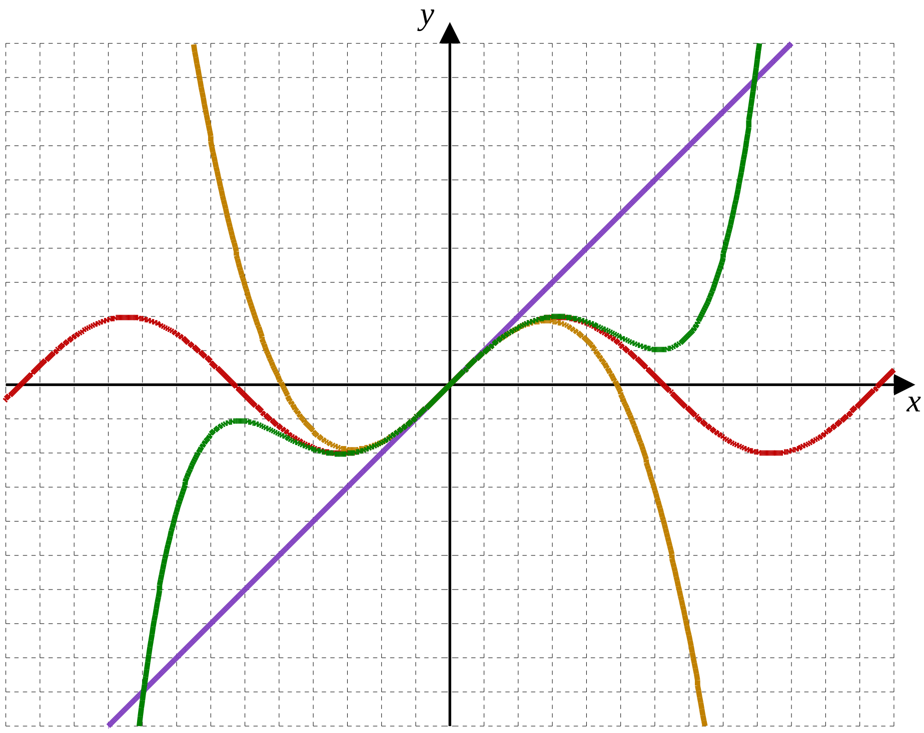
Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

Maclaurin Series



~ In search of a polynomial that will model the sine function ~

Purple : $y = x$

Gold : $y = x - \frac{x^3}{3!}$

Green : $y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

MACLAURIN SERIES

Lesson 1

Further A-Level Pure Mathematics, Core 2

Maclaurin Series

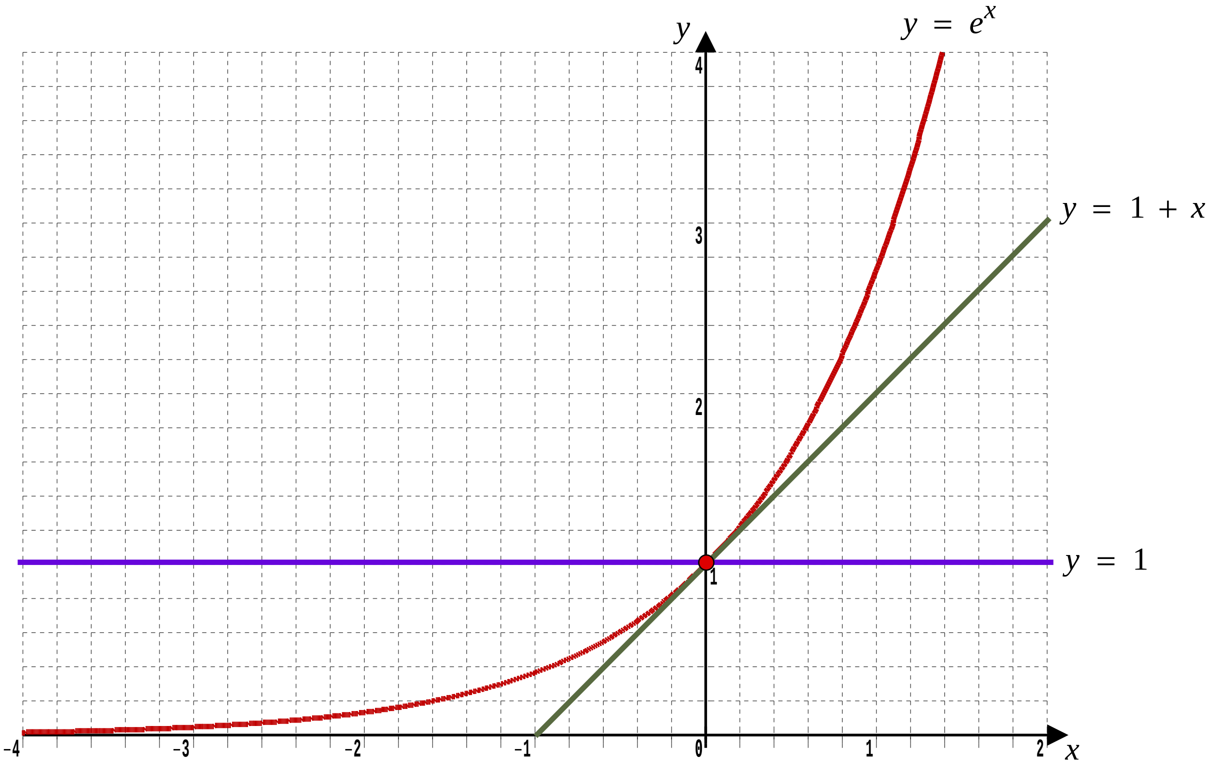
1.1 Approximation

Polynomials first caught our attention because they are straightforward to differentiate or integrate. They have many other properties that make them attractive to work with such as the fact, known as the fundamental theorem of algebra, that a polynomial of degree n has n roots over the complex numbers.

These desirables provide an incentive to approximate other functions, less easy to analyse, with polynomials. As an example, consider the exponential function. What are good polynomial approximations to this curve if we focus on where it crosses the y-axis ?

Here is how we could work up to an answer;

- The best polynomial of degree 0 is $y = 1$
- The best polynomial of degree 1 is $y = 1 + x$



The first approximation, $y = 1$, got the height where $y = e^x$ crossed the y-axis correct. The second approximation, $y = 1 + x$, got both the height the gradient of the exponential curve correct at that y-axis crossing point.

The third approximation will need to get point, gradient and bend correct at that key point (0, 1). Intuitively, a best quadratic curve is sought that will model the exponential and it should be the most successful yet at following the exponential curve for those points that are not too far away from (0, 1).

1.2 The Third Approximation

The exponential function has the remarkable property of being its own derivative.

$$y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x \quad \Rightarrow \quad \frac{d^2y}{dx^2} = e^x$$

and so, when $x = 0$, $y = 1$, $\dfrac{dy}{dx} = 1$ and $\dfrac{d^2y}{dx^2} = 1$

Let the best quadratic approximation be,

$$y = a_0 + a_1 x + a_2 x^2$$

It is required that $y = 1$ when $x = 0$ and so $a_0 = 1$

Differentiating the best quadratic approximation gives,

$$\frac{dy}{dx} = a_1 + 2 a_2 x$$

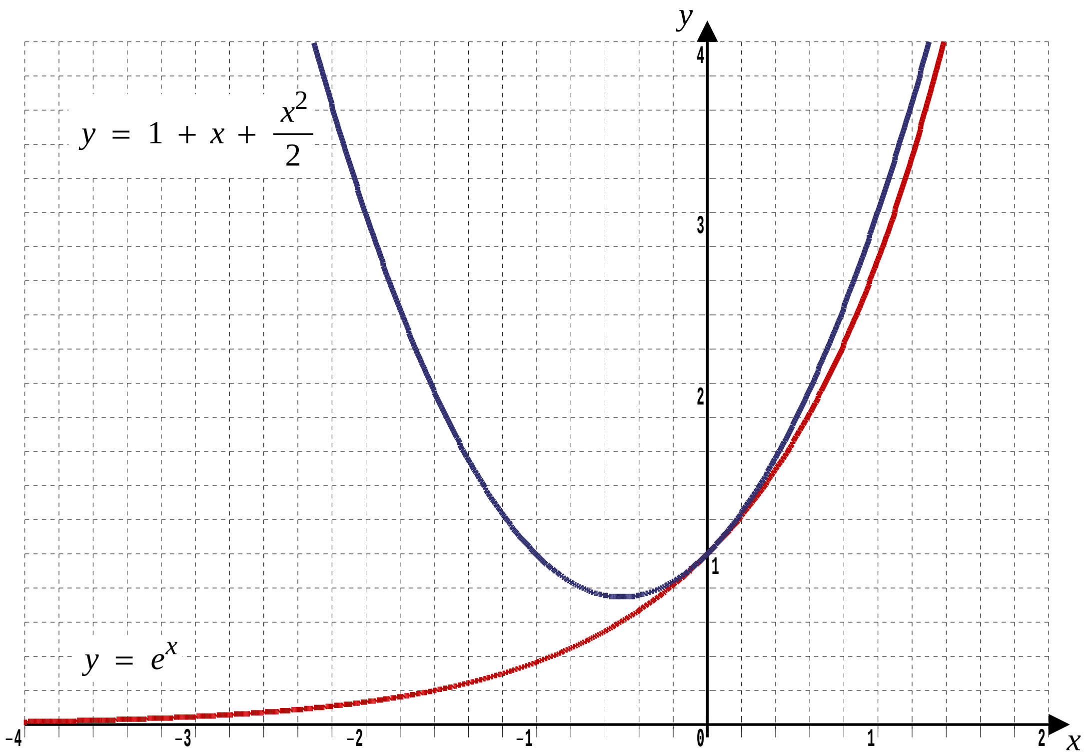
It is required that $\dfrac{dy}{dx} = 1$ when $x = 0$ and so $a_1 = 1$

Differentiating the best quadratic approximation twice gives.

$$\frac{d^2y}{dx^2} = 2 a_2$$

It is required that $\dfrac{d^2y}{dx^2} = 1$ when $x = 0$ and so $a_2 = \dfrac{1}{2}$

The third approximation is thus, $y = 1 + x + \dfrac{x^2}{2}$



1.3 The Fourth Approximation

The approximations studied so far are,

$$y_0 = 1$$

$$y_1 = 1 + x$$

$$y_2 = 1 + x + \frac{x^2}{2}$$

Notice that each higher order approximation contains those previous to it.
In consequence, for the fourth approximation the working can be shortened;

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

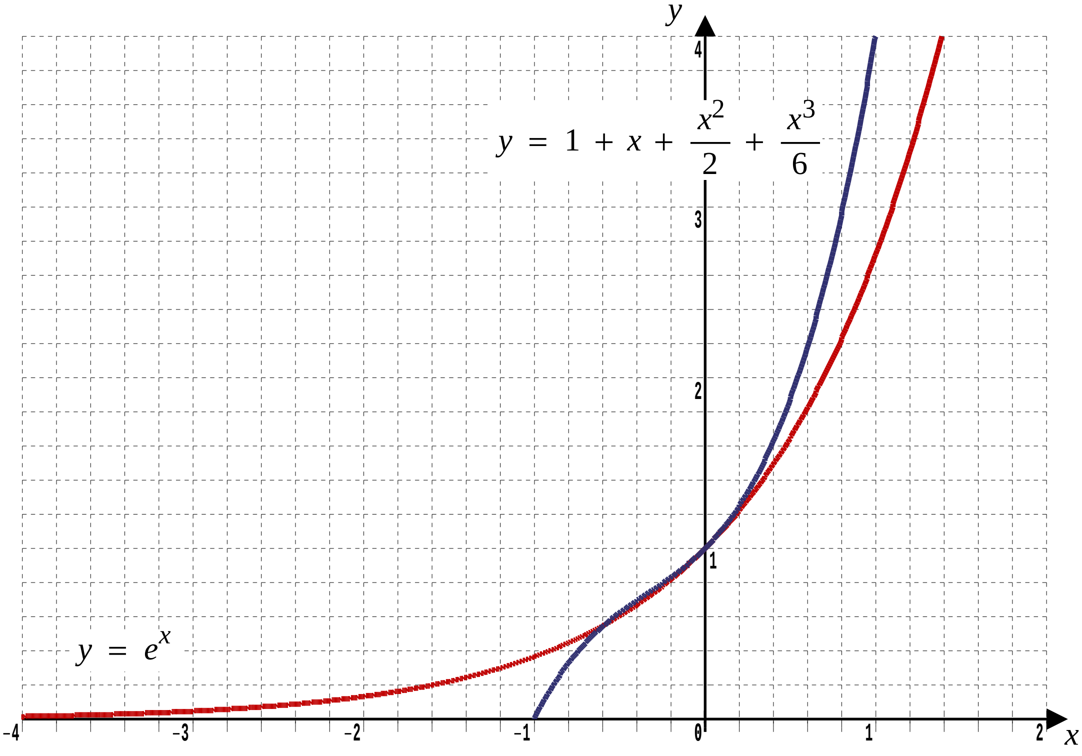
$$\frac{dy}{dx} = \ldots + 3 a_3 x^2$$

$$\frac{d^2y}{dx^2} = \ldots + 6 a_3 x$$

$$\frac{d^3y}{dx^3} = \ldots + 6 a_3$$

It is required that $\dfrac{d^3y}{dx^3} = 1$ when $x = 0$ and so $a_3 = \dfrac{1}{6}$

The fourth approximation is thus, $y = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$



Continuing likewise, the exponential function about (0, 1) can be approximated ever more accurately by taking more terms from the infinite series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad x \in \mathbb{R}$$

1.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

A general polynomial of degree four, a quartic, can be expressed in the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

Complete the table to show the first, second, third and fourth derivatives of y

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$
$\dfrac{dy}{dx} =$
$\dfrac{d^2y}{dx^2} =$
$\dfrac{d^3y}{dx^3} =$
$\dfrac{d^4y}{dx^4} =$

[4 marks]

Question 2

Complete the table to show the first, second, third and fourth derivatives of the function $y = cos\,x$ and also the values of those derivatives at $x = 0$

$y = cos\,x$	when $x = 0$,	$y = 1$
$\dfrac{dy}{dx} =$	when $x = 0$,	$\dfrac{dy}{dx} =$
$\dfrac{d^2y}{dx^2} =$	when $x = 0$,	$\dfrac{d^2y}{dx} =$
$\dfrac{d^3y}{dx^3} =$	when $x = 0$,	$\dfrac{d^3y}{dx^3} =$
$\dfrac{d^4y}{dx^4} =$	when $x = 0$,	$\dfrac{d^4y}{dx^4} =$

[4 marks]

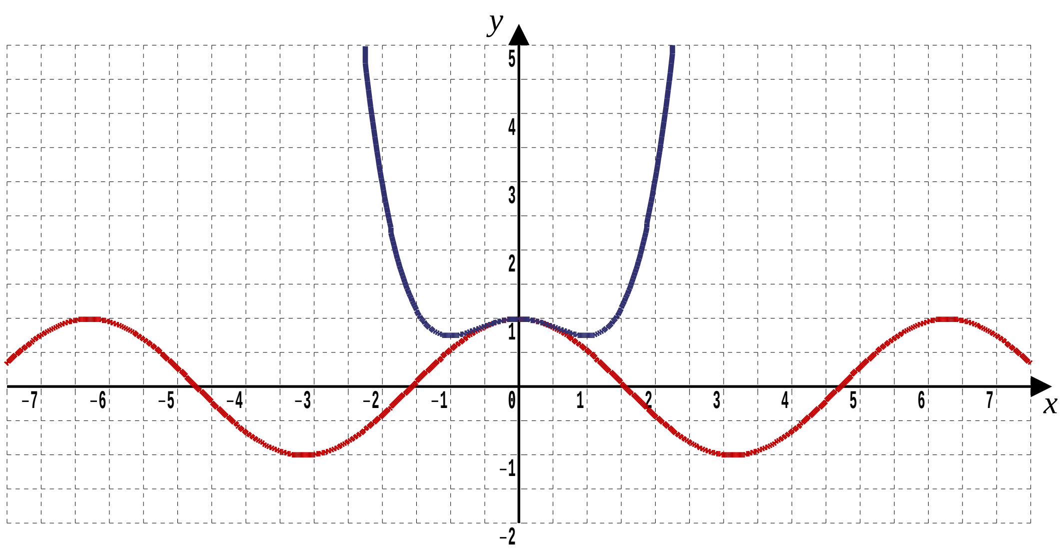
Question 3

By combining the tables of question 1 and 2, work out, in the following order, a_4 then a_3 then a_2 then a_1 and lastly a_0. Finally, write out the equation of the best quartic polynomial to represent the cosine function.

