Answer 1

(i) $$z^3 = \frac{27}{8}\text{i}$$

Begin by observing that $r = \frac{27}{8}$, $\theta = \frac{\pi}{2}$ so $w = \frac{27}{8}\text{i}$ is better expressed as,

$$w = \frac{27}{8}\left(cos\left(\frac{\pi}{2}\right) + \text{i}\,sin\left(\frac{\pi}{2}\right)\right)$$

From the Complex Roots Theorem, the roots will be given by,

$$z_k = \sqrt[3]{\frac{27}{8}}\left(cos\left(\frac{\frac{\pi}{2} + 2\pi k}{3}\right) + \text{i}\,sin\left(\frac{\frac{\pi}{2} + 2\pi k}{3}\right)\right)$$

$$= \frac{3}{2}\left(cos\left(\frac{\pi + 4\pi k}{6}\right) + \text{i}\,sin\left(\frac{\pi + 4\pi k}{6}\right)\right) \text{ for } k = 0,\ 1,\ 2$$

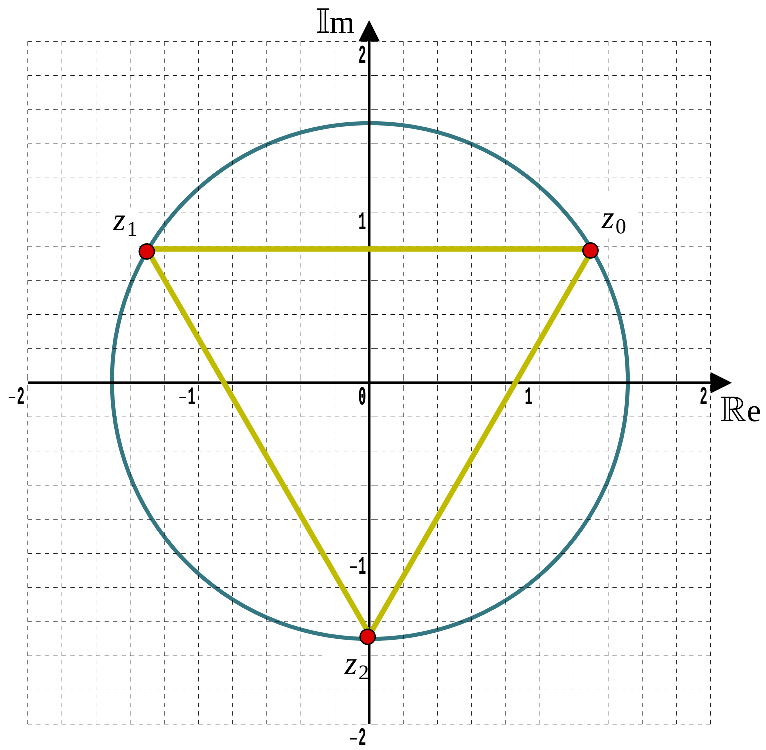
For $k = 0 : z_0 = \frac{3}{2}\left(cos\left(\frac{\pi}{6}\right) + \text{i}\,sin\left(\frac{\pi}{6}\right)\right) = \frac{3\sqrt{3}}{4} + \frac{3}{4}\text{i}$

For $k = 1 : z_1 = \frac{3}{2}\left(cos\left(\frac{5\pi}{6}\right) + \text{i}\,sin\left(\frac{5\pi}{6}\right)\right) = -\frac{3\sqrt{3}}{4} + \frac{3}{4}\text{i}$

For $k = 2 : z_2 = \frac{3}{2}\left(cos\left(\frac{3\pi}{2}\right) + \text{i}\,sin\left(\frac{3\pi}{2}\right)\right) = -\frac{3}{2}\text{i}$

[6 marks]

(ii)



$$Area\,\Delta = \frac{1}{2} \times \frac{6\sqrt{3}}{4} \times \frac{9}{4} = \frac{27\sqrt{3}}{16} \qquad \text{(About 2.923)}$$

[5 marks]

Answer 2

$$z^3 = 4\sqrt{2} - 4\sqrt{2}\,\mathrm{i} = 4\sqrt{2}\,(1 - \mathrm{i})$$

Observe that $r = 8$ and $\theta = -\dfrac{\pi}{4}$ so $w = 4\sqrt{2} - 4\sqrt{2}\,\mathrm{i}$ becomes,

$$w = 8\left(cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\, sin\left(-\frac{\pi}{4}\right)\right)$$

