1.5 Answers

1.5.1 Solution to 1.3 Example (Teaching Video)

(i)

$$f(x) = x^3 - x$$

$$\therefore\ f(-x) = (-x)^3 - (-x)$$

$$= -x^3 + x$$

$$= -(x^3 - x)$$

$$= -f(x) \qquad \therefore\ f(x) \text{ is an odd function} \quad \square$$

[2 marks]

(ii)

$$g(x) = \left| x^2 - 1 \right| - 1$$

$$\therefore\ g(-x) = \left| (-x)^2 - 1 \right| - 1$$

$$= \left| x^2 - 1 \right| - 1$$

$$= g(x) \qquad \therefore\ g(x) \text{ is an even function} \quad \square$$

[2 marks]

1.5.2 Solution to 1.4 Exercise

Answer 1

$$n(x) = \frac{3x^4 + 1}{x}$$

$$\Rightarrow \quad n(-x) = \frac{3(-x)^4 + 1}{(-x)}$$

$$= -\frac{3x^4 + 1}{x}$$

$$= -n(x)$$

$$\therefore\ n(x) \text{ is an odd function} \quad \square$$

[3 marks]

Answer 2

$$h(x) = \sqrt{1 + x + x^2} + \sqrt{1 - x + x^2}$$

$$\Rightarrow h(-x) = \sqrt{1 + (-x) + (-x)^2} + \sqrt{1 - (-x) + (-x)^2}$$

$$= \sqrt{1 - x + x^2} + \sqrt{1 + x + x^2}$$

$$= \sqrt{1 + x + x^2} + \sqrt{1 - x + x^2}$$

$$= h(x)$$

$$\therefore\ h(x) \text{ is an even function} \quad \square$$

[4 marks]

Answer 3

(i) The graph of $f(x) = sin\,x$ has half-turn rotational symmetry about the origin which is the hallmark of an odd function.

[1 mark]

(ii)

$$f(x) = sin\,x$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\Rightarrow f(-x) = sin(-x)$$

$$= (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \ldots$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots$$

$$= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

$$= -f(x)$$

$$\therefore f(x) \text{ is an odd function} \quad \square$$

[4 marks]

Answer 4

$$v(x) = x\,\frac{2^x + 1}{2^x - 1}$$

$$\Rightarrow v(-x) = (-x)\,\frac{2^{-x} + 1}{2^{-x} - 1}$$

$$= -x\,\frac{2^{-x} + 1}{2^{-x} - 1} \times \frac{2^x}{2^x}$$

$$= -x\,\frac{1 + 2^x}{1 - 2^x}$$

$$= x\,\frac{2^x + 1}{2^x - 1}$$

$$= v(x)$$

$$\therefore v(x) \text{ is an even function} \quad \square$$

[4 marks]

Answer 5

$$
\begin{aligned}
s(x) &= \frac{(1 + e^{x})^2}{e^{x}} \\
\Rightarrow \; s(-x) &= \frac{(1 + e^{-x})^2}{e^{-x}} \\
&= \frac{1 + 2e^{-x} + e^{-2x}}{e^{-x}} \\
&= \frac{1}{e^{-x}} + \frac{2e^{-x}}{e^{-x}} + \frac{e^{-2x}}{e^{-x}} \\
&= e^{x} + 2 + e^{-x} \\
&= \frac{e^{2x}}{e^{x}} + \frac{2e^{x}}{e^{x}} + \frac{1}{e^{x}} \\
&= \frac{e^{2x} + 2e^{x} + 1}{e^{x}} \\
&= \frac{(e^{x} + 1)^2}{e^{x}} \\
&= \frac{(1 + e^{x})^2}{e^{x}} \\
&= s(x)
\end{aligned}
$$

$\therefore \; s(x)$ is an even function $\quad \square$

[5 marks]

Answer 6

$$
\begin{aligned}
L(x) &= ln\left(\frac{0.5 + x}{0.5 - x}\right) \\
\Rightarrow \; L(-x) &= ln\left(\frac{0.5 + (-x)}{0.5 - (-x)}\right) \\
&= ln\left(\frac{0.5 - x}{0.5 + x}\right) \\
&= ln(0.5 - x) - ln(0.5 + x) \\
&= -\left(-ln(0.5 - x) + ln(0.5 + x)\right) \\
&= -\left(ln(0.5 + x) - ln(0.5 - x)\right) \\
&= -ln\left(\frac{0.5 + x}{0.5 - x}\right) \\
&= -L(x)
\end{aligned}
$$