If you have a graphics calculator, you may like to plot the answer to this question as one curve (blue) and the cosine curve as another (red).

Remember that, as this mathematics is a mixture of calculus and trigonometry, the units of angle must be radians.

Here is an image of what my graph plotter gives;



[6 marks]

Question 4

A general polynomial of degree five, a quintic, can be expressed in the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

Complete the table to show the first, second, third, fourth and fifth derivatives of y

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$
$\dfrac{dy}{dx} =$
$\dfrac{d^2y}{dx^2} =$
$\dfrac{d^3y}{dx^3} =$
$\dfrac{d^4y}{dx^4} =$
$\dfrac{d^5y}{dx^5} =$

[4 marks]

Question 5

Complete the table to show the first, second, third, fourth and fifth derivatives of the function $y = sin\,x$ and also the values of those derivatives at $x = 0$

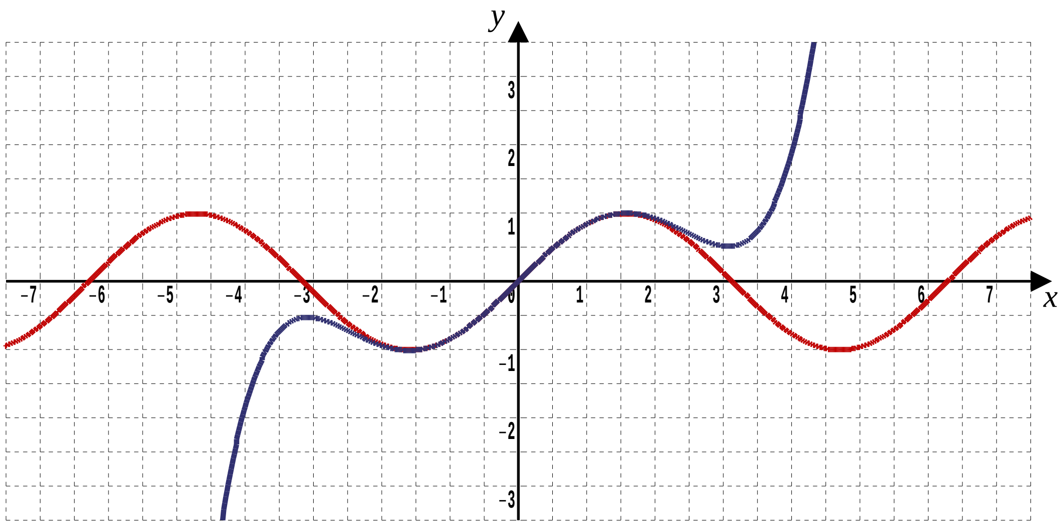
$y = sin\,x$	when $x = 0,$	$y = 0$
$\dfrac{dy}{dx} =$	when $x = 0,$	$\dfrac{dy}{dx} =$
$\dfrac{d^2y}{dx^2} =$	when $x = 0,$	$\dfrac{d^2y}{dx^2} =$
$\dfrac{d^3y}{dx^3} =$	when $x = 0,$	$\dfrac{d^3y}{dx^3} =$
$\dfrac{d^4y}{dx^4} =$	when $x = 0,$	$\dfrac{d^4y}{dx^4} =$
$\dfrac{d^5y}{dx^5} =$	when $x = 0,$	$\dfrac{d^5y}{dx^5} =$

[5 marks]

Question 6

By combining the tables of questions 4 and 5, work out, in the following order, a_5 then a_4 then a_3 then a_2 then a_1 and lastly a_0. Finally, write out the equation of the best quintic polynomial to represent the sine function.

If you have a graphics calculator, you may like to plot the answer to this question as one curve (blue) and the sine curve as another (red).



[6 marks]

Question 7

Taking care over the minus signs, and making use of the chain rule, complete the table to show the first, second, third, fourth and fifth derivatives of the function $y = ln(1 - x)$ where $x < 1$ and also the values of those derivatives at $x = 0$

$y = ln(1 - x)$	when $x = 0$,	$y = 0$
$\frac{dy}{dx} =$	when $x = 0$,	$\frac{dy}{dx} =$
$\frac{d^2y}{dx^2} =$	when $x = 0$,	$\frac{d^2y}{dx^2} =$
$\frac{d^3y}{dx^3} =$	when $x = 0$,	$\frac{d^3y}{dx^3} =$
$\frac{d^4y}{dx^4} =$	when $x = 0$,	$\frac{d^4y}{dx^4} =$
$\frac{d^5y}{dx^5} =$	when $x = 0$,	$\frac{d^5y}{dx^5} =$

[5 marks]

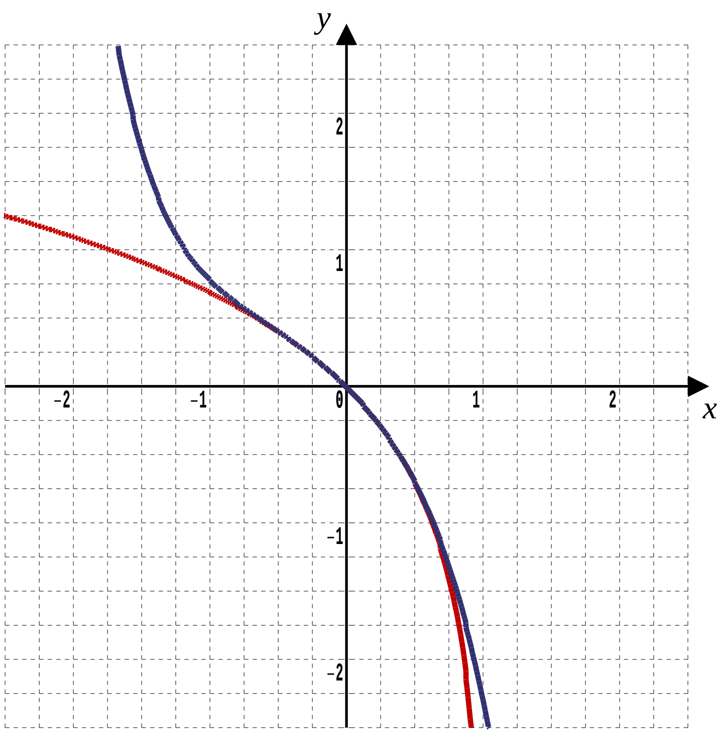
Question 8

$$y = ln(1 - x) \text{ is to be approximated by}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

By combining the tables of questions 4 and 7 work out, in the following order, a_5 then a_4 then a_3 then a_2 then a_1 and lastly a_0. Finally, write out the equation of the best quintic polynomial to represent the function $y = ln(1 - x)$

If you have a graphics calculator, you may like to plot the answer to this question as one curve (blue) and the curve $y = (1 - x)$ as another (red)



[6 marks]

1.5 Answers

Answer 1

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$\frac{dy}{dx} = a_1 + 2 a_2 x + 3 a_3 x^2 + 4 a_4 x^3$$

$$\frac{d^2y}{dx^2} = 2 a_2 + 6 a_3 x + 12 a_4 x^2$$

$$\frac{d^3y}{dx^3} = 6 a_3 + 24 a_4 x$$

$$\frac{d^4y}{dx^4} = 24 a_4$$

[4 marks]

Answer 2

$$y = \cos x \quad \text{when } x = 0, \quad y = 1$$

$$\frac{dy}{dx} = -\sin x \quad \text{when } x = 0, \quad \frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} = -\cos x \quad \text{when } x = 0, \quad \frac{d^2y}{dx^2} = -1$$

$$\frac{d^3y}{dx^3} = \sin x \quad \text{when } x = 0, \quad \frac{d^3y}{dx^3} = 0$$

$$\frac{d^4y}{dx^4} = \cos x \quad \text{when } x = 0, \quad \frac{d^4y}{dx^4} = 1$$

[4 marks]

Answer 3

$$24\,a_4 = 1 \Rightarrow a_4 = \frac{1}{24}$$

$$6\,a_3 + 24\,a_4\,x = 0 \text{ but } x = 0 \text{ so } a_3 = 0$$

$$2\,a_2 + 6\,a_3\,x + 12\,a_4\,x^2 = -1 \text{ but } x = 0 \text{ so } a_2 = -\frac{1}{2}$$

$$a_1 + 2\,a_2\,x + 3\,a_3\,x^2 + 4\,a_4\,x^3 = 0 \text{ but } x = 0 \text{ so } a_1 = 0$$

$$a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 = 1 \text{ but } x = 0 \text{ so } a_0 = 1$$

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{24} \quad \text{or} \quad y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

[6 marks]

Answer 4

The working can be shortened, as suggested in section 1.3
Here, however, is the full detail,

$y = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + a_5\,x^5$
$\frac{dy}{dx} = a_1 + 2\,a_2\,x + 3\,a_3\,x^2 + 4\,a_4\,x^3 + 5\,a_5\,x^4$
$\frac{d^2y}{dx^2} = 2\,a_2 + 6\,a_3\,x + 12\,a_4\,x^2 + 20\,a_5\,x^3$
$\frac{d^3y}{dx^3} = 6\,a_3 + 24\,a_4\,x + 60\,a_5\,x^2$
$\frac{d^4y}{dx^4} = 24\,a_4 + 120\,a_5\,x$
$\frac{d^5y}{dx^5} = 120\,a_5$

[4 marks]

Answer 5

$y = \sin x$	when $x = 0,$	$y = 0$
$\dfrac{dy}{dx} = \cos x$	when $x = 0,$	$\dfrac{dy}{dx} = 1$
$\dfrac{d^2y}{dx^2} = -\sin x$	when $x = 0,$	$\dfrac{d^2y}{dx^2} = 0$
$\dfrac{d^3y}{dx^3} = -\cos x$	when $x = 0,$	$\dfrac{d^3y}{dx^3} = -1$
$\dfrac{d^4y}{dx^4} = \sin x$	when $x = 0,$	$\dfrac{d^4y}{dx^4} = 0$
$\dfrac{d^5y}{dx^5} = \cos x$	when $x = 0,$	$\dfrac{d^5y}{dx^5} = 1$

[5 marks]

Answer 6

$$120\,a_5 = 1 \Rightarrow a_5 = \frac{1}{120}$$

$$24\,a_4 + 120\,a_5\,x = 0 \text{ but } x = 0 \text{ so } a_4 = 0$$

$$6\,a_3 + 24\,a_4\,x + 60\,a_5\,x^2 = -1 \text{ but } x = 0 \text{ so } a_3 = -\frac{1}{6}$$

$$2\,a_2 + 6\,a_3\,x + 12\,a_4\,x^2 + 20\,a_5\,x^3 = 0 \text{ but } x = 0 \text{ so } a_2 = 0$$

$$a_1 + 2\,a_2\,x + 3\,a_3\,x^2 + 4\,a_4\,x^3 + 5\,a_5\,x^4 = 1 \text{ but } x = 0 \text{ so } a_1 = 1$$

$$a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + a_5\,x^5 = 0 \text{ but } x = 0 \text{ so } a_0 = 0$$

$$y = x - \frac{x^3}{6} + \frac{x^5}{120}$$

or, more memorably,

$$y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

[6 marks]

Answer 7

$y = ln(1 - x)$	when $x = 0$,	$y = 0$
$\frac{dy}{dx} = -\frac{1}{1 - x}$	when $x = 0$,	$\frac{dy}{dx} = -1$
$\frac{d^2y}{dx^2} = -\frac{1}{(1 - x)^2}$	when $x = 0$,	$\frac{d^2y}{dx^2} = -1$
$\frac{d^3y}{dx^3} = -\frac{2}{(1 - x)^3}$	when $x = 0$,	$\frac{d^3y}{dx^3} = -2$
$\frac{d^4y}{dx^4} = -\frac{6}{(1 - x)^4}$	when $x = 0$,	$\frac{d^4y}{dx^4} = -6$
$\frac{d^5y}{dx^5} = -\frac{24}{(1 - x)^5}$	when $x = 0$,	$\frac{d^5y}{dx^5} = -24$

[5 marks]

Answer 8

$$120\,a_5 = -24 \Rightarrow a_5 = -\frac{1}{5}$$

$$24\,a_4 + 120\,a_5 x = -6 \text{ but } x = 0 \text{ so } a_4 = -\frac{1}{4}$$

$$6\,a_3 + 24\,a_4 x + 60\,a_5 x^2 = -2 \text{ but } x = 0 \text{ so } a_3 = -\frac{1}{3}$$

$$2\,a_2 + 6\,a_3 x + 12\,a_4 x^2 + 20\,a_5 x^3 = -1 \text{ but } x = 0 \text{ so } a_2 = -\frac{1}{2}$$

$$a_1 + 2\,a_2 x + 3\,a_3 x^2 + 4\,a_4 x^3 + 5\,a_5 x^4 = -1 \text{ but } x = 0 \text{ so } a_1 = -1$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 = 0 \text{ but } x = 0 \text{ so } a_0 = 0$$

$$y = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}$$

[6 marks]

2.1 Streamlining

If the process of finding the best polynomial of degree n to a given function is streamlined, the resulting general rule is termed the Maclaurin Series. It's traditionally written in function notation rather than Leibniz notation.

