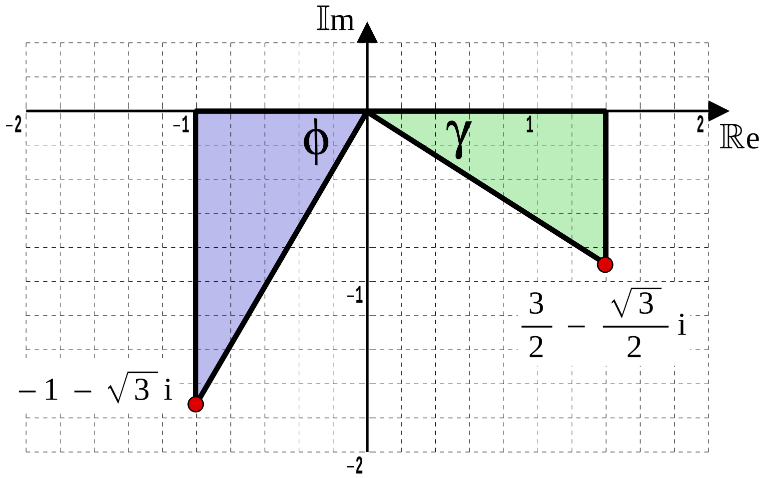
2.5 Answers

2.5.1 Solution to 2.3 Example (Teaching Video)

(i)



For the purple triangle with $z = -1 - \sqrt{3}\,\mathrm{i}$

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} \qquad \tan\phi = \frac{\sqrt{3}}{1}$$

$$= \sqrt{1 + 3} \qquad \phi = \arctan(\sqrt{3})$$

$$= 2 \qquad \phi = \frac{\pi}{3}$$

Angle required for exponential form is $-\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$

$$\therefore z = 2\,e^{-\frac{2\pi}{3}\mathrm{i}}$$

[3 marks]

For the green triangle with $z = \sqrt{3}\left(\cos\left(\frac{\pi}{6}\right) - \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right)$

As $\cos\theta$ is an even function, $\cos\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$

As $\sin\theta$ is an odd function, $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$

$$\therefore z = \sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$$

$$\therefore z = \sqrt{3}\,e^{-\frac{\pi}{6}\mathrm{i}}$$

[3 marks]

2.5.2 Solution to 2.4 Exercise

Answer 1

(i) $z_A = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \text{i}\,\sin\left(\frac{\pi}{4}\right)\right) = 1 + \text{i}$

$z_A = \sqrt{2}\, e^{\frac{\pi}{4}\text{i}}$

(ii) $z_B = \frac{3}{2}\left(\cos\left(\frac{2\pi}{3}\right) + \text{i}\,\sin\left(\frac{2\pi}{3}\right)\right) = -\frac{3}{4} + \frac{3\sqrt{3}}{4}\text{i} = -0.75 + 1.30\,\text{i}$

$z_B = \frac{3}{2}\, e^{\frac{2\pi}{3}\text{i}}$

(iii) $z_C = \sqrt{3}\left(\cos\left(\frac{5\pi}{6}\right) - \text{i}\,\sin\left(\frac{5\pi}{6}\right)\right) = -\frac{3}{2} - \frac{\sqrt{3}}{2}\text{i} = -1.5 - 0.866\,\text{i}$

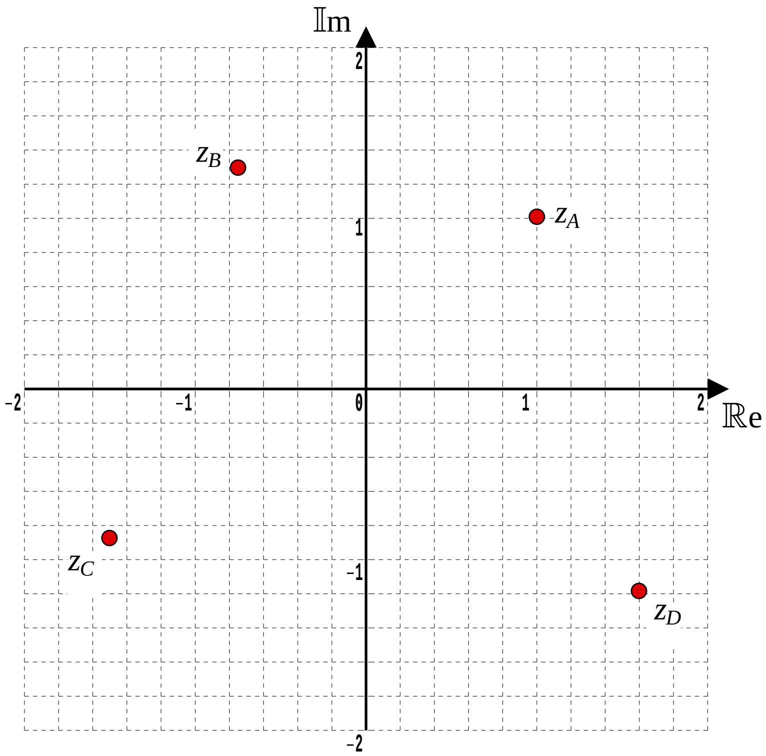
$= \sqrt{3}\left(\cos\left(-\frac{5\pi}{6}\right) + \text{i}\,\sin\left(-\frac{5\pi}{6}\right)\right)$

