6.5 Answers

6.5.1 Solution to 6.3 Example (Teaching Video)

$$\cos^4\theta = \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)$$

Apply the binomial theorem to $\left(z + \frac{1}{z}\right)^4$

$$\begin{aligned}
\left(z + \frac{1}{z}\right)^4 &= 1 \times z^4 \times \left(\frac{1}{z}\right)^0 \\
&\quad + 4 \times z^3 \times \left(\frac{1}{z}\right)^1 \\
&\quad + 6 \times z^2 \times \left(\frac{1}{z}\right)^2 \\
&\quad + 4 \times z^1 \times \left(\frac{1}{z}\right)^3 \\
&\quad + 1 \times z^0 \times \left(\frac{1}{z}\right)^4 \\
\therefore\ \left(z + \frac{1}{z}\right)^4 &= z^4 + 4z^2 + 6 + 4\left(\frac{1}{z^2}\right) + \left(\frac{1}{z^4}\right) \\
(2\cos\theta)^4 &= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6 \\
16\cos^4\theta &= 2\cos 4\theta + 8\cos 2\theta + 6 \\
8\cos^4\theta &= \cos 4\theta + 4\cos 2\theta + 3 \\
\cos^4\theta &= \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right) \qquad \square
\end{aligned}$$

[4 marks]

6.5.2 Solutions to 6.4 Exercise

Answer 1

(i)

$$\left(z + \frac{1}{z} \right)^2 = \left(z^2 + \frac{1}{z^2} \right) + 2$$

$$(2 \cos\theta)^2 = 2 \cos 2\theta + 2$$

$$4 \cos^2\theta = 2 \cos 2\theta + 2$$

$$\cos^2\theta = \frac{1}{2} \cos 2\theta + \frac{1}{2}$$

[2 marks]

(ii)

$$\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos 2\theta + 1 \, d\theta$$

$$= \frac{1}{2} \left[\frac{1}{2} \sin 2\theta + \theta \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left[0 + \frac{\pi}{2} - 0 - 0 \right]$$

$$= \frac{\pi}{4}$$

[2 marks]

(iii)

$$\left(z - \frac{1}{z} \right)^2 = \left(z^2 + \frac{1}{z^2} \right) - 2$$

$$(2\text{i} \sin\theta)^2 = 2 \cos 2\theta - 2$$

$$-4 \sin^2\theta = 2 \cos 2\theta - 2$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$$

$$\int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{6}} 1 - \cos 2\theta \, d\theta$$

$$= \frac{1}{2} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2} \left[\frac{\pi}{6} - \frac{1}{2} \times \frac{\sqrt{3}}{2} - 0 + 0 \right]$$

$$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

$$= \frac{2\pi - 9\sqrt{3}}{24}$$

[3 marks]

Answer 2

(i)

$$\left(z - \frac{1}{z}\right)^3 = z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$$

$$= \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)$$

$$(2\mathrm{i}\, sin\, \theta)^3 = 2\mathrm{i}\, sin\, 3\theta - 3 \times 2\mathrm{i}\, sin\, \theta$$

$$-8\mathrm{i}\, sin^3\, \theta = 2\mathrm{i}\, sin\, 3\theta - 6\mathrm{i}\, sin\, \theta$$

$$sin^3\, \theta = -\frac{1}{4}\, sin\, 3\theta + \frac{3}{4}\, sin\, \theta$$

That is, $p = -\frac{1}{4}$ and $q = \frac{3}{4}$

[4 marks]

(ii)

$$\text{LHS} = sin\left(\frac{\pi}{2} - \theta\right)$$

$$= sin\left(\frac{\pi}{2}\right) cos\, \theta - cos\left(\frac{\pi}{2}\right) sin\, \theta$$

$$= cos\, \theta$$

$$= \text{RHS} \qquad \square$$

$$\text{LHS} = sin\left(\frac{3\pi}{2} - 3\theta\right)$$

$$= sin\left(\frac{3\pi}{2}\right) cos\, 3\theta - cos\left(\frac{3\pi}{2}\right) sin\, 3\theta$$

$$= -cos\, 3\theta$$

$$= \text{RHS} \qquad \square$$

[2 marks]