The Maclaurin Series

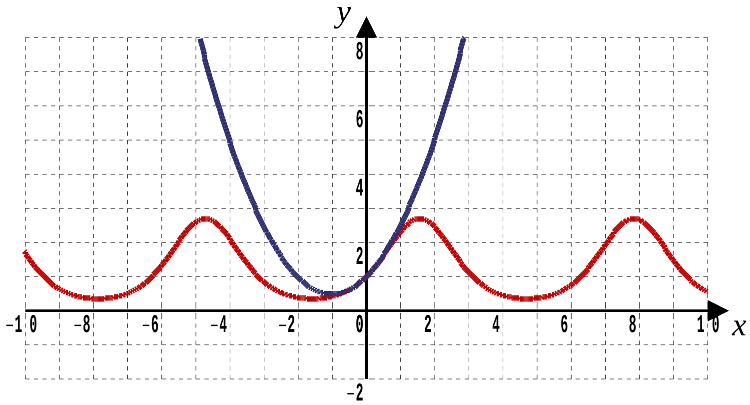
A given function, $f(x)$, may be written as the polynomial,

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \dots + \frac{f^{(r)}(0)}{r!}\,x^r + \dots$$

provided that $f(0)$, $f'(0)$, $f''(0)$, ..., $f^{(r)}(0)$, ... all have finite values.

2.2 Example

Find the first three terms of the Maclaurin series for $f(x) = e^{\sin x}$



Teaching Video : http://www.NumberWonder.co.uk/v9098/2.mp4

[6 marks]

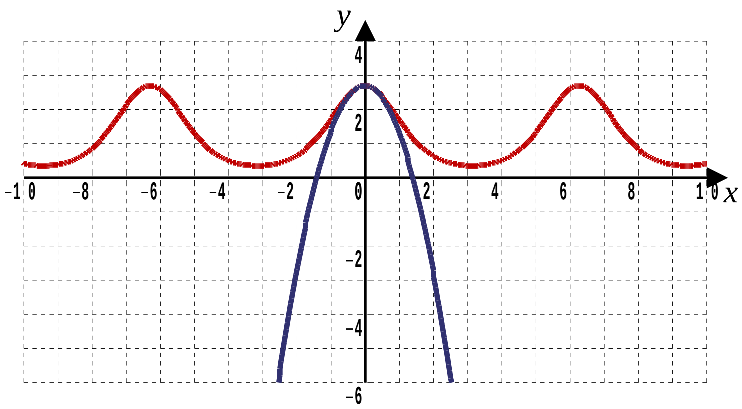
2.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Find the first three terms of the Maclaurin series for $f(x) = e^{\cos x}$

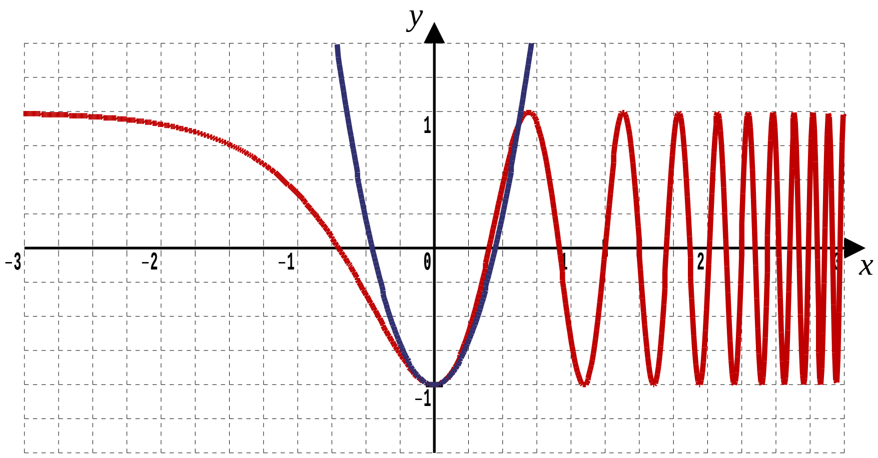


In red is the graph of $y = e^{\cos x}$ and in blue the best quadratic approximation centred on $x = 0$

[6 marks]

Question 2

Find the first three terms of the Maclaurin series for $f(x) = \cos\left(\pi e^{x}\right)$

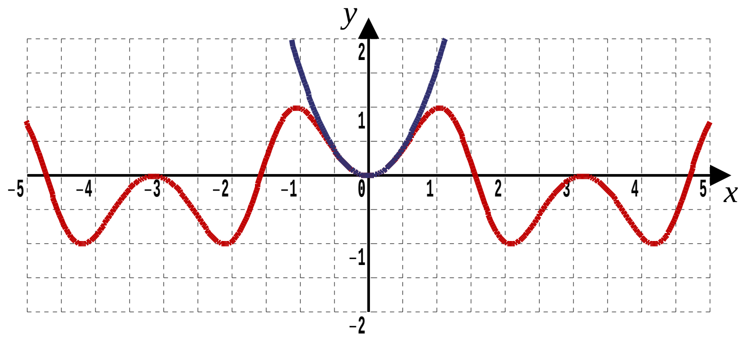


In red is the graph of $y = \cos\left(\pi e^{x}\right)$ and in blue the best quadratic approximation centred on $x = 0$

[6 marks]

Question 3

Find the first three terms of the Maclaurin series for $f(x) = \sin(\pi \cos x)$

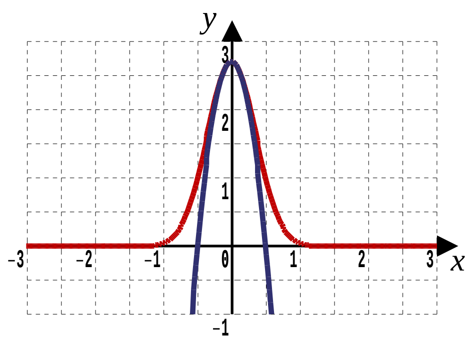


In red is the graph of $y = \sin(\pi \cos x)$ and in blue the best quadratic approximation centred on $x = 0$

[6 marks]

Question 4

Find the first three terms of the Maclaurin series for $f(x) = e^{1-x^2}$



In red is the graph of $y = e^{1-x^2}$ and in blue the best quadratic approximation centred on $x = 0$

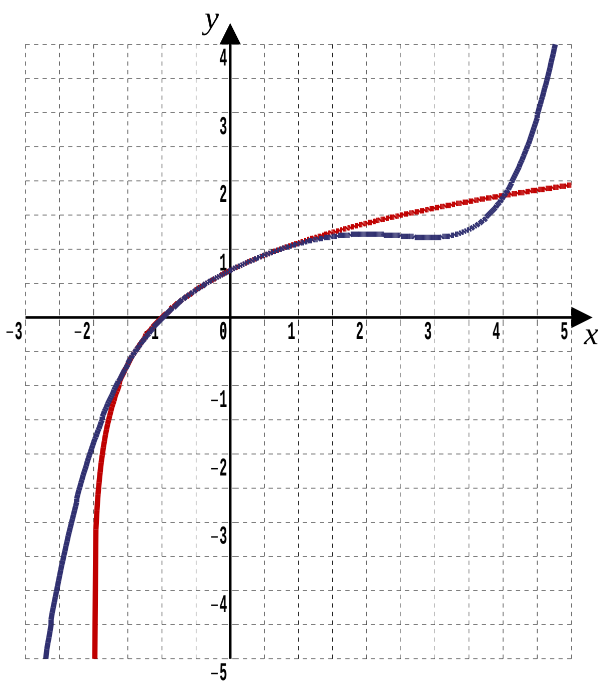
[6 marks]

Question 5

(i) Complete the table to show the first, second, third, fourth and fifth derivatives of the function $y = ln(2 + x)$ where $x > -2$
Also work out values of those derivatives at $x = 0$

$f(x) = ln(2 + x)$	when $x = 0,$	$f(0) =$
$f'(x) =$	when $x = 0,$	$f'(0) =$
$f''(x) =$	when $x = 0,$	$f''(0) =$
$f'''(x) =$	when $x = 0,$	$f'''(0) =$
$f^{(4)}(x) =$	when $x = 0,$	$f^{(4)}(0) =$
$f^{(5)}(x) =$	when $x = 0,$	$f^{(5)}(0) =$

(ii) Hence determine the best quintic polynomial approximation to the function $y = ln(2 + x)$ for values of x that are close to zero.

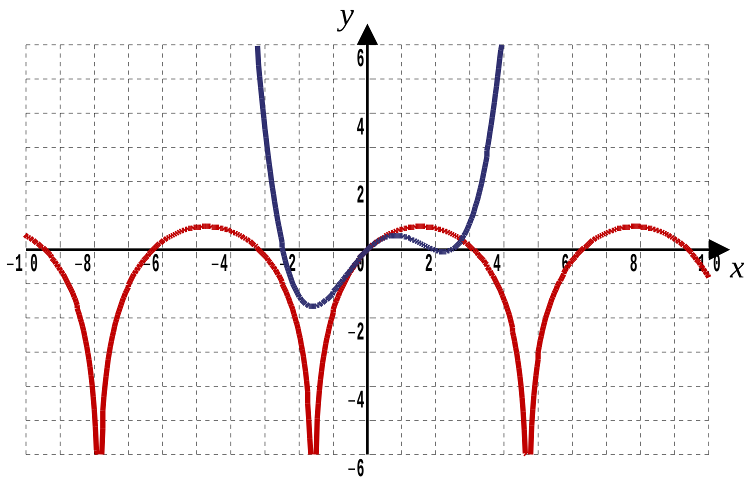


In red is the graph of $y = ln(2 + x)$ and in blue the best quintic polynomial approximation on $x = 0$

[8 marks]

Question 6

Showing full working, determine the best quartic polynomial approximation to the function $y = ln(1 + sin\,x)$ for values of x that are close to zero.



In red is the graph of $y = ln(1 + sin\,x)$ and in blue the best quartic polynomial approximation on $x = 0$

[10 marks]

2.4 Answers

2.4.1 Solution to 2.2 Example (Teaching Video)

$$\begin{aligned} f(x) &= e^{\sin x} && \Rightarrow f(0) = 1 \\ f'(x) &= e^{\sin x}\cos x && \Rightarrow f'(0) = 1 \\ f''(x) &= e^{\sin x}(-\sin x) + e^{\sin x}\cos x\cos x && \\ &= e^{\sin x}(\cos^2 x - \sin x) && \Rightarrow f''(0) = 1 \end{aligned}$$

By Maclaurin,

$$f(x) \approx f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2$$

$$f(x) \approx 1 + x + \frac{x^2}{2}$$

[6 marks]

2.4.2 Solutions to 2.3 Exercise

Answer 1

$$\begin{aligned} f(x) &= e^{\cos x} && \Rightarrow f(0) = e \\ f'(x) &= e^{\cos x}(-\sin x) && \Rightarrow f'(0) = 0 \\ f''(x) &= e^{\cos x}(-\cos x) + e^{\cos x}(-\sin x)(-\sin x) && \\ &= e^{\cos x}\left(\sin^2 x - \cos x\right) && \Rightarrow f''(0) = -e \end{aligned}$$

By Maclaurin,

$$f(x) \approx f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2$$

$$f(x) \approx e - \frac{e\,x^2}{2}$$

[6 marks]

Answer 2

$$\begin{aligned} f(x) &= \cos\left(\pi e^x\right) && \Rightarrow f(0) = -1 \\ f'(x) &= -\sin\left(\pi e^x\right)\pi e^x && \Rightarrow f'(0) = 0 \\ f''(x) &= -\sin\left(\pi e^x\right)\pi e^x - \cos\left(\pi e^x\right)\pi e^x\,\pi e^x && \\ &= \pi e^x\left(-\sin\left(\pi e^x\right) - \pi e^x\cos\left(\pi e^x\right)\right) && \Rightarrow f''(0) = \pi^2 \end{aligned}$$

By Maclaurin,

$$f(x) \approx f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2$$

$$f(x) \approx -1 + \frac{\pi^2}{2}x^2$$

[6 marks]

Answer 3

$$f(x) = \sin(\pi \cos x) \qquad \Rightarrow f(0) = 0$$

$$f'(x) = \cos(\pi \cos x)\,\pi\,(-\sin x) \qquad \Rightarrow f'(0) = 0$$

$$f''(x) = \cos(\pi \cos x)\,\pi\,(-\cos x) + (-\sin(\pi \cos x)\,\pi\,(-\sin x)\,\pi\,(-\sin x))$$

$$= -\pi \cos x \cos(\pi \cos x) - \pi^2 \sin^2 x \sin(\pi \cos x) \qquad \Rightarrow f''(0) = \pi$$

By Maclaurin, $f(x) \approx f(0) + f'(0)\,x + \dfrac{f''(0)}{2!}x^2$

$$f(x) \approx \frac{\pi}{2}x^2$$

[6 marks]

Answer 4

$$f(x) = e^{1-x^2} \qquad \Rightarrow f(0) = e$$

$$f'(x) = e^{1-x^2}(-2x) \qquad \Rightarrow f'(0) = 0$$

$$f''(x) = e^{1-x^2}(-2) + e^{1-x^2}(-2x)(-2x)$$

$$= -2e^{1-x^2} + 4x^2 e^{1-x^2} \qquad \Rightarrow f''(0) = -2e$$

By Maclaurin, $f(x) \approx f(0) + f'(0)\,x + \dfrac{f''(0)}{2!}x^2$

$$f(x) \approx e - e x^2$$

[6 mark]

Answer 5

(i)

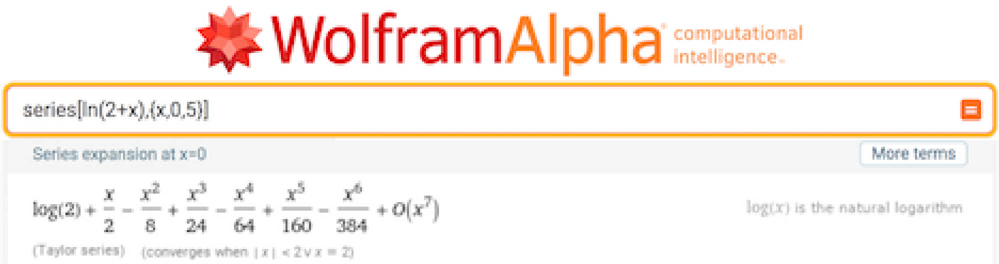
$f(x) = \ln(2 + x)$	when $x = 0$,	$f(0) = \ln 2$
$f'(x) = \dfrac{1}{2 + x}$	when $x = 0$,	$f'(0) = \dfrac{1}{2}$
$f''(x) = -\dfrac{1}{(2 + x)^2}$	when $x = 0$,	$f''(0) = -\dfrac{1}{4}$
$f'''(x) = \dfrac{2}{(2 + x)^3}$	when $x = 0$,	$f'''(0) = \dfrac{1}{4}$
$f^{(4)}(x) = -\dfrac{6}{(2 + x)^4}$	when $x = 0$,	$f^{(4)}(0) = -\dfrac{3}{8}$
$f^{(5)}(x) = \dfrac{24}{(2 + x)^5}$	when $x = 0$,	$f^{(5)}(0) = \dfrac{3}{4}$

