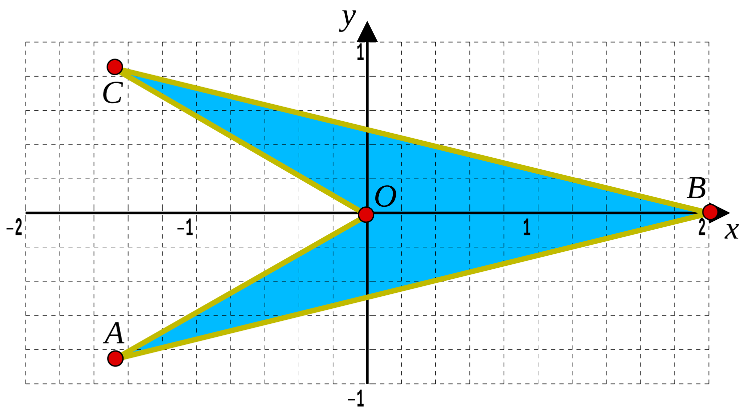
8.5 Answers

8.5.1 Solutions to 8.3 Example (Teaching Video)



The dart can be thought of as lying, not in the Cartesean plane but in the complex plane; the point $B(\,2,\;\;0\,)$ can then be written in exponential form as $2e^{0\mathrm{i}}$

The spiral similarity can be written in exponential form as $\dfrac{\sqrt{3}}{2}\,e^{\pm\frac{5\pi}{6}\mathrm{i}}$

Points A and C are thus found by the following multiplication,

$$\frac{\sqrt{3}}{2}\,e^{\pm\frac{5\pi}{6}} \times 2e^{0\mathrm{i}} = \sqrt{3}\,e^{\pm\frac{5\pi}{6}\mathrm{i}}$$

$$= \sqrt{3}\left(cos\left(\pm\frac{5\pi}{6}\right) + \mathrm{i}\,sin\left(\pm\frac{5\pi}{6}\right)\right)$$

as *cos* is an even function,

and *sin* is an odd function this can be written as

$$= \sqrt{3}\left(cos\left(\frac{5\pi}{6}\right) \pm \mathrm{i}\,sin\left(\frac{5\pi}{6}\right)\right)$$

$$= -\frac{3}{2} \pm \frac{\sqrt{3}}{2}\,\mathrm{i}$$

Although we've been working in the complex plane, the question was originally set in the Cartesean plane; our answer should be given in Cartesean coordinates.

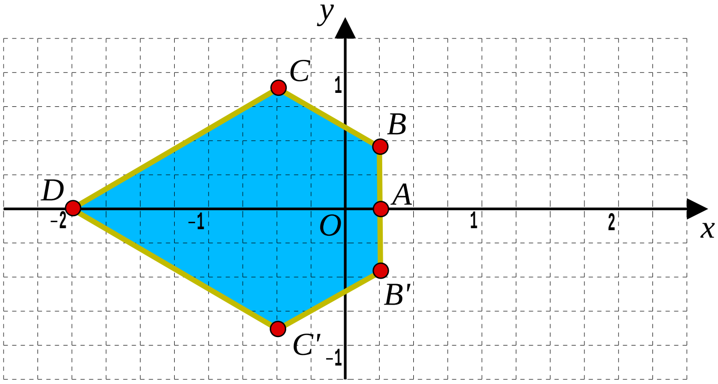
Thus, with the help of the diagram, $A\left(-\dfrac{3}{2}.\;-\;\dfrac{\sqrt{3}}{2}\right)$ and $C\left(-\dfrac{3}{2},\;\;\dfrac{\sqrt{3}}{2}\right)$

The area of the dart is given by $2 \times \dfrac{1}{2} \times 2 \times \dfrac{\sqrt{3}}{2} = \sqrt{3}$ units^2

[6 marks]

8.5.2 Solutions to 8.4 Exercise

Answer 1



The spiral similarity can be written as $2\,e^{\frac{\pi}{3}\mathrm{i}}$ and this is applied first to A which, viewed as a complex number, can be written in exponential form as $\frac{1}{4}\,e^{0\mathrm{i}}$

B has coordinates given by,

$$\begin{aligned} 2\,e^{\frac{\pi}{3}\mathrm{i}} \times \frac{1}{4}\,e^{0\mathrm{i}} &= \frac{1}{2}\,e^{\frac{\pi}{3}\mathrm{i}} \\ &= \frac{1}{2}\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right) \\ &= \frac{1}{4} + \frac{\sqrt{3}}{4}\mathrm{i} \qquad \Leftrightarrow \qquad B\left(\frac{1}{4},\ \frac{\sqrt{3}}{4}\right) \end{aligned}$$