$z_C = \sqrt{3}\, e^{-\frac{5\pi}{6}\text{i}}$

(iv) $z_D = 2\left(\cos\left(\frac{\pi}{5}\right) - \text{i}\,\sin\left(\frac{\pi}{5}\right)\right) = 1.62 - 1.18\,\text{i}$

$= 2\left(\cos\left(-\frac{\pi}{5}\right) + \text{i}\,\sin\left(-\frac{\pi}{5}\right)\right)$

$z_D = 2\, e^{-\frac{\pi}{5}\text{i}}$



[3, 3, 3, 3 marks]

Answer 2

(i) $z_A = \frac{6}{5} + \frac{8}{5}\,\mathrm{i}$

$$|z| = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{8}{5}\right)^2} \qquad \tan\phi = \frac{8}{5} \div \frac{6}{5} = \frac{4}{3}$$

$$= \sqrt{\frac{36}{25} + \frac{64}{25}} = 2 \qquad \phi = \arctan\left(\frac{4}{3}\right) = 0.927$$

$$\theta = \phi \qquad \therefore z = 2\,e^{0.927\,\mathrm{i}}$$

(ii) $z_B = -\frac{3}{2} + \frac{1}{2}\,\mathrm{i}$

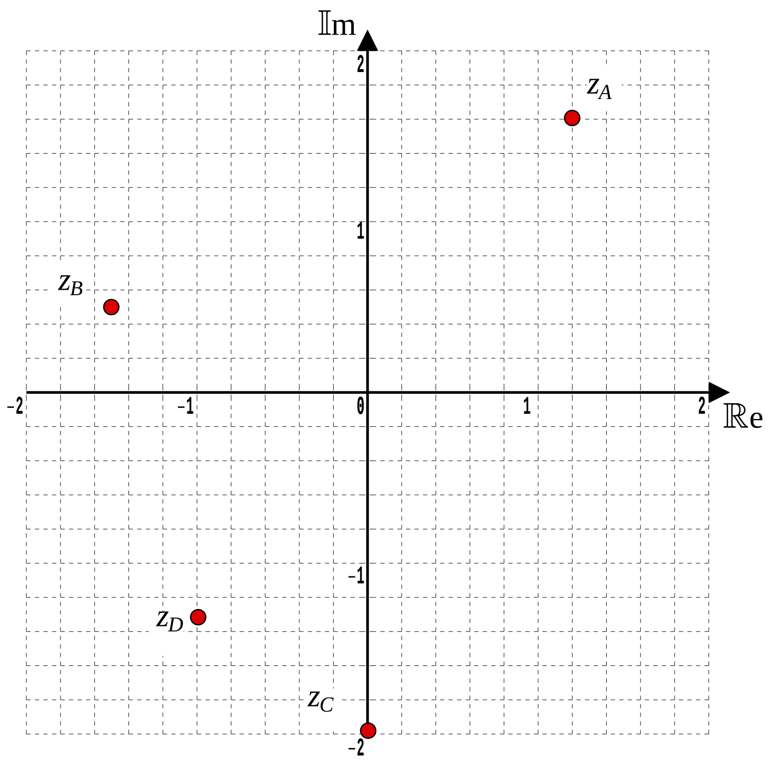
$$|z| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \qquad \tan\phi = \frac{1}{2} \div \frac{3}{2} = \frac{1}{3}$$

$$= \sqrt{\frac{9}{4} + \frac{1}{4}} = \frac{\sqrt{10}}{2} \qquad \phi = \arctan\left(\frac{1}{3}\right) = 0.321$$

$$\theta = \pi - \phi = 2.82 \qquad \therefore z = \frac{\sqrt{10}}{2}\,e^{2.82\,\mathrm{i}}$$

