3.4 Answers

Answer 1

Prove that, $z_1 z_2 = r_1 r_2 e^{i(\theta 1 + \theta 2)}$

$$
\begin{aligned}
\text{LHS} &= z_1 z_2 \\
&= r_1 e^{i\theta 1} \times r_2 e^{i\theta 2} \\
&= r_1 (\cos\theta_1 + i\sin\theta_1) r_2 (\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2 \left(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2\right) \\
&= r_1 r_2 (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\
&= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\
&= r_1 r_2 e^{i(\theta 1 + \theta 2)} \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

Answer 2

Prove that $\frac{1}{z} = e^{-i\theta}$

$$
\begin{aligned}
\text{LHS} &= \frac{1}{z} \\
&= \frac{1}{e^{-i\theta}} \\
&= \frac{1}{\cos\theta + i\sin\theta} \\
&= \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} \\
&= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i\cos\theta\sin\theta + i\sin\theta\cos\theta - i^2\sin^2\theta} \\
&= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} \\
&= \cos\theta - i\sin\theta \\
&= \cos(-\theta) + i\sin(-\theta) \\
&= e^{-i\theta} \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

Answer 3

(i) $2\sqrt{3}\, e^{\frac{\pi}{2}\mathrm{i}} \times 3\sqrt{3}\, e^{\frac{\pi}{3}\mathrm{i}} = 18\, e^{\left(\frac{3\pi}{6} + \frac{2\pi}{6}\right)\mathrm{i}} = 18\, e^{\frac{5\pi}{6}\mathrm{i}}$

[4 marks]

(ii) $\dfrac{\sqrt{8}\, e^{\frac{3\pi}{4}\mathrm{i}}}{\sqrt{2}\, e^{-\frac{2\pi}{3}\mathrm{i}}} = 2\, e^{\left(\frac{3\pi}{4} + \frac{2\pi}{3}\right)\mathrm{i}} = 2\, e^{\frac{17\pi}{12}\mathrm{i}} = 2\, e^{-\frac{7\pi}{12}\mathrm{i}}$

[4 marks]

(iii) $\sqrt{6}\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right) \times \sqrt{3}\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$

$= \sqrt{6} \times \sqrt{3}\, e^{-\frac{\pi}{12}\mathrm{i}} e^{\frac{\pi}{3}\mathrm{i}} = 3\sqrt{2}\, e^{\frac{\pi}{4}\mathrm{i}}$

[4 marks]

Answer 4

$$\frac{(\cos 11\theta + \mathrm{i}\sin 11\theta)(\cos 5\theta + \mathrm{i}\sin 5\theta)}{\cos 7\theta + \mathrm{i}\sin 7\theta} = \frac{e^{11\theta\mathrm{i}} \times e^{5\theta\mathrm{i}}}{e^{7\theta\mathrm{i}}}$$

$$= e^{9\theta\mathrm{i}}$$

$$= \cos 9\theta + \mathrm{i}\sin 9\theta$$

[3 marks]

Answer 5

(i) $z = -9 + 3\sqrt{3}\,\mathrm{i}$

$|z| = \sqrt{(-9)^2 + \left(3\sqrt{3}\right)^2}$ $\qquad \tan\phi = \dfrac{3\sqrt{3}}{9}$

$= \sqrt{81 + 27} = 6\sqrt{3}$ $\qquad \phi = \arctan\left(\dfrac{\sqrt{3}}{3}\right) = \dfrac{\pi}{6}$

$\theta = \pi - \phi \qquad \therefore z = 6\sqrt{3}\, e^{\frac{5\pi}{6}\mathrm{i}}$

[3 marks]

(ii) $w = \sqrt{3}\, e^{\frac{7\pi}{12}\mathrm{i}}$

[3 marks]

(iii) $zw = 6\sqrt{3}\, e^{\frac{5\pi}{6}\mathrm{i}} \times \sqrt{3}\, e^{\frac{7\pi}{12}\mathrm{i}} = 18\, e^{\frac{17\pi}{12}\mathrm{i}} = 18\, e^{-\frac{7\pi}{12}\mathrm{i}}$