C has coordinates given by,

$$\begin{aligned} 2\,e^{\frac{\pi}{3}\mathrm{i}} \times \frac{1}{2}\,e^{\frac{\pi}{3}\mathrm{i}} &= e^{\frac{2\pi}{3}\mathrm{i}} \\ &= \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right) \\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i} \qquad \Leftrightarrow \qquad C\left(-\frac{1}{2},\ \frac{\sqrt{3}}{2}\right) \end{aligned}$$

D has coordinates given by,

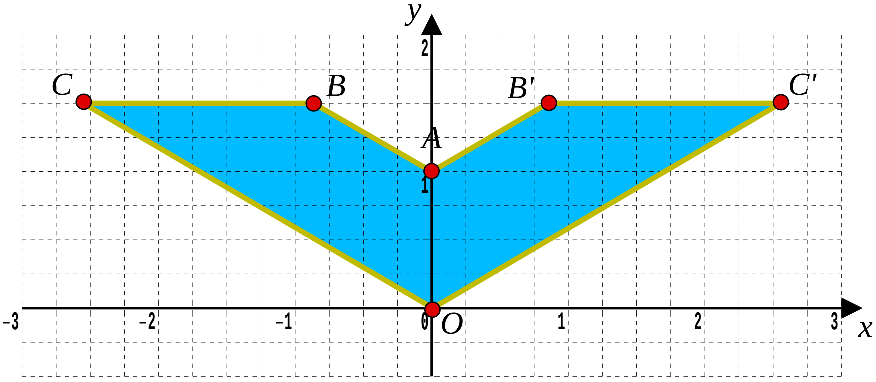
$$\begin{aligned} 2\,e^{\frac{\pi}{3}\mathrm{i}} \times e^{\frac{2\pi}{3}\mathrm{i}} &= 2\,e^{\pi\mathrm{i}} \\ &= -2 \qquad \Leftrightarrow \qquad D(-2,\ 0) \end{aligned}$$

The area can be found in various ways, I cut it vertically CC' into a triangle plus a trapezium;

$$\begin{aligned} Area &= \frac{1}{2} \times \sqrt{3} \times \frac{3}{2} + \frac{1}{2}\left(\sqrt{3} + \frac{\sqrt{3}}{2}\right) \times \frac{3}{4} \\ &= \frac{3\sqrt{3}}{4} + \frac{9\sqrt{3}}{16} \\ &= \frac{21\sqrt{3}}{16} \end{aligned}$$

[7 marks]

Answer 2



The spiral similarity can be written as $\sqrt{3}\, e^{\frac{\pi}{6}i}$ and this is applied first to A which can be thought of as a complex number written in exponential form as $e^{\frac{\pi}{2}i}$

B has coordinates given by,

$$\begin{aligned}\sqrt{3}\, e^{\frac{\pi}{6}i} \times e^{\frac{\pi}{2}i} &= \sqrt{3}\; e^{\frac{2\pi}{3}i} \\ &= \sqrt{3}\left(\cos\left(\frac{2\pi}{3}\right) + i\,\sin\left(\frac{2\pi}{3}\right)\right) \\ &= -\frac{\sqrt{3}}{2} + \frac{3}{2}i \qquad \Leftrightarrow \qquad B\left(-\frac{\sqrt{3}}{2},\ \frac{3}{2}\right)\end{aligned}$$

C has coordinates given by,

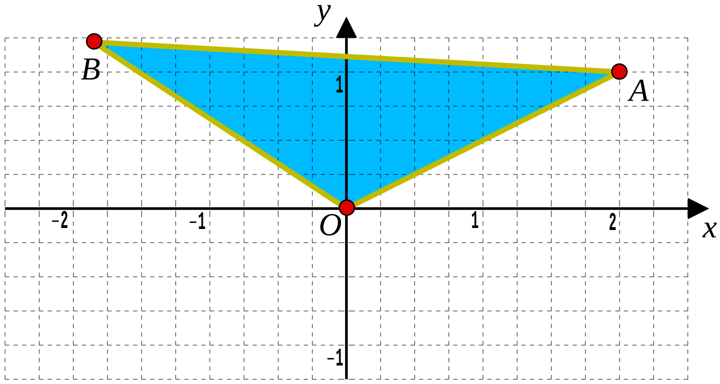
$$\begin{aligned}\sqrt{3}\, e^{\frac{\pi}{6}i} \times \sqrt{3}\, e^{\frac{2\pi}{3}i} &= 3\, e^{\frac{5\pi}{6}i} \\ &= 3\left(\cos\left(\frac{5\pi}{6}\right) + i\,\sin\left(\frac{5\pi}{6}\right)\right) \\ &= -\frac{3\sqrt{3}}{2} + \frac{3}{2}i \qquad \Leftrightarrow \qquad C\left(-\frac{3\sqrt{3}}{2},\ \frac{3}{2}\right)\end{aligned}$$

