4.3 Answers

Answer 1

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right)$$

$$\therefore\ z^5 = (\sqrt{2}\,)^5\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right)^5$$

$$= 4\sqrt{2}\left(\cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right)$$

$$= -2\sqrt{6} + 2\sqrt{2}\,\mathrm{i}$$

[3 marks]

Answer 2

$$w = 2\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$$

$$\therefore\ w^4 = 2^4\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)^4$$

$$= 16\left(\cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right)\right)$$

$$= 16\left(\cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right) \qquad \text{can skip this step}$$

$$= 16\left(\cos\left(\frac{2\pi}{3}\right) - \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right) \qquad \text{can skip this step}$$

$$= -8 - 8\sqrt{3}\,\mathrm{i}$$

[3 marks]

Answer 3

$$v = \sqrt{3}\left(\cos\left(\frac{3\pi}{4}\right) - \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right)$$

$$= \sqrt{3}\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right) \qquad \text{being pedantic !}$$

$$\therefore\ v^6 = (\sqrt{3}\,)^6\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)^6$$

$$= 27\left(\cos\left(\frac{18\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{18\pi}{4}\right)\right)$$

$$= 27\left(\cos\left(\frac{9\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{9\pi}{2}\right)\right)$$

$$= 27\left(\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$$

$$= 0 - 27\,\mathrm{i}$$

[3 marks]

Answer 4
(i)

$$z = -2 + \left(2\sqrt{3}\right) i$$

$$|z| = \sqrt{(-2)^2 + \left(2\sqrt{3}\right)^2} \qquad \tan\phi = \frac{2\sqrt{3}}{2}$$

$$= \sqrt{4 + 12} \qquad \phi = \arctan\left(\sqrt{3}\right)$$

$$= 4 \qquad \phi = \frac{\pi}{3}$$

Angle required for polar form is $\pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

$$\therefore z = 4\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

[3 marks]

(ii)

$$z^6 = 4^6\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)^6$$

$$= 4096\left(\cos(4\pi) + i\sin(4\pi)\right)$$

$$= 4096$$

[2 marks]

Answer 5

Let $z = 3 + \sqrt{3}\, i$

$$|z| = \sqrt{(3)^2 + \left(\sqrt{3}\right)^2} \qquad \tan\phi = \frac{\sqrt{3}}{3}$$

$$= \sqrt{9 + 3} \qquad \phi = \arctan\left(\frac{\sqrt{3}}{3}\right)$$

$$= 2\sqrt{3} \qquad \phi = \frac{\pi}{6}$$

Angle required for polar form is unchanged $\dfrac{\pi}{6}$

$$z = 2\sqrt{3}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

$$z^5 = \left(2\sqrt{3}\right)^5\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)^5$$

$$= 32 \times 9\sqrt{3}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$

$$= 288\sqrt{3}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$

$$= -432 + 144\sqrt{3}\, i$$

[4 marks]

Answer 6

$$\left(r\,e^{\mathrm{i}\theta}\right)^n = r^n\,e^{\mathrm{i}n\theta}$$

$$\begin{aligned}
\text{LHS} &= \left(r\,e^{\mathrm{i}\theta}\right)^n \\
&= r^n\left(e^{\mathrm{i}\theta}\right)^n \\
&= r^n\left(\cos\theta + \mathrm{i}\sin\theta\right)^n \\
&= r^n\left(\cos(n\theta) + \mathrm{i}\sin(n\theta)\right) \\
&= r^n\,e^{\mathrm{i}n\theta} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 7

$$\frac{\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right)^6}{\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right)^{11}} = \cos\left(\frac{\pi}{4}\right) - \mathrm{i}\sin\left(\frac{\pi}{4}\right)$$

$$\begin{aligned}
\text{LHS} &= \frac{\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right)^6}{\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right)^{11}} \\
&= \left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right)^6 \left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right)^{-11} \\
&= \left(\cos\left(\frac{18\pi}{4}\right) + \mathrm{i}\sin\left(\frac{18\pi}{4}\right)\right)\left(\cos\left(-\frac{11\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{11\pi}{4}\right)\right) \\
&= e^{\frac{18\pi}{4}\mathrm{i}} \times e^{-\frac{11\pi}{4}\mathrm{i}} \\
&= e^{\frac{7\pi}{4}\mathrm{i}} \\
&= e^{-\frac{\pi}{4}\mathrm{i}} \\
&= \cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{4}\right) - \mathrm{i}\sin\left(\frac{\pi}{4}\right) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[6 marks]