(ii) $$f(x) \approx \ln 2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} - \frac{x^4}{64} + \frac{x^5}{160}$$

[8 marks]

Wolfram Alpha can do Maclaurin series.
To check your answer to question 5,

$$series[\ \ln(2 + x),\ \{x,\ 0,\ 5\}\ ]$$



Answer 6

$f(x) = \ln(1 + \sin x)$	when $x = 0$,	$f(0) = 0$
$f'(x) = \dfrac{\cos x}{1 + \sin x}$	when $x = 0$,	$f'(0) = 1$
$f''(x) = -\dfrac{1}{1 + \sin x}$	when $x = 0$,	$f''(0) = -1$
$f'''(x) = -\dfrac{\cos x}{(1 + \sin x)^2}$	when $x = 0$,	$f'''(0) = -1$
$f^{(4)}(x) = \dfrac{\sin x - \sin^2 x + 2\cos^2 x}{(1 + \sin x)^3}$	when $x = 0$,	$f^{(4)}(0) = 2$

$$f(x) \approx x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12}$$

[10 marks]

3.1 Famous Maclaurin Series

Several Maclaurin series are quotable and may be used without showing how they are derived. In the examination they are given in the formulae booklet;

Quotable Maclaurin Series Expansions

- $e^x = 1 + x + \dfrac{x^2}{2!} + \dots + \dfrac{x^r}{r!} + \dots$ valid for all x
- $\ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dots + (-1)^{r+1}\dfrac{x^r}{r} + \dots$ $-1 < x \leqslant 1$
- $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots + (-1)^r \dfrac{x^{2r+1}}{(2r+1)!} + \dots$ valid for all x
- $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dots + (-1)^r \dfrac{x^{2r}}{(2r)!} + \dots$ valid for all x
- $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots. + (-1)^r \dfrac{x^{2r+1}}{2r+1} + \dots$ $-1 < x \leqslant 1$

In addition the binomial expansions of $f(x) = (1 + x)^n$ where n is fractional or negative and $|x| < 1$ are also the Maclaurin series expansion of $f(x)$.
Thus the binomial theorem gives further Maclaurin series.

One worth knowing is the geometric series expansion of $\dfrac{1}{1 - x}$ which is,

- $(1 - x)^{-1} = 1 + x + x^2 + \dots + x^r + \dots$ $-1 < x < 1$

3.2 Convergence

In obtaining polynomial approximations to functions, what is sought is the situation in which the first few terms of the polynomial provide a good model of the function. Then, “throwing away” the infinite number of other, higher order terms, will make minimal difference to the approximation model for values of x close to $x = 0$. If the Maclaurin series is convergent for all values of x, such as is the case with the functions $\sin x$ and $\cos x$, for example, then taking more terms yields a polynomial that follows the function over an increasing domain.

However, for some Maclaurin series, convergence is not valid for all x. Instead it is valid only for a restricted domain. Going beyond these values means that no higher order terms can be thrown away because they are no longer insignificant. No approximation of the full series can then be made. This is why some of the series in the table “Quotable Maclaurin Series Expansions” have a restricted domain. The restriction indicates the values of x for which truncating the series yields a valid approximating polynomial.

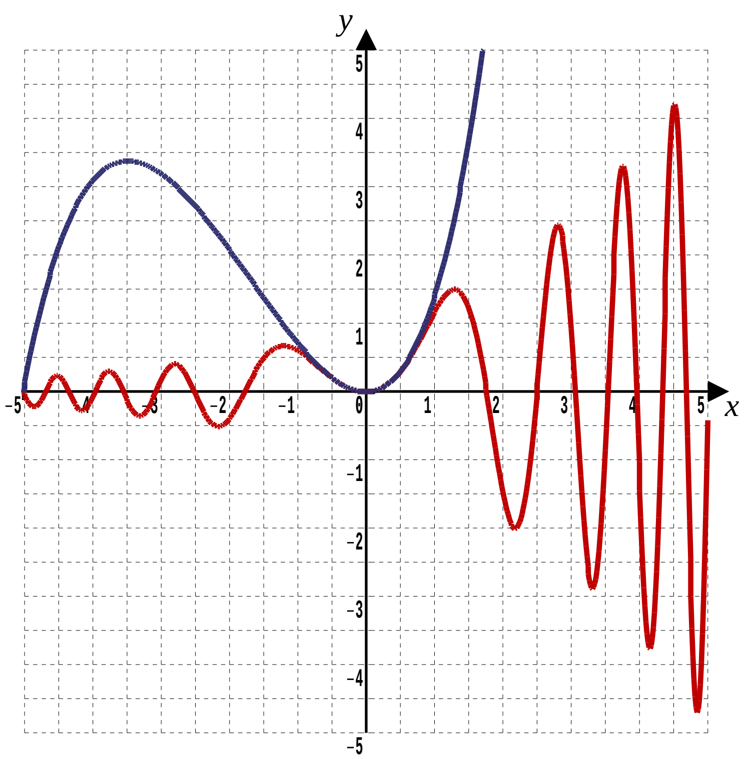
3.3 Substitution

Given an unfamiliar function, and the need to obtain its Maclaurin series, time can often be saved by looking to see if it is sufficiently similar to any of the series in the table “Quotable Maclaurin Series Expansions” to allow a simple substitution to be made. This can side step having to do the several differentiations that would otherwise be required to obtain an approximating polynomial.

Example

Find the best quintic approximation to $f(x) = e^{\frac{x}{\pi}} sin\left(x^2\right)$ centred on $x = 0$

Teaching Video : http://www.NumberWonder.co.uk/v9098/3.mp4



In red is the graph of $y = e^{\frac{x}{\pi}} sin\left(x^2\right)$ and in blue the best quintic polynomial approximation on $x = 0$

[6 marks]

3.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

(i) By means of a suitable substitution into the Maclaurin series for e^x, find a quintic polynomial approximation to the function $g(x) = \dfrac{1}{e^x}$

(ii) State the values of x for which the expansion is valid.

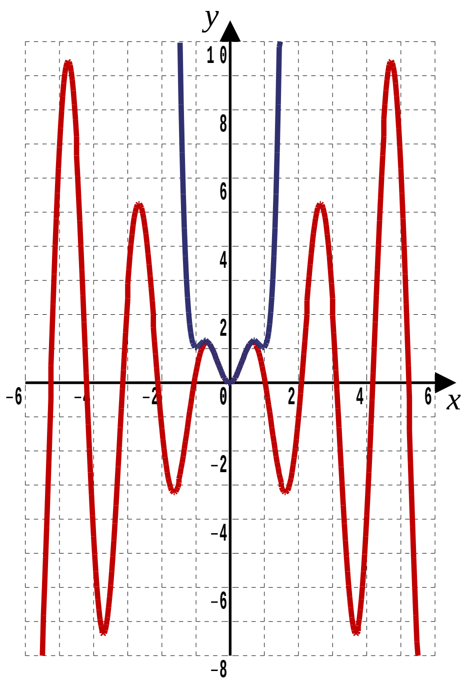
[3 marks]

Question 2

(i) By means of a suitable substitution into the Maclaurin series for $\sin x$, find a sextic (degree 6) polynomial approximation to the function

$$h(x) = 2x \sin(3x)$$

(ii) State the values of x for which the expansion is valid.



In red is the graph of the function $h(x) = 2x \sin(3x)$
In blue is the sextic polynomial approximation on $x = 0$

[3 marks]

Question 3

(i) The Maclaurin series for *arctan x* states that,

- $$arctan\, x \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \; .... \; + (-1)^r \frac{x^{2r+1}}{2r+1} + \; ... \qquad -1 < x \leqslant 1$$

By substituting $x = 1$ into both sides of this Maclaurin series deduce an astonishing formula for π in terms of a multiple of an alternating series of reciprocal odd numbers. This result is known as the Leibniz formula for π although it was first discovered long before Leibniz by the Indian mathematician Madhava of Sangamagrama in the 14th century.

[2 marks]

(ii) Determine the value of π given by using the first five terms of the series.
The Leibniz formula for π converges extremely slowly.
To get π accurate to 10 decimal places requires five billion terms !

[2 marks]

Question 4

By means of a suitable substitution into the geometric series,

- $$(1 - x)^{-1} = 1 + x + x^2 + \; ... \; + x^r + \; ... \qquad \text{valid for } -1 < x < 1$$

find an octic (degree 8) polynomial approximation to $g(x) = \dfrac{1}{1 + x^2}$

[3 marks]

Question 5

Further A-Level Examination Question from June 2009, FP2, Q1 (a) (MEI)

(i) Use the Maclaurin series for $ln(1 + x)$ and $ln(1 - x)$ to obtain the first three non-zero terms in the Maclaurin series for $ln\left(\frac{1 + x}{1 - x}\right)$

State the range of validity of this series.

[4 marks]

(ii) Find the value of x for which $\frac{1 + x}{1 - x} = 3$

Hence find an approximation to $ln\ 3$, to three decimal places.

[4 marks]

Question 6

(i) Use a standard Maclaurin series, together with a suitable substitution, to find a quartic polynomial approximation for the function,

$$f(x) = (1 - 3x)\, ln(1 + 2x)$$

(ii) State the values of x for which the expansion is valid

[4 marks]

Question 7

(i) By writing $2 + 3x$ as $2\left(1 + \frac{3x}{2}\right)$, use a standard Maclaurin series, together with a suitable substitution, to find a quintic polynomial approximation to, $g(x) = 64x\, ln(2 + 3x)$

(ii) State the values of x for which the expansion is valid

[4 marks]

Question 8

(i) Write down the first five non-zero terms in the series expansion of $e^{-\frac{x^2}{2}}$

[3 marks]

(ii) Using your result from part (a), find an approximate value for

$$\int_{-1}^{1} e^{-\frac{x^2}{2}}\, dx$$

giving your answer to 3 decimal places

[3 marks]

Question 9

Further A-Level Examination Question from January 2010, FP2, Q2 (c) (MEI)

Write down the Maclaurin series for e^t as far as the term in t^2

Hence show that, for t close to zero,

$$\frac{t}{e^t - 1} \approx 1 - \frac{1}{2}t$$

[5 marks]

3.5 Answers

3.5.1 Solution to 3.3.Example (Teaching Video)

From the standard results that, $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots$

and $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots$

$$e^{\frac{x}{\pi}} \sin(x^2) = \left(1 + \frac{x}{\pi} + \frac{x^2}{2\pi^2} + \frac{x^3}{6\pi^3} + \dots\right)\left(x^2 - \frac{x^6}{6} + \dots\right)$$

$$= x^2 + \frac{x^3}{\pi} + \frac{x^4}{2\pi^2} + \frac{x^5}{6\pi^3} + \dots$$

So $f(x) \approx x^2 + \frac{x^3}{\pi} + \frac{x^4}{2\pi^2} + \frac{x^5}{6\pi^3}$

[6 marks]

3.5.2 Solutions to 3.4 Exercise

Answer 1

(i) From the standard result that,

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots \qquad \text{valid for all } x$$

$$g(x) = \frac{1}{e^x}$$

$$= e^{-x}$$

$$= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \dots$$

$$\approx 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$$

(ii) It's valid for all x, but increasingly inaccurate as one moves away from its centred value of $x = 0$

[3 marks]

Answer 2

(i) From the standard result that,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{valid for all } x$$

$$\begin{aligned} h(x) &= 2x \sin(3x) \\ &= 2x\left(3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} + \ldots\right) \\ &\approx 6x^2 - 9x^4 + \frac{81}{20}x^6 \end{aligned}$$

(ii) It's valid for all *x*, but increasingly inaccurate as one moves away from its centred value of $x = 0$, as is obvious from the provided graph.

[3 marks]

Answer 3

(i) The standard result is,

- $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots. + (-1)^r \dfrac{x^{2r+1}}{2r+1} + \ldots \quad -1 < x \leqslant 1$

With $x = 1$ this yields,

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

Remembering to use radians,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right)$$

[2 marks]

(ii) Using the first five terms of the series,

$$\begin{aligned} \pi &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\right) \\ &= \frac{1052}{315} \\ &= 3.3396825\ldots \end{aligned}$$

[2 marks]

Answer 4

$$g(x) = \frac{1}{1 + x^2}$$
$$= \frac{1}{1 - \left(-x^2\right)}$$
$$= \left(1 - \left(-x^2\right)\right)^{-1}$$

Substituting $\left(-x^2\right)$ into

- $(1 - x)^{-1} = 1 + x + x^2 + \ldots + x^r + \ldots$ valid for $-1 < x < 1$

gives

$$g(x) = 1 + \left(-x^2\right) + \left(-x^2\right)^2 + \left(-x^2\right)^3 + \left(-x^2\right)^4 + \ldots$$
$$\approx 1 - x^2 + x^4 - x^6 + x^8$$

[3 marks]

Answer 5

(i) From the standard result that,

- $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \quad -1 < x \leqslant 1$

and the substitution of $(-x)$ in place of x it is deduced that,

$$ln(1 - x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \ldots \qquad -1 \leqslant x < 1$$

$$ln\left(\frac{1 + x}{1 - x}\right) = ln(1 + x) - ln(1 - x)$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right)$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \ldots$$
$$\approx 2x + \frac{2x^3}{3} + \frac{2x^5}{5} \qquad \text{valid for } -1 < x < 1$$

[4 marks]

(ii)

$$\frac{1 + x}{1 - x} = 3$$

$$1 + x = 3(1 - x)$$

$$1 + x = 3 - 3x$$

$$4x = 2$$

$$x = \frac{1}{2}$$

$$\therefore\ ln\,3 \approx 2 \times \frac{1}{2} + \frac{2}{3} \times \left(\frac{1}{2}\right)^3 + \frac{2}{5} \times \left(\frac{1}{2}\right)^5$$

$$ln\,3 \approx 1 + \frac{1}{12} + \frac{1}{80}$$

$$ln\,3 \approx 1.096$$

[4 marks]

Answer 6

(i) From the standard result that,

- $ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots + (-1)^{r+1}\dfrac{x^r}{r} + \ldots \quad -1 < x \leqslant 1$

and the substitution of ($2x$) in place of x it is deduced that,

$$ln(1 + 2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \ldots \qquad -\frac{1}{2} < x \leqslant \frac{1}{2}$$

$$f(x) = (1 - 3x)\ ln(1 + 2x)$$

$$f(x) \approx (1 - 3x)\left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4\right)$$

$$f(x) \approx 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 - 6x^2 + 6x^3 - 8x^4$$

$$f(x) \approx 2x - 8x^2 + \frac{26}{3}x^3 - 12x^4$$

(ii) It's valid for $-\dfrac{1}{2} < x \leqslant \dfrac{1}{2}$

[4 marks]