The area can be found in various ways, I thought of it as the large triangle OCC' minus the small triangle ABB',

$$\begin{aligned}Area &= \frac{1}{2} \times 2\left(-\frac{3\sqrt{3}}{2}\right) \times \frac{3}{2} - \frac{1}{2} \times 2\left(\frac{\sqrt{3}}{2}\right) \times \frac{1}{2} \\ &= \frac{9\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \\ &= 2\sqrt{3}\ \text{units}^2\end{aligned}$$

[7 marks]

Answer 3



(i) Trying to write the point A in exponential form gives $\sqrt{5}\,e^{arctan\left(\frac{1}{2}\right)}$ and the issue now is that $arctan\left(\frac{1}{2}\right)$ is now obviously expressible as an exact quantity. The result is that there is not an obvious way to obtain the exact coordinates of vertex B by this method (even although it worked so well in the previous two questions).

[2 marks]

(ii) Working on the rotation, in exponential form it would be $e^{\frac{2\pi}{3}\mathrm{i}}$ which can be turned into Cartesian form via Euler's relation;

$$e^{\frac{2\pi}{3}\mathrm{i}} = cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\,sin\left(\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i} = \frac{1}{2}\left(-1 + \sqrt{3}\,\mathrm{i}\right)$$

Now the multiplication on A can be performed,

$$\frac{1}{2}\left(-1 + \sqrt{3}\,\mathrm{i}\right)\left(2 + \mathrm{i}\right) = \frac{1}{2}\left(-2 - \mathrm{i} + 2\sqrt{3}\,\mathrm{i} + \sqrt{3}\,\mathrm{i}^2\right)$$

$$= \frac{-2 - \sqrt{3}}{2} + \frac{-1 + 2\sqrt{3}}{2}\,\mathrm{i}$$

So the exact coordinates of B are $\left(\dfrac{-2 - \sqrt{3}}{2}, \dfrac{-1 + 2\sqrt{3}}{2}\right)$

[5 marks]

(iii) Easiest method is to use the cosine rule to get the length of o

$$o^2 = a^2 + b^2 - 2ab\,cos\,O$$

$$= \left(\sqrt{5}\right)^2 + \left(\sqrt{5}\right)^2 - 2\left(\sqrt{5}\right)\left(\sqrt{5}\right)cos\left(\frac{2\pi}{3}\right)$$

$$= 5 + 5 - 10 \times \left(-\frac{1}{2}\right)$$

$$= 15 \Rightarrow o = \sqrt{15}$$

The exact perimeter is $2\sqrt{5} + \sqrt{15}$ or $\sqrt{5}\left(2 + \sqrt{3}\right)$

[2 marks]

Answer 4

(a) The get point B, rotate A by 120°, that is, $\frac{2\pi}{3}$ radians

To do this multiply by $e^{\frac{2\pi}{3}i}$

$$= cos\left(\frac{2\pi}{3}\right) + i\,sin\left(\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$= \frac{1}{2}\left(-1 + \sqrt{3}\,i\right)$$

$$(6 + 2i)\frac{1}{2}\left(-1 + \sqrt{3}\,i\right) = (3 + i)\left(-1 + \sqrt{3}\,i\right)$$

$$= -3 + 3\sqrt{3}\,i - i + \sqrt{3}\,i^2$$

$$= \left(-3 - \sqrt{3}\right) + i\left(-1 + 3\sqrt{3}\right) \quad \text{for } B$$

To get point C, rotate A by – 120°, that is, – $\frac{2\pi}{3}$ radians

To do this multiply by $e^{-\frac{2\pi}{3}i}$

$$= cos\left(-\frac{2\pi}{3}\right) + i\,sin\left(-\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$= \frac{1}{2}\left(-1 - \sqrt{3}\,i\right)$$

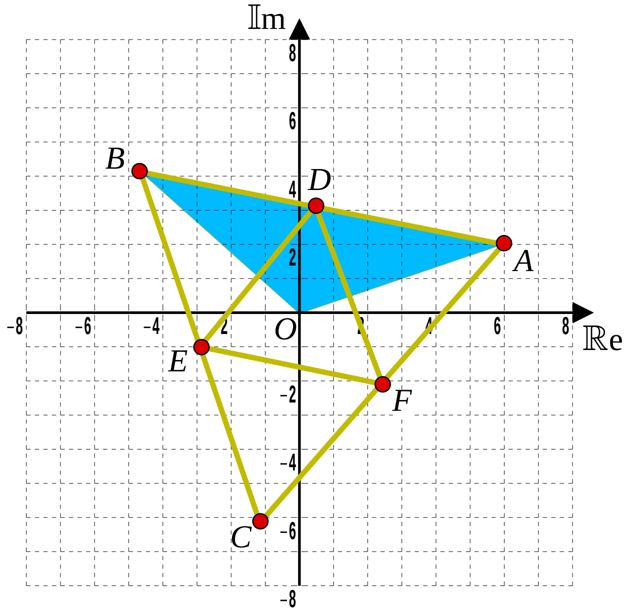
$$(6 + 2i)\frac{1}{2}\left(-1 - \sqrt{3}\,i\right) = (3 + i)\left(-1 - \sqrt{3}\,i\right)$$

$$= -3 - 3\sqrt{3}\,i - i - \sqrt{3}\,i^2$$

$$= \left(-3 + \sqrt{3}\right) + i\left(-1 - 3\sqrt{3}\right) \quad \text{for } C$$

[6 marks]

(b)



There are several ways of answering this question.
I'm going to focus on the blue isosceles triangle, three copies of which will have the same area as the large equilateral triangle ABC

$$Area\ \Delta ABC \ = \ 3 \times \frac{1}{2} \times \sqrt{6^2 + 2^2} \times \sqrt{6^2 + 2^2} \times sin\left(\frac{2\pi}{3}\right)$$

$$= \ 30\sqrt{3}$$

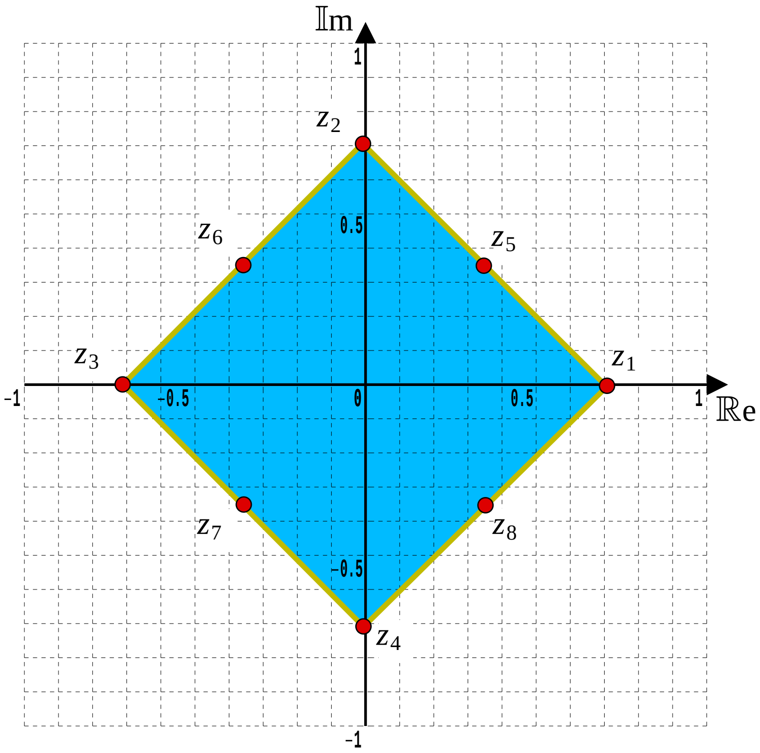
Equilateral triangle DEF has an area that is clearly one quarter of the area of the larger equilateral triangle ABC

$$Area\ \Delta DEF \ = \ \frac{1}{4} \times 30\sqrt{3}$$

$$= \ \frac{15\sqrt{3}}{2}$$

[3 marks]

Answer 5



(i) With reference to the diagram,

$$z_5 = \frac{1}{2}e^{\frac{\pi}{4}i} \qquad z_6 = \frac{1}{2}e^{\frac{3\pi}{4}i} \qquad z_7 = \frac{1}{2}e^{-\frac{3\pi}{4}i} \qquad z_8 = \frac{1}{2}e^{-\frac{\pi}{4}i}$$

As all four roots lie of a circle centre origin and radius 0.5 the equation will be of the form $z^4 = k$, for some constant k

By substituting in one of the points, z_5

$$\left(\frac{1}{2}e^{\frac{\pi}{4}i}\right)^4 = k$$

$$\frac{1}{16}e^{\pi i} = k$$

$$\therefore \; k = -\frac{1}{16}$$

The equation is thus, $z^4 = -\frac{1}{16}$

$$16z^4 + 1 = 0$$

[4 marks]

(ii) The equation cannot be of the form $a\,z^8 + b = 0$ because the roots of such an equation would all lie on the circumference of a circle centred on the origin which these eight roots do not.