$\therefore \; L(x)$ is an odd function $\quad \square$

[5 marks]

Answer 7

(i)

$$k(x) = ln\left(x + \sqrt{1 + x^2} \right)$$

$$= ln\left(x + \left(1 + x^2 \right)^{\frac{1}{2}} \right)$$

$$k'(x) = \frac{1}{\left(x + \sqrt{1 + x^2} \right)} \times \left(1 + \frac{1}{2}\left(1 + x^2 \right)^{-\frac{1}{2}} \times 2x \right)$$

$$= \frac{1}{\left(x + \sqrt{1 + x^2} \right)} \times \left(1 + \frac{x}{\sqrt{1 + x^2}} \right)$$

$$= \frac{1}{\left(x + \sqrt{1 + x^2} \right)} \times \left(\frac{\sqrt{1 + x^2}}{\sqrt{1 + x^2}} + \frac{x}{\sqrt{1 + x^2}} \right)$$

$$= \frac{1}{\left(x + \sqrt{1 + x^2} \right)} \times \frac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}}$$

$$= \frac{1}{\sqrt{1 + x^2}} \qquad \square$$

[6 marks]

(ii)

$$\Rightarrow k'(-x) = \frac{1}{\sqrt{1 + (-x)^2}}$$

$$= \frac{1}{\sqrt{1 + x^2}}$$

$$= k'(x)$$

$\therefore$ $k'(x)$ is an even function and so $k(x)$ is an odd function $\square$

[2 marks]

Answer 8
(i)

$$p(x) = ln\left(2x^3 - x \right)^2$$
$$= 2\,ln\left(2x^3 - x \right)$$
$$p'(x) = \frac{2}{\left(2x^3 - x \right)} \times \left(6x^2 - 1 \right)$$
$$= \frac{2\left(6x^2 - 1 \right)}{\left(2x^3 - x \right)}$$

$$\Rightarrow\ p'(-x) = \frac{2\left(6\ (-x)^2 - 1 \right)}{\left(2\ (-x)^3 - (-x) \right)}$$
$$= \frac{2\left(6x^2 - 1 \right)}{(-1)\left(2x^3 + x \right)}$$
$$= -\frac{2\left(6x^2 - 1 \right)}{\left(2x^3 - x \right)}$$
$$= -p'(x)$$

$\therefore\ p'(x)$ is an odd function and so $p(x)$ is an even function ☐

[4 marks]

(ii) Kevin's solution (2nd November 2022),

$$p(x) = ln\left(2x^3 - x \right)^2$$
$$= ln\left(4x^6 - 4x^4 + x^2 \right)$$

$$\Rightarrow\ p(-x) = ln\left(4(-x)^6 - 4(-x)^4 + (-x)^2 \right)$$
$$= ln\left(4x^6 - 4x^4 + x^2 \right)$$
$$= p(x)$$

$\therefore\ p(x)$ is an even function ☐

[4 marks]

Answer 9
(ii)

- The derivative of an even function is an odd function

Suppose that $f(x)$ is an even function in which case $f(-x) = f(x)$

$$\therefore \quad f(x) = f(-x)$$

differentiate both sides

$$f'(x) = f'(-x) \times (-1) \text{ by the chain rule}$$

$$f'(x) = -f'(-x)$$

Which is saying the derivative of f (which was an even function) is an odd function $\square$

[4 marks]

Answer 10

Let $g(x) = |x|$ and let $fg(x)$ be the function $h(x)$

By definition, $|(-x)| = |x|$ or, in other words, $g(-x) = g(x)$

$$\therefore \text{ As } g(-x) = g(x)$$

$$\text{it follows that } fg(-x) = fg(x)$$

$$\text{or } h(-x) = h(x)$$

which is the definition of an even function

but this is simply saying, $f(|(-x)|) = f(|x|)$

So, $f(|x|)$ is an even function

To show that $|f(x)|$ is not guaranteed to be an even function, simply find a a counter-example where it is an odd function or a counter-example where it is neither odd nor even.

[4 marks]