5.4 Answers

5.4.1 Solutions to 5.2 Example (Teaching Video)

$$(\cos\theta + \mathrm{i}\sin\theta)^2 = \cos(2\theta) + \mathrm{i}\sin(2\theta)$$ by de Moivre's theorem

$$\begin{aligned}(\cos\theta + \mathrm{i}\sin\theta)^2 &= 1 \times (\cos\theta)^2 \times (\mathrm{i}\sin\theta)^0 \\ &+ 2 \times (\cos\theta)^1 \times (\mathrm{i}\sin\theta)^1 \\ &+ 1 \times (\cos\theta)^0 \times (\mathrm{i}\sin\theta)^2 \\ &= \cos^2\theta + \mathrm{i}\,2\cos\theta\sin\theta - \sin^2\theta\end{aligned}$$ by the Binomial theorem

Equating the two expansions gives,

$$\cos(2\theta) + \mathrm{i}\sin(2\theta) = \cos^2\theta + \mathrm{i}\,2\cos\theta\sin\theta - \sin^2\theta$$

Matching the real part on the LHS with the real part on the RHS yields,

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Matching the imaginary part on the LHS with the imaginary part on the RHS yields,

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

[6 marks]

5.4.2 Solution to 5.3 Exercise

Answer 1

$$\left(cos\,\theta + i\,sin\,\theta \right)^3 = cos\,(3\theta) + i\,sin\,(3\theta) \qquad \text{by de Moivre's theorem}$$

$$\begin{aligned}\left(cos\,\theta + i\,sin\,\theta \right)^3 &= 1 \times (cos\,\theta)^3 \times \left(i\,sin\,\theta \right)^0 \qquad \text{by the Binomial theorem} \\ &+ 3 \times (cos\,\theta)^2 \times \left(i\,sin\,\theta \right)^1 \\ &+ 3 \times (cos\,\theta)^1 \times \left(i\,sin\,\theta \right)^2 \\ &+ 1 \times (cos\,\theta)^0 \times \left(i\,sin\,\theta \right)^3 \\ &= cos^3\,\theta + i\,3\,cos^2\,\theta\,sin\,\theta - 3\,cos\,\theta\,sin^2\,\theta - i\,sin^3\,\theta \\ &= cos^3\,\theta - 3\,cos\,\theta\,sin^2\,\theta + i\left(3\,cos^2\,\theta\,sin\,\theta - sin^3\,\theta \right)\end{aligned}$$

Equating the two expansions gives,

$$cos\,(3\theta) + i\,sin\,(3\theta) = cos^3\,\theta - 3\,cos\,\theta\,sin^2\,\theta + i\left(3\,cos^2\,\theta\,sin\,\theta - sin^3\,\theta \right)$$

Matching the real part on the LHS with the real part on the RHS yields,

$$\begin{aligned}cos\,(3\theta) &= cos^3\,\theta - 3\,cos\,\theta\,sin^2\,\theta \\ &= cos^3\,\theta - 3\,cos\,\theta\left(1 - cos^2\,\theta \right) \\ &= cos^3\,\theta - 3\,cos\,\theta + 3\,cos^3\,\theta \\ &= 4\,cos^3\,\theta - 3\,cos\,\theta \qquad \square\end{aligned}$$

Matching the imaginary part on the LHS with the imaginary part on the RHS yields,

$$\begin{aligned}sin\,(3\theta) &= 3\,cos^2\,\theta\,sin\,\theta - sin^3\,\theta \\ &= 3\left(1 - sin^2\,\theta \right) sin\,\theta - sin^3\,\theta \\ &= 3\,sin\,\theta - 3\,sin^2\,\theta - sin^3\,\theta \\ &= 3\,sin\,\theta - 4\,sin^2\,\theta \qquad \square\end{aligned}$$

[6 marks]

Answer 2

$$8x^3 - 6x - 1 = 0$$

let $x = cos\,\theta$ to obtain,

$$8\,cos^3\theta - 6\,cos\,\theta - 1 = 0$$

$$2\left(4\,cos^3\theta - 3\,cos\,\theta\right) = 1$$

$$4\,cos^3\theta - 3\,cos\,\theta = \frac{1}{2}$$

$$cos\,3\theta = \frac{1}{2}$$

$$3\theta = \frac{\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3},\ \ldots$$

$$\theta = \frac{\pi}{9},\ \frac{5\pi}{9},\ \frac{7\pi}{9},\ \ldots$$

$$\therefore x = 0.940,\ -0.174,\ -0.766$$

[5 marks]