[1 mark]

(iii) The factors associated with the original square are,

$$\left(z - \frac{1}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}}\right)\left(z - \frac{1}{\sqrt{2}}\mathrm{i}\right)\left(z + \frac{1}{\sqrt{2}}\mathrm{i}\right) = 0$$

$$\left(z^2 - \frac{1}{2}\right)\left(z^2 + \frac{1}{2}\right) = 0$$

$$z^4 - \frac{1}{4} = 0$$

$$4z^4 - 1 = 0$$

$\therefore$ For $P(z) = 0$

$$P(z) = \left(16z^4 + 1\right)\left(4z^4 - 1\right)$$

$$P(z) = 64z^8 + 20z^4 - 1$$

Any multiple of this will, of course, have the same roots

[4 marks]

Answer 6

(i)

$$e^{\frac{2\pi}{3}i} = cos\left(\frac{2\pi}{3}\right) + i\,sin\left(\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$$

[1 mark]

(ii)

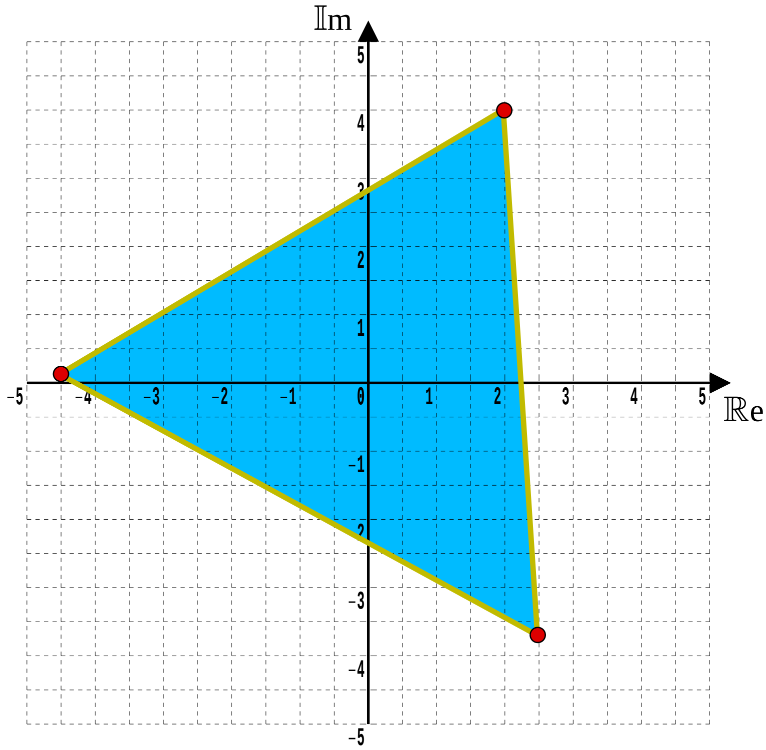
$$\frac{1}{2}\left(-1 + \sqrt{3}\,i\right)\left(2 + 4i\right) = \left(-1 + \sqrt{3}\,i\right)\left(1 + 2i\right)$$

$$= \left(-1 - 2\sqrt{3}\right) + i\left(-2 + \sqrt{3}\right)$$

$$\frac{1}{2}\left(-1 - \sqrt{3}\,i\right)\left(2 + 4i\right) = \left(-1 - \sqrt{3}\,i\right)\left(1 + 2i\right)$$

$$= \left(-1 + 2\sqrt{3}\right) + i\left(-2 - \sqrt{3}\right)$$

[6 marks]



(iii)

$$\sqrt{\left(\left(-1 + 2\sqrt{3}\right) - \left(-1 - 2\sqrt{3}\right)\right)^2 + \left(\left(-2 - \sqrt{3}\right) - \left(-2 + \sqrt{3}\right)\right)^2}$$

$$= \sqrt{\left(-1 + 2\sqrt{3} + 1 + 2\sqrt{3}\right)^2 + \left(-2 - \sqrt{3} + 2 - \sqrt{3}\right)^2}$$

$$= \sqrt{\left(4\sqrt{3}\right)^2 + \left(-2\sqrt{3}\right)^2}$$

$$= \sqrt{16 \times 3 + 4 \times 3} = \sqrt{60} = \sqrt{4 \times 15} = 2\sqrt{15} \qquad \square$$

[2 marks]