4} \\ &= -\frac{1}{2} \end{aligned}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

$$f(x) \approx \frac{\pi}{4} + \frac{x}{2} - \frac{x^2}{4}$$

[5 marks]

Answer 5

$$\sin x \times \frac{1}{\sin x} = 1$$

$$\sin x \times \csc x = 1$$

$$\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\left(\frac{1}{x} + ax + bx^3\right) \approx 1$$

$$1 + ax^2 + bx^4 - \frac{1}{6}x^2 - \frac{1}{6}ax^4 + \frac{1}{120}x^4 + O(x^6) \approx 1$$

For coefficients of x^2 ,

$$a - \frac{1}{6} = 0$$

$$a = \frac{1}{6}$$

For coefficients of x^4 ,

$$b - \frac{1}{6}a + \frac{1}{120} = 0$$

$$b = \frac{1}{6} \times \frac{1}{6} - \frac{1}{120}$$

$$= \frac{7}{360}$$

$$\text{Thus } \sec x \approx \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^2$$

[5 marks]

Answer 6

$$(a) \quad f(x) = e^{2x} \sin x \quad \Rightarrow f(0) = 0$$

$$\begin{aligned} f'(x) &= e^{2x} \cos x + 2e^{2x} \sin x \\ &= e^{2x} \cos x + 2f(x) \quad \Rightarrow f'(0) = 1 \end{aligned}$$

$$\begin{aligned} f''(x) &= e^{2x} (-\sin x) + 2e^{2x} \cos x + 2f'(x) \\ &= -f(x) + 2e^{2x} \cos x + 2f'(x) \end{aligned}$$

$$\text{From, } f'(x) = e^{2x} \cos x + 2f(x)$$

$$e^{2x} \cos x = f'(x) - 2f(x)$$

$$\begin{aligned} \therefore f''(x) &= -f(x) + 2(f'(x) - 2f(x)) + 2f'(x) \\ &= -f(x) + 2f'(x) - 4f(x) + 2f'(x) \\ &= -5f(x) + 4f'(x) \quad \Rightarrow f''(0) = 4 \end{aligned}$$

That is, $p = -5$ and $q = 4$

[5 marks]

$$\begin{aligned} (b) \quad f'''(x) &= -5f'(x) + 4f''(x) \\ &= -5f'(x) + 4(-5f(x) + 4f'(x)) \\ &= -20f(x) + 11f'(x) \quad \Rightarrow f'''(0) = 11 \end{aligned}$$

$$\begin{aligned} f^{(4)}(x) &= -20f'(x) + 11f''(x) \\ &= -20f'(x) + 11(-5f(x) + 4f'(x)) \\ &= -55f(x) + 24f'(x) \quad \Rightarrow f^{(4)}(0) = 24 \end{aligned}$$

$$\begin{aligned} f^{(5)}(x) &= -55f'(x) + 24f''(x) \\ &= -55f'(x) + 24(-5f(x) + 4f'(x)) \\ &= -120f(x) + 41f'(x) \quad \Rightarrow f^{(5)}(0) = 41 \end{aligned}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

$$f(x) \approx x + 2x^2 + \frac{11}{6}x^3 + x^4 + \frac{41}{120}x^5$$

[3 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School

It may be freely duplicated and distributed, unaltered, for non-profit educational use

In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"

© 2022 Number Wonder

Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk