11.2 Answers

Further A-Level Pure Mathematics, Core 2
Complex Numbers II

Answer 1

(i) $$g(x) = \frac{f(x) + f(-x)}{2} \Rightarrow g(-x) = \frac{f(-x) + f(x)}{2}$$

$$= \frac{f(x) + f(-x)}{2}$$

$$= g(x) \qquad \therefore g(x) \text{ is an even function}$$

[2 marks]

(ii) $$h(x) = \frac{f(x) - f(-x)}{2} \Rightarrow h(-x) = \frac{f(-x) - f(x)}{2}$$

$$= -\frac{f(x) - f(-x)}{2}$$

$$= -h(x) \qquad \therefore h(x) \text{ is an odd function}$$

[2 marks]

(iii)
$$\begin{aligned} \text{LHS} &= g(x) + h(x) \\ &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\ &= \frac{f(x) + f(-x) + f(x) - f(-x)}{2} \\ &= \frac{2f(x)}{2} \\ &= f(x) \\ &= \text{RHS} \qquad \square \end{aligned}$$

[1 mark]

(iv) Every function, $f(x)$, can be written as the sum of an even function and an odd function

[1 mark]

Answers 2

METHOD 1

$$\frac{\cos 2\theta + \mathrm{i}\sin 2\theta}{\cos 9\theta - \mathrm{i}\sin 9\theta} = \frac{\cos 2\theta + \mathrm{i}\sin 2\theta}{\cos(-9\theta) + \mathrm{i}\sin(-9\theta)}$$

(As *cos* is an even function, and *sin* is an odd function)

$$= \frac{e^{\mathrm{i}2\theta}}{e^{-\mathrm{i}9\theta}}$$

$$= e^{\mathrm{i}11\theta}$$

$$= \cos 11\theta + \mathrm{i}\sin 11\theta$$

METHOD 2

$$\frac{\cos 2\theta + \mathrm{i}\sin 2\theta}{\cos 9\theta - \mathrm{i}\sin 9\theta}$$

$$= \frac{\cos 2\theta + \mathrm{i}\sin 2\theta}{\cos 9\theta - \mathrm{i}\sin 9\theta} \times \frac{\cos 9\theta + \mathrm{i}\sin 9\theta}{\cos 9\theta + \mathrm{i}\sin 9\theta}$$

$$= \frac{\cos 9\theta\cos 2\theta - \sin 9\theta\sin 2\theta + \mathrm{i}(\sin 9\theta\cos 2\theta + \cos 9\theta\sin 2\theta)}{\cos^2 9\theta - \mathrm{i}\cos 9\theta\sin 9\theta + \mathrm{i}\sin 9\theta\cos 9\theta + \sin^2 9\theta}$$

$$= \frac{\cos(9\theta + 2\theta) + \mathrm{i}\sin(9\theta + 2\theta)}{1}$$

$$= \cos 11\theta + \mathrm{i}\sin 11\theta$$

[4 marks]

Answer 3

$$\text{LHS} = \frac{1 + e^{\mathrm{i}\theta}}{1 - e^{\mathrm{i}\theta}}$$

$$= \frac{1 + e^{\mathrm{i}\theta}}{1 - e^{\mathrm{i}\theta}} \times \frac{e^{-\mathrm{i}\frac{\theta}{2}}}{e^{-\mathrm{i}\frac{\theta}{2}}}$$

$$= \frac{e^{-\mathrm{i}\frac{\theta}{2}} + e^{\mathrm{i}\frac{\theta}{2}}}{e^{-\mathrm{i}\frac{\theta}{2}} - e^{\mathrm{i}\frac{\theta}{2}}}$$

$$= -\frac{e^{\mathrm{i}\frac{\theta}{2}} + e^{-\mathrm{i}\frac{\theta}{2}}}{e^{\mathrm{i}\frac{\theta}{2}} - e^{-\mathrm{i}\frac{\theta}{2}}}$$

$$= -\frac{2\cos\left(\frac{\theta}{2}\right)}{2\mathrm{i}\sin\left(\frac{\theta}{2}\right)} \times \frac{\mathrm{i}}{\mathrm{i}}$$

$$= \mathrm{i}\cot\left(\frac{\theta}{2}\right)$$

[5 marks]

Answer 4

(a)

$$\begin{aligned}
\text{LHS} &= z^n + \frac{1}{z^n} \\
&= e^{\mathrm{i}n\theta} + \frac{1}{e^{\mathrm{i}n\theta}} \\
&= e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta} \\
&= cos(n\theta) + \mathrm{i}\, sin(n\theta) + cos(-n\theta) + \mathrm{i}\, sin(-n\theta)
\end{aligned}$$

As *cos* is an even function and *sin* odd

$$\begin{aligned}
&= cos\, n\theta + \mathrm{i}\, sin\, n\theta + cos\, n\theta - \mathrm{i}\, sin\, n\theta \\
&= 2\, cos\, n\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[2 marks]

(b) Apply the binomial theorem to $\left(z + \frac{1}{z}\right)^4$

$$\begin{aligned}
\left(z + \frac{1}{z}\right)^4 &= 1 \times z^4 \times \left(\frac{1}{z}\right)^0 \\
&+ 4 \times z^3 \times \left(\frac{1}{z}\right)^1 \\
&+ 6 \times z^2 \times \left(\frac{1}{z}\right)^2 \\
&+ 4 \times z^1 \times \left(\frac{1}{z}\right)^3 \\
&+ 1 \times z^0 \times \left(\frac{1}{z}\right)^4
\end{aligned}$$

$$\begin{aligned}
\therefore \left(z + \frac{1}{z}\right)^4 &= z^4 + 4z^2 + 6 + 4\left(\frac{1}{z^2}\right) + \left(\frac{1}{z^4}\right) \\
(2\, cos\, \theta)^4 &= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6 \\
16\, cos^4\theta &= 2\, cos\, 4\theta + 8\, cos\, 2\theta + 6 \\
8\, cos^4\theta &= cos\, 4\theta + 4\, cos\, 2\theta + 3 \\
cos^4\theta &= \frac{1}{8}(cos\, 4\theta + 4\, cos\, 2\theta + 3) \qquad \square
\end{aligned}$$

[4 marks]

Answer 5

$$C + \mathrm{i}S = a\,(\,cos\,\theta + \mathrm{i}\,sin\,\theta\,)$$
$$+ a^2\,(\,cos\,2\theta + \mathrm{i}\,sin\,2\theta\,)$$
$$+ a^3\,(\,cos\,3\theta + \mathrm{i}\,sin\,3\theta\,)$$
$$+ \ \ldots$$
$$= a\,e^{\mathrm{i}\theta} + a^2\,e^{\mathrm{i}2\theta} + a^3\,e^{\mathrm{i}3\theta} + \ldots$$
$$= \frac{a\,e^{\mathrm{i}\theta}}{1 - a\,e^{\mathrm{i}\theta}}$$
$$= \frac{a\,cos\,\theta + a\mathrm{i}\,sin\,\theta}{1 - a\,cos\,\theta - a\mathrm{i}\,sin\,\theta} \times \frac{1 - a\,cos\,\theta + a\mathrm{i}\,sin\,\theta}{1 - a\,cos\,\theta + a\mathrm{i}\,sin\,\theta}$$
$$= \frac{a\,cos\,\theta - a^2\,cos^2\,\theta + a^2\mathrm{i}\,cos\,\theta\,sin\,\theta + a\mathrm{i}\,sin\,\theta - a^2\mathrm{i}\,sin\,\theta\,cos\,\theta - a^2\,sin^2\,\theta}{1 - a\,cos\,\theta - a\,cos\,\theta + a^2\,cos^2\,\theta + a^2 sin^2\,\theta}$$
$$= \frac{a\,cos\theta - a^2\,(\,cos^2\,\theta + sin^2\,\theta\,) + a\mathrm{i}\,sin\,\theta}{1 - 2\,cos\,\theta + a^2\,\left(cos^2\,\theta + sin^2\,\theta\right)}$$
$$= \frac{a\,cos\,\theta - a^2 + a\mathrm{i}\,sin\,\theta}{1 - 2\,cos\,\theta + a^2}$$
$$\therefore\ C = \frac{a\,cos\,\theta - a^2}{1 - 2\,cos\,\theta + a^2}$$
$$\text{and}\ \ S = \frac{a\,sin\,\theta}{1 - 2\,cos\,\theta + a^2} \qquad \square$$

[8 marks]

Answer 6

$$\text{LHS} = 1 + z + z^2 + \ldots + z^{2n-1}$$

$$= 1 + e^{\frac{\pi}{n}\mathrm{i}} + \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^2 + \ldots + \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^{2n-1}$$

$$= \frac{1\left(1 - \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^{2n}\right)}{1 - e^{\frac{\pi}{n}\mathrm{i}}}$$

$$= \frac{1 - e^{\frac{2n\pi}{n}\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}}$$

$$= \frac{1 - e^{2\pi\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}}$$

$$= \frac{1 - 1}{1 - e^{\frac{\pi}{n}\mathrm{i}}}$$

$$= \text{RHS}$$