Answer 7

(i) From the standard result that,

- $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{r+1}\frac{x^r}{r} + ... \quad -1 < x \leqslant 1$

and the substitution of $\left(\frac{3x}{2}\right)$ in place of x it is deduced that,

$$ln\left(1 + \frac{3x}{2}\right) = \left(\frac{3x}{2}\right) - \frac{1}{2}\left(\frac{3x}{2}\right)^2 + \frac{1}{3}\left(\frac{3x}{2}\right)^3 - \frac{1}{4}\left(\frac{3x}{2}\right)^4 + ...$$

$$= \frac{3x}{2} - \frac{9x^2}{8} + \frac{9x^3}{8} - \frac{81x^4}{64} + ... \qquad -\frac{2}{3} < x \leqslant \frac{2}{3}$$

$$g(x) = 64x\,ln(2 + 3x)$$

$$= 64x\,ln\left(2\left(1 + \frac{3x}{2}\right)\right)$$

$$= 64x\,ln(2) + 64x\,ln\left(1 + \frac{3x}{2}\right)$$

$$g(x) \approx 64\,ln(2)x + 64x\left(\frac{3x}{2} - \frac{9x^2}{8} + \frac{9x^3}{8} - \frac{81x^4}{64}\right)$$

$$g(x) \approx 64\,ln(2)x + 96x^2 - 72x^3 + 72x^4 - 81x^5$$

(ii) It's valid for $-\frac{2}{3} < x \leqslant \frac{2}{3}$

[4 marks]

Answer 8

(i) From the standard result that, $e^x = 1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^r}{r!} + \ldots$

$$e^{-\frac{x^2}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2!}\left(-\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x^2}{2}\right)^3 + \frac{1}{4!}\left(-\frac{x^2}{2}\right)^4 + \ldots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384} + \ldots$$

[3 marks]

(ii)

$$\int_{-1}^{1} e^{-\frac{x^2}{2}}\,dx \approx \int_{-1}^{1}\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384}\right)dx$$

$$= \left[\, x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456} \,\right]_{-1}^{1}$$

$$= \left[1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456}\right] - \left[-1 + \frac{1}{6} - \frac{1}{40} + \frac{1}{336} - \frac{1}{3456}\right]$$

$$= 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} + 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456}$$

$$= 2 \times \left[1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456}\right]$$

$$= 1.711$$

[3 marks]

Answer 9

$$e^t \approx 1 + t + \frac{t^2}{2!}$$

$$\frac{t}{e^t - 1} \approx \frac{t}{1 + t + \frac{t^2}{2!} - 1} = \frac{t}{t + \frac{t^2}{2!}}$$

$$= \frac{1}{1 + \frac{t}{2}}$$

$$= \left(1 + \frac{t}{2}\right)^{-1}$$

By the binomial theorem, $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots$

$$\therefore \left(1 + \frac{t}{2}\right)^{-1} \approx 1 + (-1)\left(\frac{t}{2}\right) = 1 - \frac{1}{2}t \qquad \square$$

[5 marks]

4.1 The Inverse Trig Functions

A few derivatives of inverse trigonometric functions are amongst those given in the examination formula booklet. Even so, an examination may well ask for one of the results given to be proven.

$f(x)$	$f'(x)$	In Formula Book ?
x^n	nx^{n-1}	No
e^x	e^x	No
$\ln x$	$\frac{1}{x}$	No
$\sin x$	$\cos x$	No
$\cos x$	$-\sin x$	No
$\tan x$	$\sec^2 x$	Yes
$\csc x$	$-\csc x \cot x$	Yes
$\sec x$	$\sec x \tan x$	Yes
$\cot x$	$-\csc^2 x$	Yes
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	Yes
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	Yes
$\arctan x$	$\frac{1}{1+x^2}$	Yes

Example #1

Prove that the derivative of $\arcsin x$ is $\dfrac{1}{\sqrt{1-x^2}}$

Teaching Video : http://www.NumberWonder.co.uk/v9028/10b.mp4

[5 marks]

4.2 The Binomial Theorem

To find the Maclaurin series for *arcsin x* use could be made of the rule,

---

The Maclaurin Series

A given function, $f(x)$, may be written as the polynomial,

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$$

provided that $f(0)$, $f'(0)$, $f''(0)$, ..., $f^{(r)}(0)$, ... all have finite values.

However, the Binomial Theorem provides an easier way.

Example #2

With the help of the Binomial Theorem, find the first four non-zero terms in the Maclaurin series for *arcsin x*

Teaching Video : http://www.NumberWonder.co.uk/v9098/4.mp4

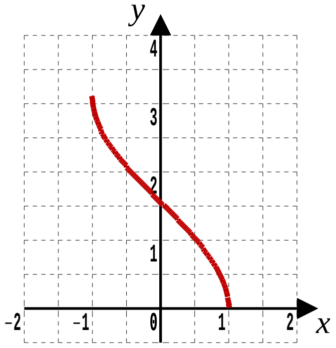
[8 marks]

4.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

With the aid of the graph, prove that the derivative of *arccos* x is $-\dfrac{1}{\sqrt{1-x^2}}$



$y = \arccos x$

[5 marks]

Question 2

(i) It is obviously true that, $\dfrac{1}{\sqrt{1-x^2}} + \left(-\dfrac{1}{\sqrt{1-x^2}}\right) = 0$

By integrating each term in this equation, deduce a simple relationship between *arcsin* x and *arccos* x

[3 marks]

(ii) Hence, use the Maclaurin series for *arcsin* x to deduce the Maclaurin series for *arccos* x.

[2 marks]

Question 3

Prove that the derivative of *arctan x* is $\dfrac{1}{1 + x^2}$

[5 marks]

Question 4

With the help of the Binomial Theorem, find the first four non-zero terms in the Maclaurin series for *arctan x*

[8 marks]

Question 5

Prove that the derivative of *arccot x* is $-\dfrac{1}{1 + x^2}$

[5 marks]

Question 6

(i) It is obviously true that, $\dfrac{1}{1 + x^2} + \left(-\dfrac{1}{1 + x^2}\right) = 0$

By integrating each term in this equation, deduce a simple relationship between *arctan x* and *arccot x*

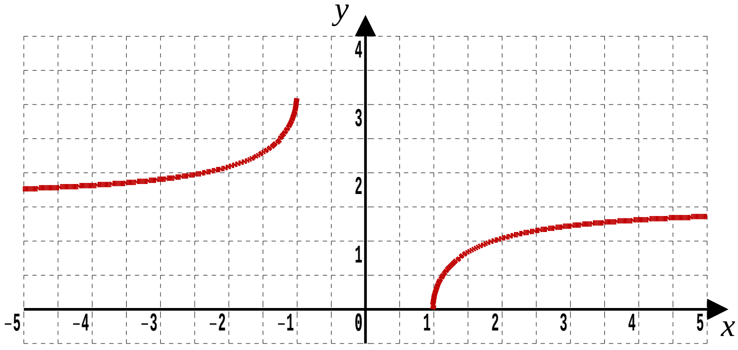
[3 marks]

(ii) Hence, use the Maclaurin series for *arctan x* to deduce the Maclaurin series for *arccot x*.

[2 marks]

Question 7

With the aid of the graph, prove that the derivative of *arcsec* x is $\dfrac{1}{x\sqrt{x^2 - 1}}$



The graph of $y = arcsec\ x$

[5 marks]

Question 8

Give a reason why a Maclaurin series for *arcsec* x will not exist.

[2 marks]

4.4 Answers

4.4.1 Solution to 4.1 Example #1 (Teaching Video)

A Proof that the derivative of $arcsin\, x$ is $\dfrac{1}{\sqrt{1 - x^2}}$

$$y = arcsin\, x$$

$$\therefore x = sin\, y$$

$$\frac{dx}{dy} = cos\, y$$

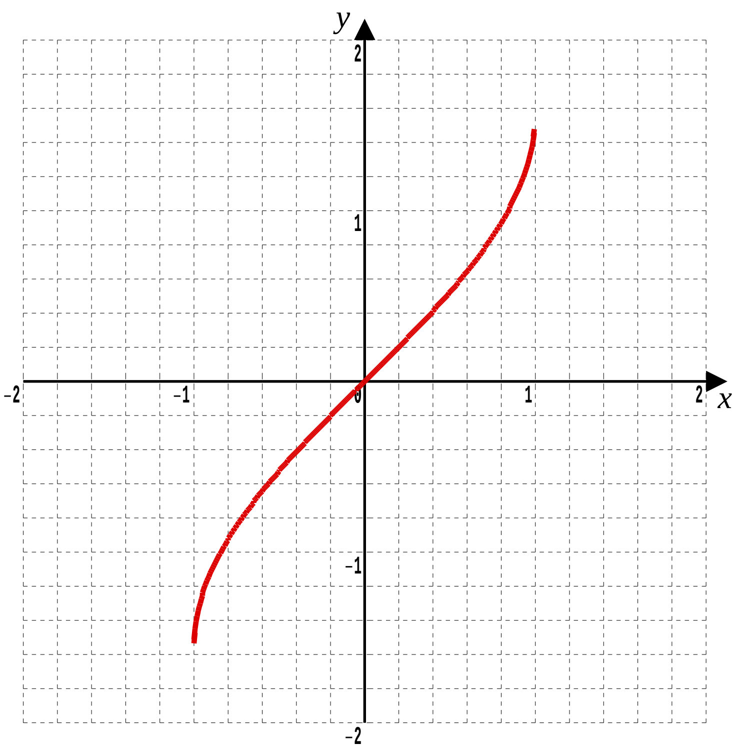
Now use the fact that $cos^2 y + sin^2 y = 1$ to get...

$$\frac{dx}{dy} = \pm\sqrt{1 - sin^2 y}$$

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1 - x^2}}$$

From the graph of $y = arcsin\, x$ we can see that the gradient is always positive

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \qquad \square$$



[5 marks]

4.4.2 Solution to 4.2 Example #2 (Teaching Video)

With the help of the Binomial Theorem, find the first four non-zero terms in the Maclaurin series for *arcsin x*

The Binomial Theorem begins;

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

and it's known that the derivative of $arcsin\,x$ is $\dfrac{1}{\sqrt{1-x^2}}$

$$\frac{1}{\sqrt{1-x^2}}$$

$$= \left(1-x^2\right)^{-\frac{1}{2}}$$

$$= \left(1+\left(-x^2\right)\right)^{-\frac{1}{2}}$$

$$= 1 + \left(-\frac{1}{2}\right)\left(-x^2\right) + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-x^2\right)^2 + \frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-x^2\right)^3 + \ldots$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \ldots \qquad \text{valid for } -1 < x < 1$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \ldots\,dx$$

$$arcsin\,x + c = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots$$

As $arcsin\,x = 0$ when $x = 0$, the constant of the integration, $c = 0$

$$\therefore\ arcsin\,x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots \qquad \text{valid for } -1 < x < 1$$

Note :

In fact the series is valid for $-1 \leqslant x \leqslant 1$ but proving that is outwith the course.

[8 marks]

4.4.3 Solutions to 4.3 Exercise

Answer 1

$$y = arccos\, x$$
$$\therefore x = cos\, y$$
$$\frac{dx}{dy} = -sin\, y$$

Now use the fact that $cos^2 y + sin^2 y = 1$ to get...

$$\frac{dx}{dy} = \pm\sqrt{1 - cos^2 y}$$
$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1 - x^2}}$$

From the graph of $y = arccos\, x$ it can be seen that the gradient is always negative

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \qquad \square$$

[5 marks]

Answer 2

(i)

$$\frac{1}{\sqrt{1 - x^2}} + \left(-\frac{1}{\sqrt{1 - x^2}}\right) = 0$$
$$\int \frac{1}{\sqrt{1 - x^2}}\, dx + \int \left(-\frac{1}{\sqrt{1 - x^2}}\right) dx = \int 0\, dx$$
$$arcsin\, x + arccos\, x = c$$

When $x = 0$, $arcsin\, x = 0$ and $arccos\, x = \frac{\pi}{2}$

$$\Rightarrow \quad c = \frac{\pi}{2}$$
$$\therefore arcsin\, x + arccos\, x = \frac{\pi}{2}$$

[3 marks]

(ii) Given that, from Example #2,

$$arcsin\, x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots \qquad \text{valid for } -1 < x < 1$$

the part (i) relationship gives,

$$arccos\, x = \frac{\pi}{2} - arcsin\, x$$
$$= \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots\right) \text{ valid for } -1 < x < 1$$

Note :

In fact the series is valid for $-1 \leqslant x \leqslant 1$ but proving that is outwith the course.

[2 marks]

Answer 3

$$y = arctan\, x$$

$$\therefore\ x = tan\, y$$

$$\frac{dx}{dy} = sec^2\, y$$

Now use the fact that $1 + tan^2\, y = sec^2\, y$ to get...

$$\frac{dx}{dy} = 1 + tan^2\, y$$

$$\frac{dx}{dy} = 1 + x^2$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2} \qquad \square$$

[5 marks]

Answer 4

With the help of the Binomial Theorem, find the first four non-zero terms in the Maclaurin series for *arcsin x*

The Binomial Theorem begins;

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ ...$$

and it's known that the derivative of $arctan\, x$ is $\dfrac{1}{1 + x^2}$

$$\frac{1}{1 + x^2} = \left(1 + x^2\right)^{-1}$$

$$= 1 + (-1)\left(x^2\right) + \frac{1}{2!}(-1)(-2)\left(x^2\right)^2 + \frac{1}{3!}(-1)(-2)(-3)\left(x^2\right)^3 + \ ...$$

$$= 1 - x^2 + x^4 - x^6 + \ ... \qquad \text{valid for } -1 < x < 1$$

$$\int \frac{1}{1 + x^2}\, dx = \int 1 - x^2 + x^4 - x^6 + \ ...\ dx$$

$$arctan\, x + c = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ ...$$

As $arctan\, x = 0$ when $x = 0$, the constant of the integration, $c = 0$

$$\therefore\ arctan\, x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ ... \qquad \text{valid for } -1 < x < 1$$

Note :

In fact the series is valid for $-1 \leqslant x \leqslant 1$ but proving that is outwith the course.