Answer 3

$$cos\,5\theta + 5\,cos\,3\theta = 0$$

$$\left(16\,cos^5\theta - 20\,cos^3\theta + 5\,cos\,\theta\right) + 5\left(4\,cos^3\theta - 3\,cos\,\theta\right) = 0$$

$$16\,cos^5\theta - 20\,cos^3\theta + 5\,cos\,\theta + 20\,cos^3\theta - 15\,cos\,\theta = 0$$

$$16\,cos^5\theta - 10\,cos\,\theta = 0$$

$$2\,cos\,\theta\left(8\,cos^4\theta - 5\right) = 0$$

Either $cos\,\theta = 0 \Rightarrow \theta = \frac{\pi}{2} = 1.571$

Or $cos\,\theta = \pm\sqrt[4]{\frac{5}{8}} \Rightarrow \theta = 0.475,\ 2.666$

That is, $\theta \in \{0.475,\ 1.571,\ 2.666\}$

[6 marks]

Answer 4
(a)

$$\left(\cos\theta + \mathrm{i}\sin\theta \right)^5 = \cos(5\theta) + \mathrm{i}\sin(5\theta) \qquad \text{by de Moivre's theorem}$$

$$\begin{aligned}
\left(\cos\theta + \mathrm{i}\sin\theta \right)^3 = \;& 1 \times (\cos\theta)^5 \times (\mathrm{i}\sin\theta)^0 \qquad \text{by the Binomial theorem} \\
& + 5 \times (\cos\theta)^4 \times (\mathrm{i}\sin\theta)^1 \\
& + 10 \times (\cos\theta)^3 \times (\mathrm{i}\sin\theta)^2 \\
& + 10 \times (\cos\theta)^2 \times (\mathrm{i}\sin\theta)^3 \\
& + 5 \times (\cos\theta)^1 \times (\mathrm{i}\sin\theta)^4 \\
& + 1 \times (\cos\theta)^0 \times (\mathrm{i}\sin\theta)^5 \\
= \;& \cos^5\theta + \mathrm{i}\,5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\
& - \mathrm{i}\,10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta
\end{aligned}$$

Equating the two expansions gives,

$$\begin{aligned}
\cos(5\theta) + \mathrm{i}\sin(5\theta) = \;& \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\
& + \mathrm{i}\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \right)
\end{aligned}$$

Matching the imaginary part on the LHS with the imaginary part on the RHS yields,

$$\begin{aligned}
\sin(5\theta) &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\
&= 5\left(1 - \sin^2\theta\right)^2 \sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta \\
&= 5\left(1 - 2\sin^2\theta + \sin^4\theta\right)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\
&= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\
&= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta \\
&= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta \qquad \square
\end{aligned}$$

[5 marks]

(b)

$$16x^5 - 20x^3 + 5x + \frac{1}{2} = 0$$

let $x = \sin\theta$ to obtain,

$$16\sin^5\theta - 20\sin^3\theta + 5\sin\theta + \frac{1}{2} = 0$$

$$\sin 5\theta + \frac{1}{2} = 0$$

$$\sin 5\theta = -\frac{1}{2} \qquad \text{(OK to do the next part in degrees)}$$

$$5\theta = -2\pi - \frac{\pi}{6}, -\pi + \frac{\pi}{6}, -\frac{\pi}{6}, \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$

$$5\theta = -\frac{13\pi}{6}, -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\theta = -\frac{13\pi}{30}, -\frac{5\pi}{30}, -\frac{\pi}{30}, \frac{7\pi}{30}, \frac{11\pi}{30}$$

$$x = -0.978, -0.5, -0.105, 0.669, 0.914$$

[5 marks]

(c)

$$\int_0^{\frac{\pi}{4}} \left(4\sin^5\theta - 5\sin^3\theta\right) d\theta = \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(\sin 5\theta - 5\sin\theta\right) d\theta$$

$$= \frac{1}{4}\left[-\frac{\cos 5\theta}{5} + 5\cos\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[\frac{\sqrt{2}}{10} + \frac{5\sqrt{2}}{2}\right] - \frac{1}{4}\left[-\frac{1}{5} + 5\right]$$

$$= \frac{1}{4}\left[\frac{26\sqrt{2}}{10}\right] - \frac{1}{4}\left[\frac{24}{5}\right]$$

$$= \frac{13\sqrt{2}}{20} - \frac{6}{5}$$

[4 marks]

Answer 5

(i)