(iii) $z_C = -2\,\mathrm{i}$

$$|z| = 2 \qquad \theta = -\frac{\pi}{2} \qquad \therefore z = 2\,e^{-\frac{\pi}{2}\,\mathrm{i}}$$



(iv) $z_D = -1 - \dfrac{4}{3}\,\text{i}$

$$|z| = \sqrt{\left(-\frac{3}{3}\right)^2 + \left(-\frac{4}{3}\right)^2} \qquad \tan\phi = \frac{4}{3} \div 1 = \frac{4}{3}$$

$$= \sqrt{\frac{9}{9} + \frac{16}{9}} = \frac{5}{3} \qquad \phi = \arctan\left(\frac{4}{3}\right) = 0.927$$

$$\theta = -\pi + \phi = -2.21 \qquad \therefore z = \frac{5}{3}\,e^{-2.21\,\text{i}}$$

[3, 3, 3, 3 marks]

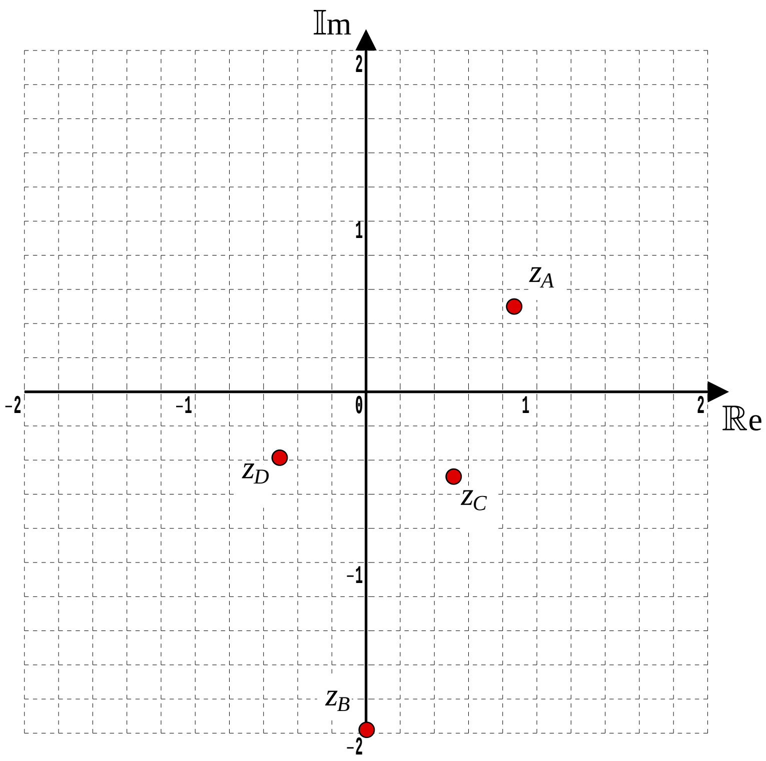
Answer 3

(i) $z_A = e^{\frac{\pi}{6}\text{i}} = \cos\left(\frac{\pi}{6}\right) + \text{i}\sin\left(\frac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\,\text{i}$

(ii) $z_B = 2\,e^{\frac{3\pi}{2}\text{i}} = 2\,e^{-\frac{\pi}{2}\text{i}} = 2\left(\cos\left(-\frac{\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{2}\right)\right) = 0 - 2\,\text{i}$

(iii) $z_C = \dfrac{\sqrt{2}}{2}\,e^{-\frac{\pi}{4}\text{i}} = \dfrac{\sqrt{2}}{2}\left(\cos\left(-\frac{\pi}{4}\right) + \text{i}\sin\left(-\frac{\pi}{4}\right)\right) = \dfrac{1}{2} - \dfrac{1}{2}\,\text{i}$

(iv) $z_D = \dfrac{2}{3}\,e^{-\frac{4\pi}{5}\text{i}} = \dfrac{2}{3}\left(\cos\left(-\frac{4\pi}{5}\right) + \text{i}\sin\left(-\frac{4\pi}{5}\right)\right) = -0.539 - 0.392\,\text{i}$



[3, 3, 3, 3 marks]

Answer 4

(i) $z_A = \sqrt{3}\ e^{\frac{3\pi}{2}\mathrm{i}} = \sqrt{3}\ e^{-\frac{\pi}{2}\mathrm{i}}$