[8 marks]

Answer 5

$$y = arccot\, x$$

$$\therefore x = cot\, y$$

$$\frac{dx}{dy} = -csc^2 y$$

Now use the fact that $cot^2 y + 1 = csc^2 y$ to get...

$$\frac{dx}{dy} = (-1)\left(cot^2 y + 1 \right)$$

$$\frac{dx}{dy} = (-1)\left(x^2 + 1 \right)$$

$$\frac{dy}{dx} = -\frac{1}{1 + x^2} \qquad \square$$

[5 marks]

Answer 6

(i)
$$\frac{1}{1 + x^2} + \left(-\frac{1}{1 + x^2} \right) = 0$$

$$\int \frac{1}{1 + x^2}\, dx + \int \left(-\frac{1}{1 + x^2} \right) dx = \int 0\, dx$$

$$arctan\, x + arccot\, x = c$$

When $x = 0$, $arctan\, x = 0$ and $arccot\, x = \frac{\pi}{2} \Rightarrow \quad c = \frac{\pi}{2}$

$$\therefore arctan\, x + arccot\, x = \frac{\pi}{2}$$

Note :

$$arccot\,(0) = X \Rightarrow cot\,(X) = 0 \Rightarrow cos\,(X) = 0$$

$$\Rightarrow arccos\,(0) = X \Rightarrow X = \frac{\pi}{2}$$

[3 marks]

(ii) Given that, from Question 4,

$arctan\, x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ valid for $-1 < x < 1$

the part (i) relationship gives,

$$arccot\, x = \frac{\pi}{2} - arctan\, x$$

$$= \frac{\pi}{2} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \right) \text{ valid for } -1 < x < 1$$

Note :

In fact the series is valid for $-1 \leqslant x \leqslant 1$

but proving that is beyond the Further A-Level course.

[2 marks]

Answer 7

$$y = arcsec\, x$$
$$\therefore x = sec\, y$$
$$\frac{dx}{dy} = sec\, y\;\; tan\, y$$

Now use the fact that $1 + tan^2 y = sec^2 y$ to get...

$$\frac{dx}{dy} = \pm sec\, y \sqrt{sec^2 y - 1}$$
$$\frac{dx}{dy} = \pm x \sqrt{x^2 - 1}$$
$$\frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}}$$

From the graph of $y = arcsec\, x$ it can be seen that the gradient is always positive

(Each of its two parts are an increasing function)

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}} \qquad \square$$

[5 marks]

Answer 8

The quickest reason to give to explain why no Maclaurin series can exist for *arcsec x* is simply to observe that the function does not have a crossing point of the *y*-axis, the graph having been provided in question 7.

Another reason that is easily given is to observe that for a Maclaurin series to exist, all of $f(0)$, $f'(0)$, $f''(0)$, ..., $f^{(r)}(0)$, ... must have finite values, a fact clearly stated when the theorem "Maclaurin Series" was given.
So, simply observe that *arcsec*(0) does not exist or, using the derivative from question 7, that $f'(0)$ is undefined due to a division by zero.

[2 marks]

Graph Plotting

To draw the graphs of *arcsec x*, *arccsc x* and *arccot x*, or calculate their value at a particular point, use the relationships,

$$arcsec\, x = arccos\left(\frac{1}{x}\right)$$
$$arccsc\, x = arcsin\left(\frac{1}{x}\right)$$
$$arccot\, x = arctan\left(\frac{1}{x}\right)$$

5.1 Techniques

When the function for which a Maclaurin expansion is sought includes an exponential, the calculation of,

$$f(0),\ f'(0),\ f''(0),\ \ldots f^{(r)}(0)$$

often has a repetitive component.

A technique to speed up such calculations is the subject of this lesson.

To illustrate the method a simple example will be used that could be done by other methods that, in this case, would be more efficient. However, Exercise 5.3 will look at situations where the “other method” will fail, or runs into algebraic complications. A key skill in this topic is an ability to switch between solution methods when one approach runs into technical issues.

5.2 Example

Find a sextic approximation for the function $f(x) = x\,e^{x}$ centred on $x = 0$

Teaching Video : http://www.NumberWonder.co.uk/v9098/5.mp4

[8 marks]

5.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 54

Question 1

(i) Given that $f(x) = e^{1-2x}$ show that $f'''(x) = -8f(x)$

[3 marks]

(ii) Deduce a general result for $f^{(n)}(x)$

[2 marks]

(iii) Deduce a general result for $f^{(n)}(0)$

[1 mark]

(iv) Determine a sextic polynomial approximation to $f(x)$ centred on $x = 0$

[3 marks]

(v) Substitute $(1 - 2x)$ into the standard Maclaurin series for the exponential function. That is,

- $e^x = 1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^r}{r!} + \ldots$ valid for all x

There is no need to expand any brackets.

[1 mark]

(vi) If the brackets in part(v) were now to be expanded, the Maclaurin series would not be obtained in a manageable way. The substitution method fails. Briefly explain why.

[2 marks]

(vii) Part (vi) illustrates a problem that can arise with the substitution method of obtaining a Maclaurin series from one of the standard expansions in the examination formula book. In this case, a workaround would be to rewrite the function as $f(x) = e \times e^{-2x}$
Show how this now allows the substitution method to work and obtain the same series as was found in part (iv).
Note that in other such questions a simple workaround may not be possible.

[2 marks]

(viii) By letting $x = 0$ in your part (v) answer, obtain a remarkable result involving the reciprocals of factorials.

[2 marks]

Question 2

Given that, $f(x) = e^{3x} - e^{-3x}$

(i) Show that $f''(x) = 3^2 f(x)$

[2 marks]

(ii) Show that $f^{(4)}(x) = 3^4 f(x)$

[2 marks]

(iii) Hence explain why the Maclaurin series for $f(x)$ will not contain any terms where x is even powered.

[2 marks]

(iv) Find the first three non-zero terms of the Maclaurin series for $f(x)$ giving each coefficient in its simplest form.

[3 marks]

(v) Find an expression the the n^{th} non-zero term of the Maclaurin series for $f(x)$

[2 marks]

Question 3

Further A-Level Examination Question from 2018 Mock Core 1 Paper, Q1, (Edexcel)

Given that $f(x) = e^{2x} \cos x$

(a) Show that $f''(x) = p f(x) + q f'(x)$

where p and q are integers to be determined.

[5 marks]

(b) Hence find the Maclaurin series for $f(x)$, in ascending powers of x, up to and including the term in x^5, giving each coefficient in its simplest form.

[3 marks]

Question 4

Given that, $f(x) = g(x) + h(x)$ where $g(x) = e^x \sin x$ and $h(x) = e^x \cos x$

obtain expressions for $f'(x)$, $f''(x)$, ... $f^{(5)}(x)$ each in the form $p\,g(x) + q\,h(x)$ where p and q for each expression are to be determined.

Hence find the Maclaurin series for

Hence find the Maclaurin series for $f(x)$, in ascending powers of x, up to and including the terms in x^5 giving each coefficient in its simplest form.

[8 marks]

Question 5

Given that $f(x) = ln(1 + cos 2x)$ $\quad 0 \leqslant x < \dfrac{\pi}{2}$

Show that,

(i) $\quad f'(x) = -2 \tan x$

[2 marks]

(ii) $\quad f''''(x) = -\left(f'''(x)\, f'(x) + \left(f''(x)^2 \right) \right)$

[5 marks]

(iii) Find the Maclaurin series expansion of $f(x)$, in ascending powers of x, up to and including the term in x^4

[4 marks]

5.4 Answers

5.4.1 Solution to 5.2 Example

$$\begin{aligned}
f(x) &= x\,e^x & &\Rightarrow f(0) = 0 \\
f'(x) &= x\,e^x + e^x & & \\
&= f(x) + e^x & &\Rightarrow f'(0) = 1 \\
f''(x) &= [\, f'(x) \,] + e^x & & \\
&= [\, f(x) + e^x \,] + e^x & & \\
&= f(x) + 2\,e^x & &\Rightarrow f''(0) = 2 \\
f'''(x) &= [\, f'(x) \,] + 2\,e^x & & \\
&= [\, f(x) + e^x \,] + 2\,e^x & & \\
&= f(x) + 3\,e^x & &\Rightarrow f'''(0) = 3 \\
& \quad \ldots & & \\
f^{(n)}(x) &= f(x) + n\,e^x & &\Rightarrow f^{(n)}(0) = n
\end{aligned}$$

(All set for a proof by induction !)

The Maclaurin Series

A given function, $f(x)$, may be written as the polynomial,

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \ldots + \frac{f^{(r)}(0)}{r!}\,x^r + \ldots$$

provided that $f(0)$, $f'(0)$, $f''(0)$, ..., $f^{(r)}(0)$, ... all have finite values.

$$f(x) \approx x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120}$$

[8 marks]

5.4.2 Solutions to 5.3 Exercise

Answer 1

(i)

$$f(x) = e^{1-2x}$$
$$f'(x) = e^{1-2x}(-2)$$
$$= -2f(x)$$
$$f''(x) = -2\,[\,f'(x)\,]$$
$$= -2\,[-2f(x)\,]$$
$$= 4f(x)$$
$$f'''(x) = 4\,[\,f'(x)\,]$$
$$= 4\,[-2f(x)\,]$$
$$= -8f(x) \qquad \square$$

[3 marks]

(ii)

$$f^{(n)}(x) = (-1)^n\, 2^n f(x)$$

[2 marks]

(iii)

As $f(0) = e, \quad f^{(n)} = (-1)^n\, 2^n\, e$

[1 mark]

(iv)

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!} + \ldots$$
$$= e - 2e\,x + \frac{4e}{2}x^2 - \frac{8e}{6}x^3 + \frac{16e}{24}x^4 - \frac{32e}{120}x^5 + \frac{64e}{720}x^6 - \ldots$$
$$\approx e - 2e\,x + 2e\,x^2 - \frac{4e}{3}x^3 + \frac{2e}{3}x^4 - \frac{4}{15}x^5 + \frac{4e}{45}x^6$$

[3 marks]

(v) From $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots$ valid for all x

with x replaced with $(1 - 2x)$ comes,

$$e^{1-2x} = 1 + (1 - 2x) + \frac{1}{2!}(1 - 2x)^2 + \frac{1}{3!}(1 - 2x)^3 + \ldots$$

[1 mark]

(vi) Every bracket when expanded gives a number, and so the first term of the Maclaurin series is obtained as an infinite series rather than a single number. It is,

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

The same issue arises in trying to determine the coefficient of x or, indeed, the coefficient of any power of x in the series.

[2 marks]

(vii)

$$f(x) = e^{1-2x}$$
$$= e^{1} e^{-2x}$$
$$= e\left(1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + \frac{(-2x)^5}{5!} + \frac{(-2x)^6}{6!} + \ldots\right)$$
$$= e\left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 + \frac{4}{45}x^6 + \ldots\right)$$
$$\approx e - 2ex + 2ex^2 - \frac{4e}{3}x^3 + \frac{2e}{3}x^4 - \frac{4e}{15}x^5 + \frac{4e}{45}x^6$$

Which is exactly the same sextic as was found in part (iv) ☐

[2 marks]

(viii) From part (v),

$$e^{1-2x} = 1 + (1 - 2x) + \frac{1}{2!}(1 - 2x)^2 + \frac{1}{3!}(1 - 2x)^3 + \ldots$$

with $x = 0$

$$e^{1} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots\ldots$$

[2 marks]

Answer 2

(i)

$$f(x) = e^{3x} - e^{-3x}$$
$$f'(x) = 3e^{3x} + 3e^{-3x}$$
$$= 3\left(e^{3x} + e^{-3x}\right)$$
$$f''(x) = 3\left(3e^{3x} - 3e^{-3x}\right)$$
$$= 3^2\left(e^{3x} - e^{-3x}\right)$$
$$= 3^2 f(x) \quad ☐$$

[2 marks]

(ii)

$$f'''(x) = 3^2 f'(x)$$
$$f^{(4)}(x) = 3^2 \left[f''(x) \right]$$
$$= 3^2 \left[3^2 f(x) \right]$$
$$= 3^4 f(x) \quad ☐$$

[2 marks]

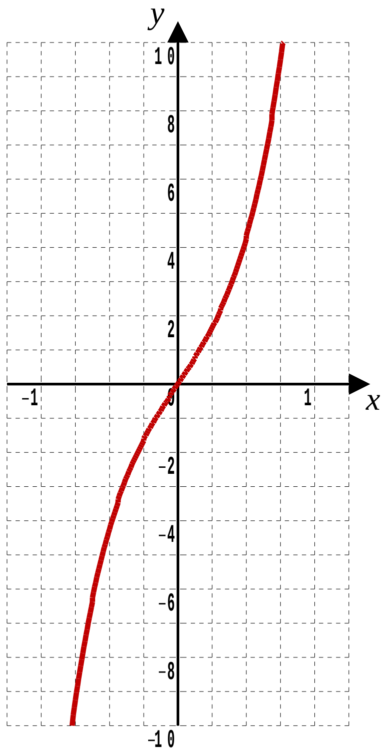
(iii) A reasonable deduction is that, $f^{(2n)}(x) = 3^{2n} f(x)$

As $f(x) = 0$ it follows that $f^{(2n)}(0) = 0$

In the Maclaurin series for $f(x)$ the coefficients of all the even powers of x are thus zero. That is, the series will not contain even powers of x ☐

An alternative argument can be made involving odd functions.