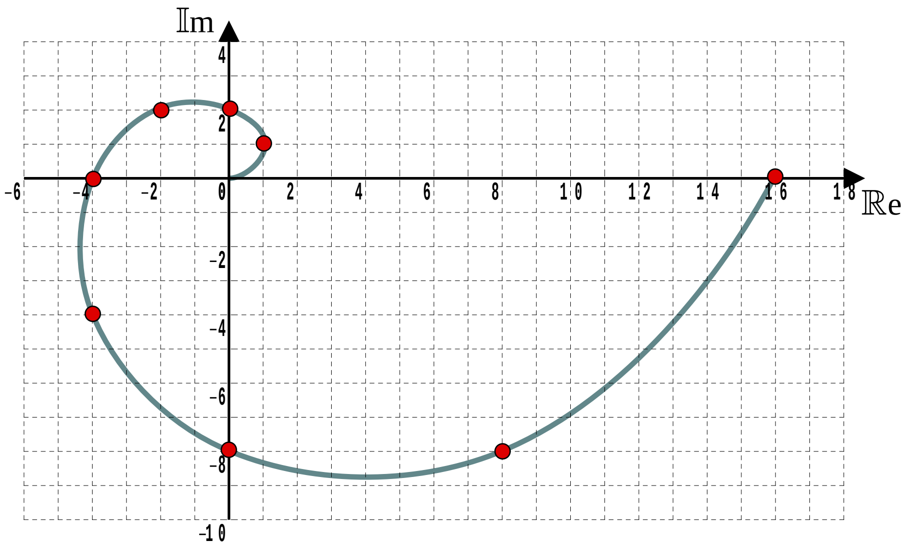
[3 marks]

(iv) $\dfrac{z}{w} = \dfrac{6\sqrt{3}\, e^{\frac{5\pi}{6}\mathrm{i}}}{\sqrt{3}\, e^{\frac{7\pi}{12}\mathrm{i}}} = 6\, e^{\frac{\pi}{4}\mathrm{i}}$

[3 marks]

Answer 6

n	z^n (exponential form) $r\,e^{i\theta}$	z^n (Cartesian form) $x + i\,y$
1	$\sqrt{2}\;e^{\frac{\pi}{4}i}$	$1 + \quad i$
2	$2\,e^{\frac{2\pi}{4}i}$	$0 + 2\,i$
3	$2\sqrt{2}\;e^{\frac{3\pi}{4}i}$	$-2 + 2\,i$
4	$4\,e^{\frac{4\pi}{4}i}$	$-4 + 0\,i$
5	$4\sqrt{2}\;e^{\frac{5\pi}{4}i}$	$-4 - 4\,i$
6	$8\,e^{\frac{6\pi}{4}i}$	$-8 + 0\,i$
7	$8\sqrt{2}\;e^{\frac{7\pi}{4}i}$	$8 - 8\,i$
8	$16\,e^{\frac{8\pi}{4}i}$	$16 + 0\,i$



Each successive multiplication by $\sqrt{2}\;e^{\frac{\pi}{4}i}$ is rotating the point by $\dfrac{\pi}{4}$ radians and increasing its distance from the origin by a length scale factor of $\sqrt{2}$

[7 marks]

Answer 7

(i)

$$f(\theta) = \frac{\cos\theta + \mathrm{i}\sin\theta}{e^{\mathrm{i}\theta}}$$

$$f'(\theta) = \frac{e^{\mathrm{i}\theta}(-\sin\theta + \mathrm{i}\cos\theta) - \mathrm{i}\,e^{\mathrm{i}\theta}(\cos\theta + \mathrm{i}\sin\theta)}{\left(e^{\mathrm{i}\theta}\right)^2}$$

$$= \frac{e^{\mathrm{i}\theta}(-\sin\theta + \mathrm{i}\cos\theta - \mathrm{i}\cos\theta - \mathrm{i}^2\sin\theta)}{\left(e^{\mathrm{i}\theta}\right)^2}$$

$$= \frac{e^{\mathrm{i}\theta}(-\sin\theta + \mathrm{i}\cos\theta - \mathrm{i}\cos\theta + \sin\theta)}{\left(e^{\mathrm{i}\theta}\right)^2}$$

$$= \frac{e^{\mathrm{i}\theta}\,(0)}{\left(e^{\mathrm{i}\theta}\right)^2}$$

$$= 0$$

[4 marks]

(ii)

$$f(0) = \frac{\cos(0) + \mathrm{i}\sin(0)}{e^{\mathrm{i}0}}$$

$$= \frac{1 + 0}{1}$$

$$= 1$$

[1 mark]

(iii)

$$f(x) = 1$$

$$\frac{\cos\theta + \mathrm{i}\sin\theta}{e^{\mathrm{i}\theta}} = 1$$

$$\therefore\ e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta \qquad \square$$

[1 mark]