From the Complex Roots Theorem, the roots will be given by,

$$z_k = \sqrt[3]{8}\left(cos\left(\frac{-\frac{\pi}{4} + 2\pi k}{3}\right) + \mathrm{i}\, sin\left(\frac{-\frac{\pi}{4} + 2\pi k}{3}\right)\right)$$

$$= 2\left(cos\left(\frac{-\pi + 8\pi k}{12}\right) + \mathrm{i}\, sin\left(\frac{-\pi + 8\pi k}{12}\right)\right) \text{ for } k = 0,\ 1,\ 2$$

For $k = 0 : z_0 = 2\left(cos\left(\dfrac{-\pi}{12}\right) + \mathrm{i}\, sin\left(\dfrac{-\pi}{12}\right)\right)$

For $k = 1 : z_1 = 2\left(cos\left(\dfrac{7\pi}{12}\right) + \mathrm{i}\, sin\left(\dfrac{7\pi}{12}\right)\right)$

For $k = 2 : z_2 = 2\left(cos\left(\dfrac{5\pi}{4}\right) + \mathrm{i}\, sin\left(\dfrac{5\pi}{4}\right)\right)$

$$= 2\left(cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\, sin\left(-\frac{3\pi}{4}\right)\right)$$

[6 marks]

Answer 3

$$z^3 = -32 - 32\mathrm{i}\sqrt{3} = 32(-1 - \sqrt{3}\,\mathrm{i})$$

Observe that $r = 64$ and $\theta = -\dfrac{2\pi}{3}$ so $w = -32 - 32\mathrm{i}\sqrt{3}$ becomes,

$$w = 64\left(cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\, sin\left(-\frac{2\pi}{3}\right)\right)$$

From the Complex Roots Theorem, the roots will be given by,

$$z_k = \sqrt[3]{64}\left(cos\left(\frac{-\frac{2\pi}{3} + 2\pi k}{3}\right) + \mathrm{i}\, sin\left(\frac{-\frac{2\pi}{3} + 2\pi k}{3}\right)\right)$$

$$= 4\left(cos\left(\frac{-2\pi + 6\pi k}{9}\right) + \mathrm{i}\, sin\left(\frac{-2\pi + 6\pi k}{9}\right)\right) \text{ for } k = 0,\ 1,\ 2$$

For $k = 0 : z_0 = 4\left(cos\left(\dfrac{-2\pi}{9}\right) + \mathrm{i}\, sin\left(\dfrac{-2\pi}{9}\right)\right) = 4\,e^{-\frac{2\pi}{9}\mathrm{i}}$

For $k = 1 : z_1 = 4\left(cos\left(\dfrac{4\pi}{9}\right) + \mathrm{i}\, sin\left(\dfrac{4\pi}{9}\right)\right) = 4e^{\frac{4\pi}{9}\mathrm{i}}$

For $k = 2 : z_2 = 4\left(cos\left(\dfrac{10\pi}{9}\right) + \mathrm{i}\, sin\left(\dfrac{10\pi}{9}\right)\right)$

$$= 4\left(cos\left(-\frac{8\pi}{9}\right) + \mathrm{i}\, sin\left(-\frac{8\pi}{9}\right)\right) = 4e^{-\frac{8\pi}{9}\mathrm{i}}$$

[6 marks]

Answer 4

(i)

$$z^5 - 1 = 0 \Leftrightarrow z^5 = 1$$

Observe that $r = 1$ and $\theta = 0$ so $w = 1$ becomes,

$$w = 1\left(cos\left(0 \right) + \mathrm{i}\, sin\left(0 \right) \right)$$

From the Complex Roots Theorem, the roots will be given by,

$$z_k = \sqrt[5]{1}\left(cos\left(\frac{0 + 2\pi k}{5} \right) + \mathrm{i}\, sin\left(\frac{0 + 2\pi k}{5} \right) \right)$$

$$= \left(cos\left(\frac{2\pi k}{5} \right) + \mathrm{i}\, sin\left(\frac{2\pi k}{5} \right) \right) \text{ for } k = 0,\ 1,\ 2,\ 3,\ 4$$

For $k = 0 : z_0 = \left(cos\left(0 \right) + \mathrm{i}\, sin\left(0 \right)\right)$

For $k = 1 : z_1 = \left(cos\left(\frac{2\pi}{5} \right) + \mathrm{i}\, sin\left(\frac{2\pi}{5} \right)\right)$

For $k = 2 : z_2 = \left(cos\left(\frac{4\pi}{5} \right) + \mathrm{i}\, sin\left(\frac{4\pi}{5} \right)\right)$

For $k = 3 : z_2 = \left(cos\left(\frac{6\pi}{5} \right) + \mathrm{i}\, sin\left(\frac{6\pi}{5} \right)\right)$

$$= \left(cos\left(-\frac{4\pi}{5} \right) + \mathrm{i}\, sin\left(-\frac{4\pi}{5} \right)\right)$$