It is simply that, as $f(x)$ is an odd function it can contain only odd powers of x. That is, the series will not contain even powers of x ☐



Let $f(x) = e^{3x} - e^{-3x}$ in which case;

$$
\begin{aligned}
f(-x) &= e^{3(-x)} - e^{-3(-x)} \\
&= e^{-3x} - e^{3x} \\
&= -e^{3x} + e^{-3x} \\
&= -\left(e^{3x} - e^{-3x}\right) \\
&= -f(x)
\end{aligned}
$$

$\therefore\ f(x)$ is an odd function

The graph of the odd function $f(x) = e^{3x} - e^{-3x}$

[2 marks]

(iv) $f'(x) = 3\left(e^{3x} + e^{-3x}\right) \Rightarrow f'(0) = 3 \times 2$ 1st term

$f'''(x) = 3^2 f'(x) \Rightarrow f'''(0) = 3^3 \times 2$ 2nd term

$f^{(5)}(x) = 3^4 f'(x) \Rightarrow f^{(5)}(0) = 3^5 \times 2$ 3rd term

$$\therefore\ f(x) \approx 6x + 9x^3 + \frac{81}{20}x^5$$

[3 marks]

(v) n^{th} term will be $3^{2n-1} \times 2 \times \dfrac{1}{(2n-1)!} x^{2n-1}$

[2 marks]

Answer 3

(a)

$$f(x) = e^{2x} \cos x \qquad \Rightarrow f(0) = 1$$

$$f'(x) = e^{2x}(-\sin x) + 2e^{2x}\cos x$$

$$= -e^{2x}\sin x + 2f(x) \qquad \Rightarrow f'(0) = 2$$

$$f''(x) = -e^{2x}\cos x + \left(-2e^{2x}\right)\sin x + 2f'(x)$$

$$= -f(x) - 2e^{2x}\sin x + 2f'(x)$$

From, $f'(x) = -e^{2x}\sin x + 2f(x)$

$$e^{2x}\sin x = 2f(x) - f'(x)$$

$$\therefore \; f''(x) = -f(x) - 2\left(2f(x) - f'(x)\right) + 2f'(x)$$

$$= -f(x) - 4f(x) + 2f'(x) + 2f'(x)$$

$$= -5f(x) + 4f'(x) \qquad \Rightarrow f''(0) = 3$$

That is, $p = -5$ and $q = 4$

[5 marks]

(b)

$$f'''(x) = -5f'(x) + 4f''(x)$$

$$= -5f'(x) + 4\left(-5f(x) + 4f'(x)\right)$$

$$= -20f(x) + 11f'(x) \qquad \Rightarrow f'''(0) = 2$$

$$f^{(4)}(x) = -20f'(x) + 11f''(x)$$

$$= -20f'(x) + 11\left(-5f(x) + 4f'(x)\right)$$

$$= -55f(x) + 24f'(x) \qquad \Rightarrow f^{(4)}(0) = -7$$

$$f^{(5)}(x) = -55f'(x) + 24f''(x)$$

$$= -55f'(x) + 24\left(-5f(x) + 4f'(x)\right)$$

$$= -120f(x) + 41f'(x) \qquad \Rightarrow f^{(5)}(0) = -38$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$$

$$f(x) \approx 1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3 - \frac{7}{24}x^4 - \frac{19}{60}x^5$$

[3 marks]

Answer 4

$$g(x) = e^x \sin x \qquad\qquad \Rightarrow\ g(0) = 0$$

$$\therefore\ g'(x) = e^x \cos x + e^x \sin x$$

$$= h(x) + g(x) \qquad \text{(Note also } g'(x) = f(x)\text{)}$$

$$h(x) = e^x \cos x \qquad\qquad \Rightarrow\ h(0) = 1$$

$$h'(x) = e^x(-\sin x) + e^x \cos x$$

$$= h(x) - g(x)$$

$$f'(x) = g'(x) + h'(x)$$

$$= h(x) + g(x) + h(x) - g(x)$$

$$= 2\,h(x) \qquad\qquad \Rightarrow\ f'(0) = 2$$

$$f''(x) = 2\,h'(x)$$

$$= 2\,h(x) - 2\,g(x) \qquad\qquad \Rightarrow\ f''(0) = 2$$

$$f'''(x) = 2\,h'(x) - 2\,g'(x)$$

$$= 2\,h(x) - 2\,g(x) - 2\,h(x) - 2\,g(x)$$

$$= -4\,g(x) \qquad\qquad \Rightarrow\ f'''(0) = 0$$

$$f^{(4)}(x) = -4\,g'(x)$$

$$= -4\,h(x) - 4\,g(x) \qquad\qquad \Rightarrow\ f^{(4)}(0) = -4$$

$$f^{(5)}(x) = -4\,h'(x) - 4\,g'(x)$$

$$= -4\,h(x) + 4\,g(x) - 4\,h(x) - 4\,g(x)$$

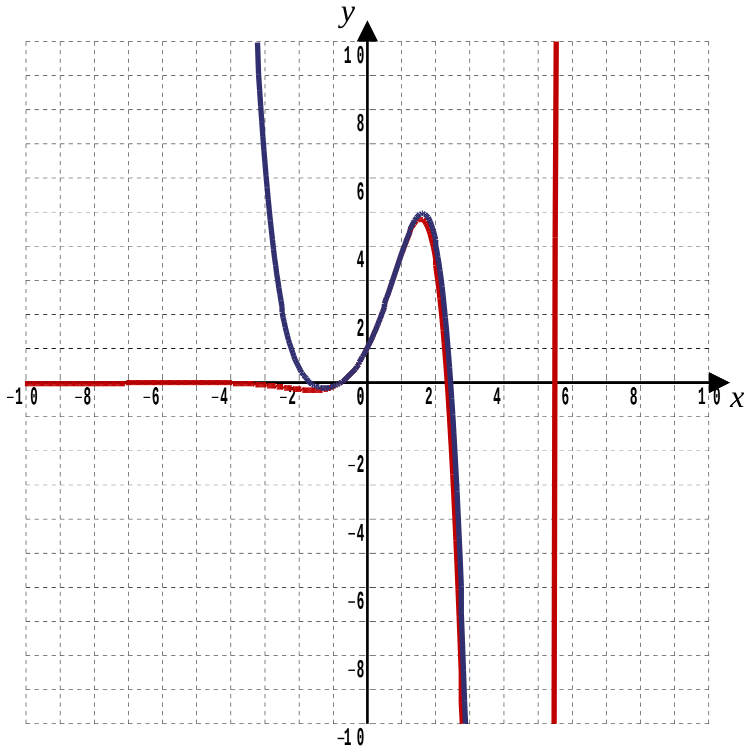
$$= -8\,h(x) \qquad\qquad \Rightarrow\ f^{(5)}(0) = -8$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

$$f(x) \approx 1 + 2x + x^2 - \frac{1}{6}x^4 - \frac{1}{15}x^5$$

[8 marks]

Answer 4 : Graph



Answer 4 : Checking Coefficients using Wolfram Alpha

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,0}]

$$1$$

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,1}]

$$2$$

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,2}]

$$1$$

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,3}]

$$0$$

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,4}]

$$-\frac{1}{6}$$

SeriesCoefficient[$exp(x)(sin(x) + cos(x))$, {x,0,5}]

$$-\frac{1}{15}$$

Answer 5

(i) Mr Hansen's Method

$$f\,(x) \;=\; ln\,(\,1 \;+\; cos\,2x\,)$$

$$f'\,(x) \;=\; \frac{1}{1 \;+\; cos\,2x} \;\times\; (-\,sin\,2x\,) \;\times\; 2$$

$$=\; -\,2 \;\times\; \frac{sin\,2x}{1 \;+\; cos\,2x}$$

$$=\; -\,2 \;\times\; \frac{2\,sin\,x\,cos\,x}{cos^2\,x \;+\; sin^2\,x \;+\; cos^2\,x \;-\; sin^2\,x}$$

$$=\; -\,2 \;\times\; \frac{2\,sin\,x\,cos\,x}{2\,cos^2\,x}$$

$$=\; -\,2 \;\times\; \frac{sin\,x}{cos\,x}$$

$$=\; -\,2\,tan\,x \qquad \square$$

Sheldon's Method

$$f\,(x) \;=\; ln\,(\,1 \;+\; cos\,2x\,)$$

$$=\; ln\left(\,cos^2\,x \;+\; sin^2\,x \;+\; cos^2\,x \;-\; sin^2\,x\,\right)$$

$$=\; ln\left(\,2\,cos^2\,x\,\right)$$

$$=\; ln\,2 \;+\; ln\left(\,cos^2\,x\,\right)$$

$$=\; ln\,2 \;+\; 2\,ln\,(\,cos\,x\,)$$

$$f'\,(x) \;=\; 0 \;+\; \frac{2}{cos\,x} \;\times\; (-\,sin\,x\,)$$

$$=\; -\,2\,tan\,x \qquad \square$$

[2 marks]

(ii)

$$f''\,(x) \;=\; -\,2\,sec^2\,x$$

$$f'''\,(x) \;=\; -\,2 \;\times\; 2\,sec\,x\,sec\,x\,tan\,x$$

$$=\; -\,1 \;\times\; \left(-\,2\,sec^2\,x\,\right)\left(-\,2\,tan\,x\,\right)$$

$$=\; -\,1\,f''\,(x)\;f'\,(x)$$

$$f''''\,(x) \;=\; -\,f''\,(x)\;f''\,(x) \;-\; f'''\,(x)\,f'\,(x)$$

$$=\; -\left(\,f'''\,(x)\;f'\,(x) \;+\; \left(\,f''\,(x)^2\right)\,\right)$$

[5 marks]

(iii)

$$f(0) = \ln 2$$

$$f'(0) = 0$$

$$f''(0) = -2$$

$$f'''(0) = 0$$

$$f''''(0) = -4$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \ldots + \frac{f^{(r)}(0)}{r!}\,x^r + \ldots$$

$$f(x) \approx \ln 2 - x^2 - \frac{1}{6}\,x^4$$

[8 marks]

Lesson 6

6.1 Revision

6.2 Useful Information

6.2.1 A Table of Functions and their Derivatives

$f(x)$	$f'(x)$	In Formula Book ?
x^n	$n\,x^{n-1}$	No
e^x	e^x	No
$\ln x$	$\frac{1}{x}$	No
$\sin x$	$\cos x$	No
$\cos x$	$-\sin x$	No
$\tan x$	$\sec^2 x$	Yes
$\csc x$	$-\csc x \cot x$	Yes
$\sec x$	$\sec x \tan x$	Yes
$\cot x$	$-\csc^2 x$	Yes
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	Yes
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	Yes
$\arctan x$	$\frac{1}{1+x^2}$	Yes
$\text{arccot}\, x$	$-\frac{1}{1+x^2}$	No
$\text{arcsec}\, x$	$\frac{1}{x\sqrt{x^2-1}}$	No
$\text{arccsc}\, x$	$-\frac{1}{x\sqrt{x^2-1}}$	No

6.2.2 The Trigonometric Addition Formula

The Addition Formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

6.2.3 Standard Maclaurin Series

Quotable Maclaurin Series Expansions

- $e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^r}{r!} + ...$ valid for all x
- $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{r+1}\frac{x^r}{r} + ...$ $-1 < x \leqslant 1$
- $sin\, x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + ...$ valid for all x
- $cos\, x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + (-1)^r \frac{x^{2r}}{(2r)!} + ...$ valid for all x
- $arctan\, x = x - \frac{x^3}{3} + \frac{x^5}{5} - .... + (-1)^r \frac{x^{2r+1}}{2r+1} + ...$ $-1 < x \leqslant 1$

6.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 38

Question 1

Use the addition formula for $sin(A + B)$ and the series expansions of $sin\, x$ and $cos\, x$ to show that,

$$sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + ...\right)$$

[4 marks]

Question 2

Given that $f(x) = \ln \sec x$

(a) show that $f'(x) = \tan x$

[2 marks]

(b) find the values of $f'(0)$, $f''(0)$, $f'''(0)$ and $f''''(0)$

[2 marks]

(c) express $\ln \sec x$ as a series in ascending powers of x up to and and including the term in x^4

[2 marks]

(d) show that using the first two non-zero terms of the Maclaurin series for $\ln \sec x$, with $x = \frac{\pi}{4}$, gives a value for $\ln 2$ of $\frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$

[2 marks]

Question 3

Further A-Level Examination Question from June 2007, FP2, Q1(c) (MEI) edited

$$f(x) = arccos(2x)$$

(i) Write down $f'(x)$

[2 marks]

(ii) A special case of the binomial theorem is that,

$$(1 + x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \ldots \qquad -1 < x \leqslant 1$$

Show how to use this to expand $f'(x)$, and hence find the Maclaurin series for $f(x)$ in ascending powers of x up to and including the term in x^5

[4 marks]

Question 4

Obtain the first three terms of the Maclaurin series for $f(x) = arccot(1 + x)$

[5 marks]

Question 5

Further A-Level Examination Question from January 2009, FP2, Q1(a)(ii) (MEI)

The Maclaurin expansion of *sec x* begins

$$1 + a x^2 + b x^4$$

where a and b are constants.

Explain why, for sufficiently small x,

$$\left(1 - \frac{1}{2} x^2 + \frac{1}{24} x^4\right)\left(1 + a x^2 + b x^4\right) \approx 1$$

and hence find the values of a and b

[5 marks]

Question 6

Given that $f(x) = ln\left(x + \sqrt{1 + x^2}\right)$

(i) Show that $\sqrt{1 + x^2}\, f'(x) = 1$

[2 marks]

(ii) Show that $\left(1 + x^2\right) f''(x) + x f'(x) = 0$

[2 marks]

(iii) Show that $\left(1 + x^2\right) f'''(x) + 3x f''(x) + f'(x) = 0$

[2 mark]

(iv) State the values of $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$

[2 marks]

(v) Give the Maclaurin series for $f(x)$ in ascending powers of x up to and including the term in x^3

[2 marks]

Answer 1

$$\begin{aligned} \sin\left(x + \frac{\pi}{4}\right) &= \sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}\,(\sin x + \cos x) \\ &= \frac{\sqrt{2}}{2}\left(\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)\right) \\ &= \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right) \\ &= \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right) \quad \square \end{aligned}$$

[4 marks]

Answer 2

(a)

$$f(x) = \ln \sec x \quad \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{\sec x} \times \sec x \tan x$$

$$= \tan x \quad \square \quad \Rightarrow f'(0) = 0$$

(b)

$$f''(x) = \sec^2 x$$

$$= 1 + \tan^2 x \quad \Rightarrow f''(0) = 1$$

$$f'''(x) = 2 \tan x \sec^2 x \quad \Rightarrow f'''(0) = 0$$

$$= 2 \tan x \left(1 + \tan^2 x\right)$$

$$= 2 \tan x + 2 \tan^3 x$$

$$f''''(x) = 2 \sec^2 x + 6 \tan^2 x \sec^2 x$$

$$= 2\left(1 + \tan^2 x\right) + 6 \tan^2 x \left(1 + \tan^2 x\right)$$

$$= 2 + 8 \tan^2 x + 6 \tan^4 x \quad \Rightarrow f''''(x) = 2$$

(c)

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$$

$$f(x) \approx \frac{x^2}{2} + \frac{x^4}{12}$$

(d) With $x = \frac{\pi}{4}$, $\ln \sec\left(\frac{\pi}{4}\right) \approx \frac{\pi^2}{32} + \frac{\pi^4}{3072}$

$$\ln \sqrt{2} \approx \frac{\pi^2}{32}\left(1 + \frac{\pi^2}{96}\right) \Rightarrow \ln 2 \approx \frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$$

[2, 2, 2, 2 marks]

Answer 3

(i) From the standard result that $\dfrac{d}{dx}(arccos\, x) = -\dfrac{1}{\sqrt{1 - x^2}}$

$$f(x) = arccos\,(2x)$$

$$f'(x) = -\frac{1}{\sqrt{1 - (2x)^2}} \times 2$$

$$= -\frac{2}{\sqrt{1 - 4x^2}}$$

[2 marks]

(ii)

$$-\frac{2}{\sqrt{1 - 4x^2}} = (-2)\left(1 + \left(-4x^2\right)\right)^{-\frac{1}{2}}$$

$$= (-2)\left(1 - \frac{1}{2}\left(-4x^2\right) + \frac{3}{8}\left(-4x^2\right)^2 - \ldots\right)$$

$$= (-2)\left(1 + 2x^2 + 6x^4 + \ldots\right)$$

$$f'(x) = -2 - 4x^2 - 12x^4 - \ldots$$

Integrate both sides...