$\left(\cos\theta + \mathrm{i}\sin\theta \right)^4 = \cos(4\theta) + \mathrm{i}\sin(4\theta)$ by de Moivre's theorem

$$\begin{aligned}\left(\cos\theta + \mathrm{i}\sin\theta \right)^4 &= 1 \times (\cos\theta)^4 \times (\mathrm{i}\sin\theta)^0 \quad \text{by the Binomial theorem}\\ &+ 4 \times (\cos\theta)^3 \times (\mathrm{i}\sin\theta)^1\\ &+ 6 \times (\cos\theta)^2 \times (\mathrm{i}\sin\theta)^2\\ &+ 4 \times (\cos\theta)^1 \times (\mathrm{i}\sin\theta)^3\\ &+ 1 \times (\cos\theta)^0 \times (\mathrm{i}\sin\theta)^4\\ &= \cos^4\theta + \mathrm{i}\,4\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta\\ &\quad - \mathrm{i}\,4\cos\theta\sin^3\theta + \sin^4\theta\end{aligned}$$

Equating the two expansions gives,

$$\begin{aligned}&\cos(4\theta) + \mathrm{i}\sin(4\theta)\\ &\quad = \cos^4\theta + \mathrm{i}\,4\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - \mathrm{i}\,4\cos\theta\sin^3\theta + \sin^4\theta\end{aligned}$$

Matching the imaginary part on the LHS with the imaginary part on the RHS yields,

$$\sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \qquad \square$$

[4 marks]

(ii)

Matching the real part on the LHS with the real part on the RHS yields,

$$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\begin{aligned}\text{LHS} &= \tan 4\theta\\ &= \frac{\sin 4\theta}{\cos 4\theta}\\ &= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \times \frac{\frac{1}{\cos^4\theta}}{\frac{1}{\cos^4\theta}}\\ &= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}\\ &= \text{RHS} \qquad \square\end{aligned}$$

[4 marks]

(iii)

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

Let $x = tan\,\theta$ in which case,

$$tan^4\theta + 4\,tan^3\theta - 6\,tan^2\theta - 4\,tan\,\theta + 1 = 0$$

$$tan^4\theta - 6\,tan^2\theta + 1 = 4\,tan\,\theta - 4\,tan^3\theta$$

$$\frac{4\,tan\,\theta - 4\,tan^3\theta}{1 - 6\,tan^2\theta + tan^4\theta} = 1$$

$$tan\,4\theta = 1$$

$$4\theta = \frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{9\pi}{4},\ \frac{13\pi}{4},\ \ldots$$

$$\theta = \frac{\pi}{16},\ \frac{5\pi}{16},\ \frac{9\pi}{16},\ \frac{13\pi}{16},\ \ldots$$

$$x = 0.20,\ 1.50,\ -5.03,\ -0.67$$

[5 marks]

Answer 6

$$(\cos\theta + i\sin\theta)^7 = \cos(7\theta) + i\sin(7\theta)$$ by de Moivre's theorem

$$\begin{aligned}(\cos\theta + i\sin\theta)^7 = {} & 1 \times (\cos\theta)^7 \times (i\sin\theta)^0 \\ & + 7 \times (\cos\theta)^6 \times (i\sin\theta)^1 \\ & + 21 \times (\cos\theta)^5 \times (i\sin\theta)^2 \\ & + 35 \times (\cos\theta)^4 \times (i\sin\theta)^3 \\ & + 35 \times (\cos\theta)^3 \times (i\sin\theta)^4 \\ & + 21 \times (\cos\theta)^2 \times (i\sin\theta)^5 \\ & + 7 \times (\cos\theta)^1 \times (i\sin\theta)^6 \\ & + 1 \times (\cos\theta)^0 \times (i\sin\theta)^7 \\ = {} & \cos^7\theta + i\,7\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - i\,35\cos^4\theta\sin^3\theta \\ & + 35\cos^3\theta\sin^4\theta + i\,21\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - \sin^7\theta\end{aligned}$$ by the Binomial theorem

Equating the two expansions gives,

$$\begin{aligned}& \cos(7\theta) + i\sin(7\theta) \\ & \quad = \cos^7\theta + i\,7\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - i\,35\cos^4\theta\sin^3\theta \\ & \quad + 35\cos^3\theta\sin^4\theta + i\,21\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta\end{aligned}$$

Matching the real part on the LHS with the real part on the RHS yields,

$$\begin{aligned}& \cos(7\theta) \\ & = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta \\ & = \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2 - 7\cos\theta(1-\cos^2\theta)^3 \\ & = \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1 - 2\cos^2\theta + \cos^4\theta) \\ & \qquad - 7\cos\theta(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta) \\ & = \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta \\ & \qquad - 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta \\ & = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta \qquad \square\end{aligned}$$

[6 marks]