Answer 1

(i)

$$\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right) = x^2+1-x^2 \quad \text{difference of two squares}$$
$$= 1$$

[1 mark]

(ii) Odd

[1 mark]

(iii)

$$f(x) = ln\left(x+\sqrt{x^2+1}\right)$$
$$\therefore \quad f(-x) = ln\left((-x)+\sqrt{(-x)^2+1}\right)$$
$$= ln\left((-x)+\sqrt{x^2+1}\right)$$
$$= ln\left(\sqrt{x^2+1}-x\right)$$
$$= ln\left(\frac{\sqrt{x^2+1}-x}{1}\times\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)$$
$$= ln\left(\frac{1}{\sqrt{x^2+1}+x}\right) \quad \text{using the part (i) result}$$
$$= ln\left(\sqrt{x^2+1}+x\right)^{-1}$$
$$= -ln\left(x+\sqrt{x^2+1}\right)$$
$$= -f(x)$$

which is the definition of an odd function

$\therefore$ $f(x)$ is an odd function ☐

[4 marks]

Answer 2

METHOD 1

$$\frac{\cos 5\theta + \mathrm{i}\sin 5\theta}{\cos 2\theta + \mathrm{i}\sin 2\theta} = \frac{e^{\mathrm{i}5\theta}}{e^{\mathrm{i}2\theta}}$$

$$= e^{\mathrm{i}3\theta}$$

$$= \cos 3\theta + \mathrm{i}\sin 3\theta$$

METHOD 2

$$\frac{\cos 5\theta + \mathrm{i}\sin 5\theta}{\cos 2\theta + \mathrm{i}\sin 2\theta}$$

$$= \frac{\cos 5\theta + \mathrm{i}\sin 5\theta}{\cos 2\theta + \mathrm{i}\sin 2\theta} \times \frac{\cos 2\theta - \mathrm{i}\sin 2\theta}{\cos 2\theta - \mathrm{i}\sin 2\theta}$$

$$= \frac{\cos 5\theta\cos 2\theta + \sin 5\theta\sin 2\theta + \mathrm{i}(\sin 5\theta\cos 2\theta - \cos 5\theta\sin 2\theta)}{\cos^2 2\theta - \mathrm{i}\cos 2\theta\sin 2\theta + \mathrm{i}\sin 2\theta\cos 2\theta + \sin^2 2\theta}$$

$$= \frac{\cos(5\theta - 2\theta) + \mathrm{i}\sin(5\theta - 2\theta)}{1}$$

$$= \cos 3\theta + \mathrm{i}\sin 3\theta$$

[4 marks]

Answer 3

$$\text{LHS} = \frac{e^{\mathrm{i}\theta} - 1}{e^{\mathrm{i}\theta} + 1}$$

$$= \frac{e^{\mathrm{i}\theta} - 1}{e^{\mathrm{i}\theta} + 1} \times \frac{e^{-\mathrm{i}\frac{\theta}{2}}}{e^{-\mathrm{i}\frac{\theta}{2}}}$$

$$= \frac{e^{\mathrm{i}\frac{\theta}{2}} - e^{-\mathrm{i}\frac{\theta}{2}}}{e^{\mathrm{i}\frac{\theta}{2}} + e^{-\mathrm{i}\frac{\theta}{2}}}$$

$$= \frac{2\mathrm{i}\sin\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)}$$

$$= \mathrm{i}\tan\left(\frac{\theta}{2}\right) = \text{RHS} \qquad \square$$

[5 marks]

Answer 4

(i) $(cos\,\theta + \mathrm{i}\,sin\,\theta)^4 = 1 \times (cos\,\theta)^4 \times (\mathrm{i}\,sin\,\theta)^0$ Binomial theorem

$$+ 4 \times (cos\,\theta)^3 \times (\mathrm{i}\,sin\,\theta)^1$$

$$+ 6 \times (cos\,\theta)^2 \times (\mathrm{i}\,sin\,\theta)^2$$

$$+ 4 \times (cos\,\theta)^1 \times (\mathrm{i}\,sin\,\theta)^3$$

$$+ 1 \times (cos\,\theta)^0 \times (\mathrm{i}\,sin\,\theta)^4$$

$$cos(4\theta) + \mathrm{i}\,sin(4\theta) = cos^4\theta + \mathrm{i}\,4\,cos^3\theta\,sin\,\theta - 6\,cos^2\theta\,sin^2\theta$$

$$- \mathrm{i}\,4\,cos\,\theta\,sin^3\theta + sin^4\theta$$

Matching real parts : $cos(4\theta) = cos^4\theta - 6\,cos^2\theta\,sin^2\theta + sin^4\theta$

Matching imaginary parts : $sin(4\theta) = 4\,cos^3\theta\,sin\,\theta - 4\,cos\,\theta\,sin^3\theta$

$$\text{LHS} = cot\,4\theta$$

$$= \frac{cos\,4\theta}{sin\,4\theta}$$

$$= \frac{cos^4\theta - 6\,cos^2\theta\,sin^2\theta + sin^4\theta}{4\,cos^3\theta\,sin\,\theta - 4\,cos\,\theta\,sin^3\theta} \times \frac{\frac{1}{cos^4\theta}}{\frac{1}{cos^4\theta}}$$

$$= \frac{1 - 6\,tan^2\theta + tan^4\theta}{4\,tan\,\theta - 4\,tan^3\theta}$$

$$= \frac{1 - 6\,tan^2\theta + tan^4\theta}{4\,tan\,\theta\,(1 - tan^2\theta)} = \text{RHS} \qquad \square$$

[5 marks]

(ii)