$$f(x) + c = -2x - \frac{4}{3}x^3 - \frac{12}{5}x^5 - \ldots$$

$$\text{As } f(0) = arccos\,(0)$$

$$= \frac{\pi}{2} \Rightarrow c = -\frac{\pi}{2}$$

$$\therefore\ f(x) = \frac{\pi}{2} - 2x - \frac{4}{3}x^2 - \frac{12}{5}x^5 - \ldots$$

[4 marks]

Answer 4

$$f(x) = arccot(1+x)$$

when $x = 0,\ arccot(1) = y \Rightarrow cot\, y = 1 \Rightarrow tan\, y = 1 \Rightarrow y = \frac{\pi}{4}$

$$\Rightarrow f(0) = \frac{\pi}{4}$$

$$\begin{aligned} f'(x) &= -\frac{1}{1+(1+x)^2} \\ &= -\frac{1}{1+1+2x+x^2} \\ &= -\frac{1}{2+2x+x^2} \qquad \Rightarrow f'(0) = -\frac{1}{2} \\ &= -\left(2+2x+x^2\right)^{-1} \end{aligned}$$

$$\begin{aligned} f''(x) &= \left(2+2x+x^2\right)^{-2}(2+2x) \\ &= \frac{2(1+x)}{(2+2x+x^2)^2} \qquad \Rightarrow f''(0) = \frac{2}{4} \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \frac{1}{2} \end{aligned}$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$$

$$f(x) \approx \frac{\pi}{4} - \frac{x}{2} + \frac{x^2}{4}$$

[5 marks]

Answer 5

$$cos\,x \times \frac{1}{cos\,x} = 1$$

$$cos\,x \times sec\,x = 1$$

$$\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)\left(1 + a\,x^2 + b\,x^4\right) \approx 1$$

$$1 + a\,x^2 + b\,x^4 - \frac{1}{2}x^2 - \frac{1}{2}a\,x^4 + \frac{1}{24}x^4 + O\left(x^6\right) \approx 1$$

For coefficients of x^2,

$$a - \frac{1}{2} = 0$$

$$a = \frac{1}{2}$$

For coefficients of x^4,

$$b - \frac{1}{2}a + \frac{1}{24} = 0$$

$$b = \frac{1}{2} \times \frac{1}{2} - \frac{1}{24}$$

$$= \frac{5}{24}$$

Thus $sec\,x \approx 1 + \frac{1}{2}x + \frac{5}{24}x^2$

[5 marks]

Answer 6

(i) $$f(x) = ln\left(x + \sqrt{1 + x^2} \right)$$

$$f'(x) = \frac{1}{x + \sqrt{1 + x^2}} \times \left(1 + \frac{1}{2} \left(1 + x^2 \right)^{-\frac{1}{2}} \times 2x \right)$$

$$= \frac{1}{x + \sqrt{1 + x^2}} \times \left(1 + \frac{x}{\sqrt{1 + x^2}} \right)$$

$$= \frac{1}{x + \sqrt{1 + x^2}} \times \left(\frac{\sqrt{1 + x^2}}{\sqrt{1 + x^2}} + \frac{x}{\sqrt{1 + x^2}} \right)$$

$$= \frac{1}{x + \sqrt{1 + x^2}} \times \frac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}}$$

$$= \frac{1}{\sqrt{1 + x^2}} \qquad \therefore \ \sqrt{1 + x^2}\, f'(x) = 1 \qquad \square$$

[2 marks]

(ii) $$\left(1 + x^2 \right)^{\frac{1}{2}} f'(x) = 1$$

$$\left(1 + x^2 \right)^{\frac{1}{2}} f''(x) + \frac{1}{2} \left(1 + x^2 \right)^{-\frac{1}{2}} \times 2x\, f'(x) = 0$$

$$\sqrt{1 + x^2}\, f''(x) + \frac{x}{\sqrt{1 + x^2}} f'(x) = 0$$

multiply through by $\sqrt{1 + x^2}$

$$\left(1 + x^2 \right) f''(x) + x\, f'(x) = 0 \qquad \square$$

[2 marks]

(iii)

$$\left(1 + x^2 \right) f'''(x) + 2x\, f''(x) + x\, f''(x) + f'(x) = 0$$

$$\left(1 + x^2 \right) f'''(x) + 3x\, f''(x) + f'(x) = 0$$

[2 marks]

(iv)

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1$$

[2 marks]

(v)

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!} x^2 + ... + \frac{f^{(r)}(0)}{r!} x^r + ...$$

$$f(x) \approx x - \frac{x^3}{6}$$

[2 marks]

7.1 Test

7.2 Useful Information

7.2.1 A Table of Functions and their Derivatives

$f(x)$	$f'(x)$	In Formula Book ?
x^n	$n\,x^{n-1}$	No
e^x	e^x	No
$\ln x$	$\frac{1}{x}$	No
$\sin x$	$\cos x$	No
$\cos x$	$-\sin x$	No
$\tan x$	$\sec^2 x$	Yes
$\csc x$	$-\csc x \cot x$	Yes
$\sec x$	$\sec x \tan x$	Yes
$\cot x$	$-\csc^2 x$	Yes
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	Yes
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	Yes
$\arctan x$	$\frac{1}{1+x^2}$	Yes
$\text{arccot}\, x$	$-\frac{1}{1+x^2}$	No
$\text{arcsec}\, x$	$\frac{1}{x\sqrt{x^2-1}}$	No
$\text{arccsc}\, x$	$-\frac{1}{x\sqrt{x^2-1}}$	No

7.2.2 The Trigonometric Addition Formula

The Addition Formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

7.2.3 Standard Maclaurin Series

Quotable Maclaurin Series Expansions

- $e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^r}{r!} + ...$ valid for all x
- $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{r+1}\frac{x^r}{r} + ...$ $-1 < x \leqslant 1$
- $sin\,x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^r \frac{x^{2r+1}}{(2r + 1)!} + ...$ valid for all x
- $cos\,x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + (-1)^r \frac{x^{2r}}{(2r)!} + ...$ valid for all x
- $arctan\,x = x - \frac{x^3}{3} + \frac{x^5}{5} - .... + (-1)^r \frac{x^{2r+1}}{2r + 1} + ...$ $-1 < x \leqslant 1$

7.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 35

Question 1

Use the addition formula for $cos(A - B)$ and the series expansions of $sin\,x$ and $cos\,x$ to show that,

$$cos\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + ...\right)$$

[3 marks]

Question 2

Given that $f(x) = ln(\, sec\,x + tan\,x\,)$

(a) show that $f'(x) = sec\,x$

[2 marks]

(b) find the values of $f'(0)$, $f''(0)$, and $f'''(0)$

[2 marks]

(c) express $ln\,(\, sec\,x + tan\,x\,)$ as a series in ascending powers of x up to and including the term in x^3

[2 marks]

(d) show that using the first two non-zero terms of the Maclaurin series for $ln\,(sec\,x + tan\,x)$, with $x = \frac{\pi}{4}$, gives $ln(\,\sqrt{2} + 1\,) = \frac{\pi}{4}\left(1 + \frac{\pi^2}{96}\right)$

[2 marks]

Question 3

$$f(x) = arcsin(4x)$$

(i) Write down $f'(x)$

[2 marks]

(ii) A special case of the binomial theorem is that,

$$(1 + x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \ldots \qquad -1 < x \leqslant 1$$

Show how to use this to expand $f'(x)$, and hence find the Maclaurin series for $f(x)$ in ascending powers of x up to and including the term in x^5

[4 marks]

Question 4

Further A-Level Examination Question from June 2017, FP2, Q1 (b) (MEI)

Obtain the first three terms of the Maclaurin series for $f(x) = \arctan(1 + x)$

[5 marks]

Question 5

The Laurent series expansion of $\csc x$ begins

$$\frac{1}{x} + ax + bx^3$$

where a and b are constants.

Explain why, for sufficiently small x,

$$\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\left(\frac{1}{x} + ax + bx^3\right) \approx 1$$

and hence find the values of a and b

[5 marks]

Question 6

Given that $f(x) = e^{2x} \sin x$

(a) Show that $f''(x) = p\,f(x) + q\,f'(x)$
where p and q are integers to be determined

[5 marks]

(b) Hence find the Maclaurin series for $f(x)$, in ascending powers of x, up to and including the term in x^5, giving each coefficient in its simplest form.

[3 marks]

Answer 1

$$\cos\left(x - \frac{\pi}{4}\right) = \cos x \cos\left(\frac{\pi}{4}\right) + \sin x \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2}\,(\cos x + \sin x)$$

$$= \frac{\sqrt{2}}{2}\left(\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\right)$$

$$= \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right)$$

$$= \frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right) \qquad \square$$

[3 marks]

Answer 2

(a)

$$f(x) = \ln(\sec x + \tan x) \qquad \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{\sec x + \tan x} \times \left(\sec x \tan x + \sec^2 x\right)$$

$$= \frac{1}{\sec x + \tan x} \times \sec x\,(\tan x + \sec x)$$

$$= \sec x \qquad \square \qquad \Rightarrow f'(0) = 1$$

(b)

$$f''(x) = \sec x \tan x \qquad \Rightarrow f''(0) = 0$$

$$f'''(x) = \sec x \sec^2 x + \sec x \tan x \tan x$$

$$= \sec^3 x + \sec x \tan^2 x$$

$$= \sec^3 x + \sec x\left(\sec^2 x - 1\right)$$

$$= 2\sec^3 x - \sec x \qquad \Rightarrow f'''(0) = 1$$

(c)

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$$

$$f(x) \approx x + \frac{x^3}{6}$$

(d) With $x = \frac{\pi}{4}$,

$$\ln\left(\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right) \approx \frac{\pi}{4} + \frac{\pi^3}{384}$$

$$\ln\left(\sqrt{2} + 1\right) \approx \frac{\pi}{4}\left(1 + \frac{\pi^2}{96}\right)$$

[2, 2, 2, 2 marks]

Answer 3

(i) From the standard result that $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$

$$f(x) = \arcsin(4x)$$

$$f'(x) = \frac{1}{\sqrt{1 - (4x)^2}} \times 4$$

$$= \frac{4}{\sqrt{1 - 16x^2}}$$

[2 marks]

(ii)

$$\frac{4}{\sqrt{1 - 16x^2}} = 4\left(1 + \left(-16x^2\right)\right)^{-\frac{1}{2}}$$

$$= 4\left(1 - \frac{1}{2}\left(-16x^2\right) + \frac{3}{8}\left(-16x^2\right)^2 - \ldots\right)$$

$$= 4\left(1 + 8x^2 + 96x^4 + \ldots\right)$$

$$f'(x) = 4 + 32x^2 + 384x^4 - \ldots$$

Integrate both sides...

$$f(x) + c = 4x + \frac{32}{3}x^3 + \frac{384}{5}x^5 + \ldots$$

$$\text{As } f(0) = \arcsin(0)$$

$$= 0 \Rightarrow c = 0$$

$$\therefore f(x) = 4x + \frac{32}{3}x^3 + \frac{384}{5}x^5 + \ldots$$

[4 marks]

Answer 4

$$f(x) = arctan(1 + x)$$

$$\text{when } x = 0,\ f(0) = arctan(1) \Rightarrow f(0) = \frac{\pi}{4}$$

$$f'(x) = \frac{1}{1 + (1 + x)^2}$$
$$= \frac{1}{1 + 1 + 2x + x^2}$$
$$= \frac{1}{2 + 2x + x^2} \qquad \Rightarrow f'(0) = \frac{1}{2}$$
$$= \left(2 + 2x + x^2\right)^{-1}$$

$$f''(x) = -\left(2 + 2x + x^2\right)^{-2}(2 + 2x)$$
$$= -\frac{2(1 + x)}{(2 + 2x + x^2)^2} \qquad \Rightarrow f''(0) = -\frac{2}{4}$$
$$= -\frac{1}{2}$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$$

$$f(x) \approx \frac{\pi}{4} + \frac{x}{2} - \frac{x^2}{4}$$

[5 marks]

Answer 5

$$\sin x \times \frac{1}{\sin x} = 1$$

$$\sin x \times \csc x = 1$$

$$\left(x - \frac{1}{6} x^3 + \frac{1}{120} x^5 \right)\left(\frac{1}{x} + a x + b x^3 \right) \approx 1$$

$$1 + a x^2 + b x^4 - \frac{1}{6} x^2 - \frac{1}{6} a x^4 + \frac{1}{120} x^4 + O\left(x^6 \right) \approx 1$$

For coefficients of x^2,

$$a - \frac{1}{6} = 0$$

$$a = \frac{1}{6}$$

For coefficients of x^4,

$$b - \frac{1}{6} a + \frac{1}{120} = 0$$

$$b = \frac{1}{6} \times \frac{1}{6} - \frac{1}{120}$$

$$= \frac{7}{360}$$

Thus $\sec x \approx \frac{1}{x} + \frac{1}{6} x + \frac{7}{360} x^2$

[5 marks]

Answer 6

(a)

$$f(x) = e^{2x} \sin x \qquad \Rightarrow f(0) = 0$$

$$f'(x) = e^{2x} \cos x + 2e^{2x} \sin x$$

$$= e^{2x} \cos x + 2f(x) \qquad \Rightarrow f'(0) = 1$$

$$f''(x) = e^{2x}(-\sin x) + 2e^{2x} \cos x + 2f'(x)$$

$$= -f(x) + 2e^{2x} \cos x + 2f'(x)$$

From, $f'(x) = e^{2x} \cos x + 2f(x)$

$$e^{2x} \cos x = f'(x) - 2f(x)$$

$$\therefore \ f''(x) = -f(x) + 2\left(f'(x) - 2f(x)\right) + 2f'(x)$$

$$= -f(x) + 2f'(x) - 4f(x) + 2f'(x)$$

$$= -5f(x) + 4f'(x) \qquad \Rightarrow f''(0) = 4$$

That is, $p = -5$ and $q = 4$

[5 marks]

(b)

$$f'''(x) = -5f'(x) + 4f''(x)$$

$$= -5f'(x) + 4\left(-5f(x) + 4f'(x)\right)$$

$$= -20f(x) + 11f'(x) \qquad \Rightarrow f'''(0) = 11$$

$$f^{(4)}(x) = -20f'(x) + 11f''(x)$$

$$= -20f'(x) + 11\left(-5f(x) + 4f'(x)\right)$$

$$= -55f(x) + 24f'(x) \qquad \Rightarrow f^{(4)}(0) = 24$$

$$f^{(5)}(x) = -55f'(x) + 24f''(x)$$

$$= -55f'(x) + 24\left(-5f(x) + 4f'(x)\right)$$

$$= -120f(x) + 41f'(x) \qquad \Rightarrow f^{(5)}(0) = 41$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

$$f(x) \approx x + 2x^2 + \frac{11}{6}x^3 + x^4 + \frac{41}{120}x^5$$

[3 marks]