For $k = 4 : z_2 = \left(cos\left(\frac{8\pi}{5} \right) + \mathrm{i}\, sin\left(\frac{8\pi}{5} \right)\right)$

$$= \left(cos\left(-\frac{2\pi}{5} \right) + \mathrm{i}\, sin\left(-\frac{2\pi}{5} \right)\right)$$

[6 marks]

(ii) The sum of the 5 roots is zero.
This means the real part of the sum must be zero and the imaginary part of the sum must also be zero.
Focussing on the real part,

$$cos\left(0 \right) + cos\left(\frac{2\pi}{5} \right) + cos\left(\frac{4\pi}{5} \right) + cos\left(-\frac{4\pi}{5} \right) + cos\left(-\frac{2\pi}{5} \right) = 0$$

$$1 + cos\left(\frac{2\pi}{5} \right) + cos\left(\frac{4\pi}{5} \right) + cos\left(\frac{4\pi}{5} \right) + cos\left(\frac{2\pi}{5} \right) = 0$$

...the last line is because $cos\ \theta$ is an even function...

$$2\, cos\left(\frac{2\pi}{5} \right) + 2\, cos\left(\frac{4\pi}{5} \right) = -1$$

$$cos\left(\frac{2\pi}{5} \right) + cos\left(\frac{4\pi}{5} \right) = -\frac{1}{2} \qquad \square$$

[3 marks]

Answer 5

(i) Let $v = z + 1$ in which case the equation becomes,

$$v^3 = -1$$

Observe that $r = 1$ and $\theta = \pi$ so $w = -1$ becomes,

$$w = 1\left(\cos(\pi) + \mathrm{i}\sin(\pi)\right)$$

From the Complex Roots Theorem, the roots will be given by,

$$v_k = \sqrt[3]{1}\left(\cos\left(\frac{-\pi + 2\pi k}{3}\right) + \mathrm{i}\sin\left(\frac{-\pi + 2\pi k}{3}\right)\right) \text{ for } k = 0,\ 1,\ 2$$

For $k = 0$: $v_0 = \left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

$$\therefore\ z_0 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$$

For $k = 1$: $v_1 = \left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$

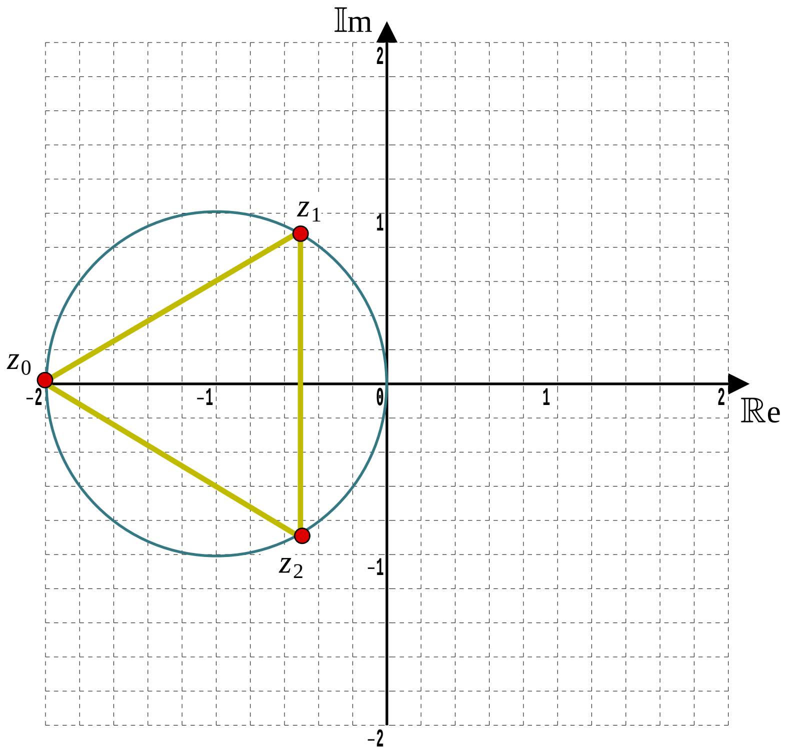
$$\therefore\ z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

For $k = 2$: $v_2 = \left(\cos(\pi) + \mathrm{i}\sin(\pi)\right) = -1$

$$z_2 = -2$$

[6 marks]

(ii)



[2 marks]

(iii) Centre $(-1,\ 0)$ That is, $-1 + 0\mathrm{i}$ and radius 1

[1 mark]