Answer 8

(i)

$$\frac{1+\sqrt{3}\,\mathrm{i}}{1-\sqrt{3}\,\mathrm{i}} = \frac{1+\sqrt{3}\,\mathrm{i}}{1-\sqrt{3}\,\mathrm{i}} \times \frac{1+\sqrt{3}\,\mathrm{i}}{1+\sqrt{3}\,\mathrm{i}}$$

$$= \frac{1+\sqrt{3}\,\mathrm{i}+\sqrt{3}\,\mathrm{i}+3\mathrm{i}^2}{1+\sqrt{3}\,\mathrm{i}-\sqrt{3}\,\mathrm{i}-3\mathrm{i}^2}$$

$$= \frac{-2+2\sqrt{3}\,\mathrm{i}}{4}$$

$$= -\frac{1}{2}+\frac{\sqrt{3}}{2}\,\mathrm{i}$$

$$|z| = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \qquad \tan\phi = \frac{\sqrt{3}}{1}$$

$$= \sqrt{\frac{1}{4}+\frac{3}{4}} \qquad \phi = \arctan\left(\sqrt{3}\right)$$

$$= 1 \qquad \phi = \frac{\pi}{3}$$

Angle required for polar form is $\pi - \phi = \dfrac{2\pi}{3}$

$$\therefore \quad \frac{1+\sqrt{3}\,\mathrm{i}}{1-\sqrt{3}\,\mathrm{i}} = e^{\frac{2\pi}{3}\mathrm{i}}$$

[4 marks]

(ii)

$$\left(\frac{1+\sqrt{3}\,\mathrm{i}}{1-\sqrt{3}\,\mathrm{i}}\right)^n = \left(e^{\frac{2\pi}{3}\mathrm{i}}\right)^n$$

$$= e^{\frac{2n\pi}{3}\mathrm{i}}$$

$$= \cos\left(\frac{2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2n\pi}{3}\right)$$

For a real positive solution, $\sin\left(\dfrac{2n\pi}{3}\right) = 0$

$$\frac{2n\pi}{3} = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots$$

$$2n\pi = 0,\ 3\pi,\ 6\pi,\ 9\pi,\ \ldots$$

$$n = 0,\ \frac{3}{2},\ 3,\ \frac{9}{2},\ \ldots$$

$\therefore$ Smallest positive integer value of n is 3 as in that case

$$\left(\frac{1+\sqrt{3}\,\mathrm{i}}{1-\sqrt{3}\,\mathrm{i}}\right)^3 = \cos(2\pi) + \mathrm{i}\sin(2\pi) = 1$$

[3 marks]

Answer 9

$$\frac{18(1-\mathrm{i})^{39}}{(1+\mathrm{i})^{41}} = 18 \times \frac{\left(\sqrt{2}\,e^{-\frac{\pi}{4}\mathrm{i}}\right)^{39}}{\left(\sqrt{2}\,e^{\frac{\pi}{4}\mathrm{i}}\right)^{41}}$$

$$= 18 \times \left(\frac{1}{\sqrt{2}}\right)^{2} \times e^{-\frac{39\pi}{4}\mathrm{i}} \times e^{-\frac{41\pi}{4}\mathrm{i}}$$

$$= 9 \times e^{-\frac{80\pi}{4}\mathrm{i}}$$

$$= 9\,e^{-20\pi\mathrm{i}}$$

$$= 9\,e^{0}$$

$$= 9$$

[5 marks]