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

Let $x = tan\,\theta$ in which case,

$$tan^4\theta + 4\,tan^3\theta - 6\,tan^2\theta - 4\,tan\,\theta + 1 = 0$$

$$tan^4\theta - 6\,tan^2\theta + 1 = 4\,tan\,\theta - 4\,tan^3\theta$$

$$\frac{1 - 6\,tan^2\theta + tan^4\theta}{4\,tan\,\theta - 4\,tan^3\theta} = 1$$

$$cot\,4\theta = 1$$

$$tan\,4\theta = 1$$

$$4\theta = \frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{9\pi}{4},\ \frac{13\pi}{4},\ \ldots$$

$$\theta = \frac{\pi}{16},\ \frac{5\pi}{16},\ \frac{9\pi}{16},\ \frac{13\pi}{16},\ \ldots$$

$$x = tan\left(\frac{\pi}{16}\right),\ tan\left(\frac{5\pi}{16}\right),\ tan\left(\frac{9\pi}{16}\right),\ tan\left(\frac{13\pi}{16}\right),\ \ldots$$

Further solutions will simply start to duplicate these four

[4 marks]

Answer 5

$$C - \mathrm{i}S = (\cos\theta + \mathrm{i}\sin\theta)$$
$$+ \frac{1}{3}(\cos 3\theta + \mathrm{i}\sin 3\theta)$$
$$+ \frac{1}{9}(\cos 5\theta + \mathrm{i}\sin 5\theta)$$
$$+ \ldots$$
$$= e^{\mathrm{i}\theta} + \frac{1}{3}e^{\mathrm{i}3\theta} + \frac{1}{9}e^{\mathrm{i}5\theta} + \ldots$$
$$= \frac{e^{\mathrm{i}\theta}}{1 - \frac{1}{3}e^{\mathrm{i}2\theta}} \times \frac{3}{3}$$
$$= \frac{3e^{\mathrm{i}\theta}}{3 - e^{\mathrm{i}2\theta}}$$
$$= \frac{3\cos\theta + 3\mathrm{i}\sin\theta}{3 - \cos 2\theta - \mathrm{i}\sin 2\theta} \times \frac{3 - \cos 2\theta + \mathrm{i}\sin 2\theta}{3 - \cos 2\theta + \mathrm{i}\sin 2\theta}$$
$$= \frac{9\cos\theta - 3\cos\theta\cos 2\theta + 3\mathrm{i}\cos\theta\sin 2\theta + 9\mathrm{i}\sin\theta - 3\mathrm{i}\sin\theta\cos 2\theta - 3\sin\theta\sin 2\theta}{9 - 3\cos 2\theta - 3\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}$$
$$= \frac{9\cos\theta - 3(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta) + 9\mathrm{i}\sin\theta + 3\mathrm{i}(\sin 2\theta\cos\theta - \cos 2\theta\sin\theta)}{10 - 6\cos 2\theta}$$
$$= \frac{9\cos\theta - 3\cos\theta + 9\mathrm{i}\sin\theta + 3\mathrm{i}\sin\theta}{10 - 6\cos 2\theta}$$
$$= \frac{6\cos\theta + 12\mathrm{i}\sin\theta}{10 - 6\cos 2\theta}$$
$$= \frac{3\cos\theta + 6\mathrm{i}\sin\theta}{5 - 3\cos 2\theta}$$
$$\therefore C = \frac{3\cos\theta}{5 - 3\cos 2\theta} \qquad \square$$
$$\text{and } S = \frac{6\sin\theta}{5 - 3\cos 2\theta}$$

[11 marks]

Answer 6

$$\begin{aligned}
\text{LHS} &= 1 + z + z^2 + \ldots + z^n \\
&= 1 + e^{\frac{\pi}{n}\mathrm{i}} + \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^2 + \ldots + \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^n \\
&= \frac{1\left(1 - \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^{n+1}\right)}{1 - e^{\frac{\pi}{n}\mathrm{i}}} \\
&= \frac{1 - e^{\frac{\pi(n+1)}{n}\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}} \\
&= \frac{1 - e^{\frac{\pi n}{n}\mathrm{i}}\, e^{\frac{\pi}{n}\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}} \\
&= \frac{1 - e^{\pi\mathrm{i}}\, e^{\frac{\pi}{n}\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}} \\
&= \frac{1 + e^{\frac{\pi}{n}\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}} \times \frac{e^{-\frac{\pi}{2n}\mathrm{i}}}{e^{-\frac{\pi}{2n}\mathrm{i}}} \\
&= \frac{e^{-\frac{\pi}{2n}\mathrm{i}} + e^{\frac{\pi}{2n}\mathrm{i}}}{e^{-\frac{\pi}{2n}\mathrm{i}} - e^{\frac{\pi}{2n}\mathrm{i}}} \\
&= -\frac{e^{\frac{\pi}{2n}\mathrm{i}} + e^{-\frac{\pi}{2n}\mathrm{i}}}{e^{\frac{\pi}{2n}\mathrm{i}} - e^{-\frac{\pi}{2n}\mathrm{i}}} \\
&= -\frac{2\cos\left(\frac{\pi}{2n}\right)}{2\mathrm{i}\sin\left(\frac{\pi}{2n}\right)} \times \frac{\mathrm{i}}{\mathrm{i}} \\
&= \mathrm{i}\cot\left(\frac{\pi}{2n}\right) \\
&= \text{RHS}
\end{aligned}$$

This makes use of the following two results from Lesson 9 : Compare with Q3

Two Trigonometric Functions, as Exponentials

- $\cos(n\theta) = \frac{1}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}\right)$
- $\sin(n\theta) = \frac{1}{2\mathrm{i}}\left(e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta}\right)$

[5 marks]