Answer 6

(a)

$$z^4 = 8\left(\sqrt{3} + \mathrm{i}\right)$$

Observe that $r = 16$ and $\theta = \dfrac{\pi}{6}$ so $w = -8\left(\sqrt{3} + \mathrm{i}\right)$ becomes,

$$w = 16\left(cos\left(\frac{\pi}{6}\right) + \mathrm{i}\, sin\left(\frac{\pi}{6}\right)\right)$$

From the Complex Roots Theorem, the roots will be given by,

$$z_k = \sqrt[4]{16}\left(cos\left(\frac{\frac{\pi}{6} + 2\pi k}{4}\right) + \mathrm{i}\, sin\left(\frac{\frac{\pi}{6} + 2\pi k}{4}\right)\right)$$

$$= 2\left(cos\left(\frac{\pi + 12\pi k}{24}\right) + \mathrm{i}\, sin\left(\frac{\pi + 12\pi k}{24}\right)\right) \text{ for } k = 0,\ 1,\ 2,\ 3$$

For $k = 0 : z_0 = 2\left(cos\left(\frac{\pi}{24}\right) + \mathrm{i}\, sin\left(\frac{\pi}{24}\right)\right) = 2e^{\frac{\pi}{24}\mathrm{i}}$

$$\therefore\ z_0 = 1.98 + 0.26\mathrm{i}$$

For $k = 1 : z_1 = 2\left(cos\left(\frac{13\pi}{24}\right) + \mathrm{i}\, sin\left(\frac{13\pi}{24}\right)\right) = 2e^{\frac{13\pi}{24}\mathrm{i}}$

$$\therefore\ z_1 = -0.26 + 1.98\mathrm{i}$$

For $k = 2 : z_2 = 2\left(cos\left(\frac{25\pi}{24}\right) + \mathrm{i}\, sin\left(\frac{25\pi}{24}\right)\right)$

$$= 2\left(cos\left(-\frac{23\pi}{24}\right) + \mathrm{i}\, sin\left(-\frac{23\pi}{24}\right)\right) = 2e^{-\frac{23\pi}{24}\mathrm{i}}$$

$$\therefore\ z_2 = -1.98 - 0.26\mathrm{i}$$

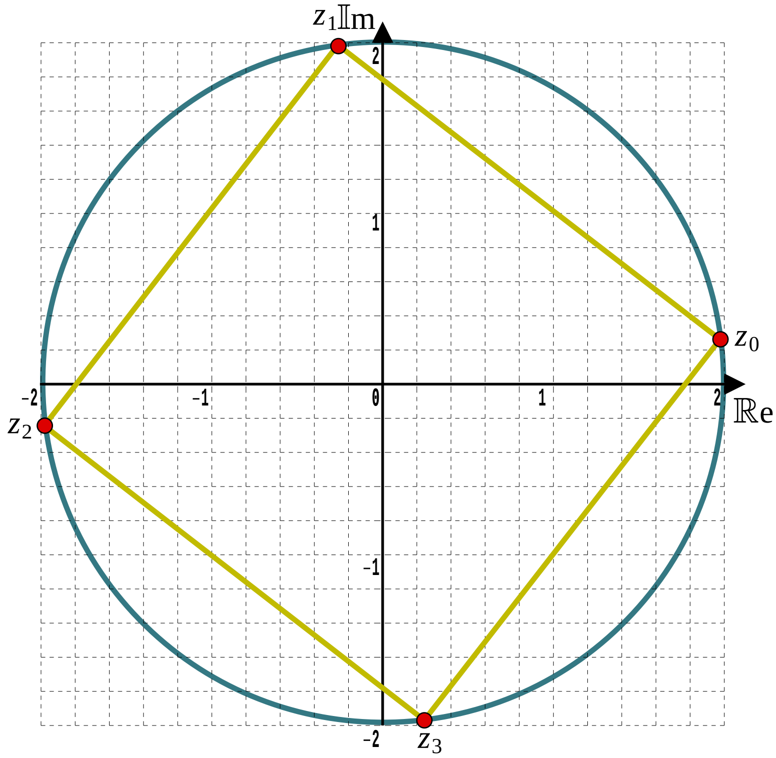
For $k = 3 : z_3 = 2\left(cos\left(\frac{37\pi}{24}\right) + \mathrm{i}\, sin\left(\frac{37\pi}{24}\right)\right)$

$$= 2\left(cos\left(-\frac{11\pi}{24}\right) + \mathrm{i}\, sin\left(-\frac{11\pi}{24}\right)\right) = 2e^{-\frac{11\pi}{24}\mathrm{i}}$$

$$\therefore\ z_3 = 0.26 - 1.98\mathrm{i}$$

[5 marks]

(b)



[2 marks]