(iii) Substitute in $\left(\frac{\pi}{2} - \theta\right)$ in place of θ in

$$sin^3\, \theta = -\frac{1}{4}\, sin\, 3\theta + \frac{3}{4}\, sin\, \theta$$

to get $$sin^3\left(\frac{\pi}{2} - \theta\right) = -\frac{1}{4}\, sin\left(\frac{3\pi}{2} - 3\theta\right) + \frac{3}{4}\, sin\left(\frac{\pi}{2} + \theta\right)$$

$$\therefore\ cos^3\, \theta = \frac{1}{4}\, cos\, 3\theta + \frac{3}{4}\, cos\, \theta$$

i.e. $r = \frac{1}{4}$ and $s = \frac{3}{4}$

[2 marks]

Answer 3

(i)

$$\left(z^2 + \frac{1}{z^2} \right)^3 = z^6 + 3z^2 + \frac{3}{z^2} + \frac{1}{z^6}$$

$$= \left(z^6 + \frac{1}{z^6} \right) + 3\left(z^2 + \frac{1}{z^2} \right)$$

$$= 2\cos 6\theta + 6\cos 2\theta$$

[3 marks]

(ii)

$$\therefore \ (2\cos 2\theta)^3 = 2\cos 6\theta + 6\cos 2\theta$$

$$\cos^3 2\theta = \frac{1}{4}\cos 6\theta + \frac{3}{4}\cos 2\theta$$

$$\text{i.e. } a = \frac{1}{4} \text{ and } b = \frac{3}{4}$$

[3 marks]

(iii)

$$\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \frac{1}{4}\int_0^{\frac{\pi}{6}} \cos 6\theta + 3\cos 2\theta \, d\theta$$

$$= \frac{1}{4}\left[\frac{1}{6}\sin 6\theta + \frac{3}{2}\sin 2\theta \right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{24} \times 0 + \frac{3}{8} \times \frac{\sqrt{3}}{2} - 0 - 0$$

$$= \frac{3\sqrt{3}}{16}$$

$$\text{i.e. } k = \frac{3}{16} \text{ (A rational constant)} \qquad \square$$

[4 marks]

Answer 4

(a)

$$\left(z - \frac{1}{z} \right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

$$(2\mathrm{i}\sin\theta)^5 = \left(z^5 - \frac{1}{z^5} \right) - \left(z^3 - \frac{1}{z^3} \right) + 10\left(z + \frac{1}{z} \right)$$

$$32\mathrm{i}\sin^5\theta = 2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta$$

$$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$$

$$\text{i.e. } a = \frac{1}{16},\ b = -\frac{5}{16} \text{ and } c = \frac{5}{8}$$

[5 marks]

(b)

$$\int_0^{\frac{\pi}{3}} sin^5\,\theta\; d\theta \;=\; \int_0^{\frac{\pi}{3}} \frac{1}{16}\,sin\,5\theta \;-\; \frac{5}{16}\,sin\,3\theta \;+\; \frac{5}{8}\,sin\,\theta\; d\theta$$

$$= \left[-\frac{1}{80}\,cos\,5\theta \;+\; \frac{5}{48}\,cos\,3\theta \;-\; \frac{5}{8}\,cos\,\theta\right]_0^{\frac{\pi}{3}}$$

$$= \left[-\frac{1}{160} \;-\; \frac{5}{48} \;-\; \frac{5}{16}\right] - \left[-\frac{1}{80} \;+\; \frac{5}{48} \;-\; \frac{5}{8}\right]$$

$$= \left[-\frac{203}{480}\right] - \left[-\frac{8}{15} \times \frac{32}{32}\right]$$

$$= \frac{53}{480} \qquad \square$$

[5 marks]