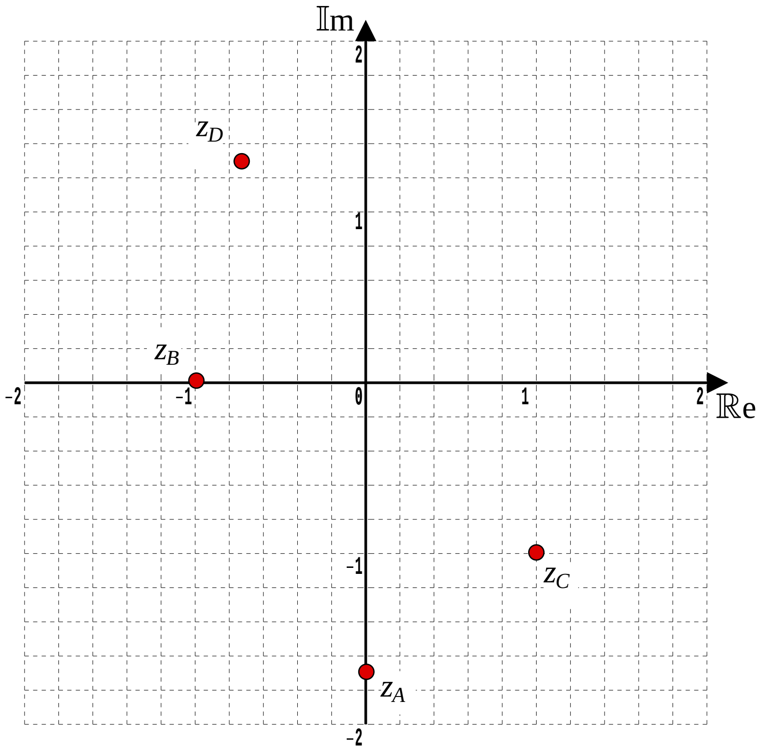
$$= \sqrt{3}\left(cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\, sin\left(-\frac{\pi}{2}\right)\right) = 0 - \sqrt{3}\ \mathrm{i}$$

(ii) $z_B = e^{\pi\mathrm{i}} = cos\ (\pi) - \mathrm{i}\, sin\,(\pi) = -1 + 0\,\mathrm{i}$

(iii) $z_C = \sqrt{2}\ e^{-\frac{17\pi}{4}\mathrm{i}} = \sqrt{2}\ e^{-\frac{\pi}{4}\mathrm{i}}$

$$= \sqrt{2}\left(cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\, sin\left(-\frac{\pi}{4}\right)\right) = 1 - \mathrm{i}$$

(iv) $z_D = \frac{3}{2}\, e^{-\frac{4\pi}{3}\mathrm{i}} = \frac{3}{2}\left(cos\left(-\frac{4\pi}{3}\right) + \mathrm{i}\, sin\left(-\frac{4\pi}{3}\right)\right) = -\frac{3}{4} + \frac{3\sqrt{3}}{4}\,\mathrm{i}$



[3, 3, 3, 3 marks]

Answer 5

(i)

$$\begin{aligned}
\text{RHS} &= \frac{1}{2\text{i}}\left(e^{\text{i}\theta} - e^{-\text{i}\theta} \right) \\
&= \frac{1}{2\text{i}}\left(\ (\cos\theta + \text{i}\sin\theta \ ) - (\cos\ (-\theta) + \text{i}\sin\ (-\theta) \ ) \ \right) \\
&= \frac{1}{2\text{i}}\left(\cos\theta + \text{i}\sin\theta \ - (\cos\theta - \text{i}\sin\theta \ ) \ \right) \\
&= \frac{1}{2\text{i}}\left(2\text{i}\sin\theta \right) \\
&= \sin\theta \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[4 marks]

(ii)

$$\cos\theta = \frac{1}{2}\left(\ e^{\text{i}\theta} + e^{-\text{i}\theta} \right)$$

$$\begin{aligned}
\text{RHS} &= \frac{1}{2}\left(e^{\text{i}\theta} + e^{-\text{i}\theta} \right) \\
&= \frac{1}{2}\left(\ (\cos\theta + \text{i}\sin\theta \ ) + (\cos\ (-\theta) + \text{i}\sin\ (-\theta) \ ) \ \right) \\
&= \frac{1}{2}\left(\cos\theta + \text{i}\sin\theta \ + (\cos\theta - \text{i}\sin\theta \ ) \ \right) \\
&= \frac{1}{2}\left(2\cos\theta \right) \\
&= \cos\theta \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[4 marks]