Answer 5

(i)

$$\left(z \;+\; \frac{1}{z}\right)^6 \;=\; z^6 \;+\; 6\,z^4 \;+\; 15\,z^2 \;+\; 20 \;+\; \frac{15}{z^2} \;+\; \frac{6}{z^4} \;+\; \frac{1}{z^6}$$

$$(\,2\,cos\,\theta\,)^6 \;=\; \left(z^6 \;+\; \frac{1}{z^6}\right) \;+\; 6\left(z^4 \;+\; \frac{1}{z^4}\right) \;+\; 15\left(z^2 \;+\; \frac{1}{z^2}\right) \;+\; 20$$

$$64\,cos^6\,\theta \;=\; 2\,cos\,6\theta \;+\; 12\,cos\,4\theta \;+\; 30\,cos\,2\theta \;+\; 20$$

$$32\,cos^6\,\theta \;=\; cos\,6\theta \;+\; 6\,cos\,4\theta \;+\; 15\,cos\,2\theta \;+\; 10$$

i.e. $p \;=\; 1,\;\; q \;=\; 6,\;\; r \;=\; 15$ and $s \;=\; 10$

[5 marks]

(ii)

$$\int_0^{\frac{\pi}{3}} cos^6\,\theta\; d\theta \;=\; \frac{1}{32}\;\int_0^{\frac{\pi}{3}} cos\,6\theta \;+\; 6\,cos\,4\theta \;+\; 15\,cos\,2\theta \;+\; 10\; d\theta$$

$$= \frac{1}{32}\left[\frac{1}{6}\,sin\,6\theta \;+\; \frac{3}{2}\,sin\,4\theta \;+\; \frac{15}{2}\,sin\,2\theta \;+\; 10\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{32}\left[0 \;-\; \frac{3\sqrt{3}}{4} \;+\; \frac{15\sqrt{3}}{4} \;+\; \frac{10\pi}{3} \;-\; 0\right]$$

$$= \frac{3\sqrt{3}}{64} \;+\; \frac{5\pi}{48}$$

[4 marks]

Answer 6

$$\begin{aligned}
\left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^2 &= \left(z - \frac{1}{z}\right)^2 \left(\left(z - \frac{1}{z}\right)\left(z + \frac{1}{z}\right)\right)^2 \\
&= \left(z - \frac{1}{z}\right)^2 \left(z^2 - \frac{1}{z^2}\right)^2 \\
&= \left(\left(z - \frac{1}{z}\right)\left(z^2 - \frac{1}{z^2}\right)\right)^2 \\
&= \left(z^3 - \frac{1}{z} - z + \frac{1}{z^3}\right)^2 \\
&= \left(\left(z^3 + \frac{1}{z^3}\right) - \left(z + \frac{1}{z}\right)\right)^2 \\
&= \left(z^3 + \frac{1}{z^3}\right)^2 - 2\left(z^3 + \frac{1}{z^3}\right)\left(z + \frac{1}{z}\right) + \left(z + \frac{1}{z}\right)^2 \\
&= z^6 + 2 + \frac{1}{z^6} - 2\left(z^4 + z^2 + \frac{1}{z^2} + \frac{1}{z^4}\right) + z^2 + 2 + \frac{1}{z^2} \\
&= z^6 + \frac{1}{z^6} + 2 - 2z^4 - \frac{2}{z^4} - 2z^2 - \frac{2}{z^2} + z^2 + 2 + \frac{1}{z^2} \\
&= z^6 + \frac{1}{z^6} - 2z^4 - \frac{2}{z^4} - z^2 - \frac{1}{z^2} + 4 \\
(2i\sin\theta)^4\ (2\cos\theta)^2 &= 2\cos 6\theta - 2 \times 2\cos 4\theta - 2\cos 2\theta + 4 \\
64\sin^4\theta\cos^2\theta &= 2\cos 6\theta - 4\cos 4\theta - 2\cos 2\theta + 4 \\
\sin^4\theta\cos^2\theta &= \frac{1}{32}\cos 6\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 2\theta + \frac{1}{16}
\end{aligned}$$

That is, $A = \dfrac{1}{32}$, $B = -\dfrac{1}{16}$, $C = -\dfrac{1}{32}$ and $D = \dfrac{1}{16}$

[6 marks]