9.5 Answers

9.5.1 Solution to 9.3 Example (Teaching Video)

$$\begin{aligned}
e^{-\frac{\theta}{2}\mathrm{i}} + e^{\frac{\theta}{2}\mathrm{i}} &= \cos\left(-\frac{\theta}{2}\right) + \mathrm{i}\sin\left(-\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right) + \mathrm{i}\sin\left(\frac{\theta}{2}\right) \\
&= \cos\left(\frac{\theta}{2}\right) - \mathrm{i}\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right) + \mathrm{i}\sin\left(\frac{\theta}{2}\right) \\
&= 2\cos\left(\frac{\theta}{2}\right)
\end{aligned}$$

To show that, $1 + e^{\theta\mathrm{i}} = 2\,e^{\frac{\theta}{2}\mathrm{i}}\cos\left(\frac{\theta}{2}\right)$

$$\begin{aligned}
\text{RHS} &= 2\,e^{\frac{\theta}{2}\mathrm{i}}\cos\left(\frac{\theta}{2}\right) \\
&= e^{\frac{\theta}{2}\mathrm{i}}\,2\cos\left(\frac{\theta}{2}\right) \\
&= e^{\frac{\theta}{2}\mathrm{i}}\left(e^{-\frac{\theta}{2}\mathrm{i}} + e^{\frac{\theta}{2}\mathrm{i}}\right) \\
&= e^{0\mathrm{i}} + e^{\theta\mathrm{i}} \\
&= 1 + e^{\theta\mathrm{i}} \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[4 marks]

9.5.2 Solutions to 9.4 Exercise

Answer 1

$$\begin{aligned} e^{-\frac{\theta}{2}\mathrm{i}} - e^{\frac{\theta}{2}\mathrm{i}} &= cos\left(-\frac{\theta}{2}\right) + \mathrm{i}\,sin\left(-\frac{\theta}{2}\right) - cos\left(\frac{\theta}{2}\right) - \mathrm{i}\,sin\left(\frac{\theta}{2}\right) \\ &= cos\left(\frac{\theta}{2}\right) - \mathrm{i}\,sin\left(\frac{\theta}{2}\right) - cos\left(\frac{\theta}{2}\right) - \mathrm{i}\,sin\left(\frac{\theta}{2}\right) \\ &= -2\mathrm{i}\,sin\left(\frac{\theta}{2}\right) \end{aligned}$$

To show that, $e^{\theta\mathrm{i}} - 1 = 2\mathrm{i}\,e^{\frac{\theta}{2}\mathrm{i}}\,sin\left(\frac{\theta}{2}\right)$

$$\begin{aligned} \text{RHS} &= 2\mathrm{i}\,e^{\frac{\theta}{2}\mathrm{i}}\,sin\left(\frac{\theta}{2}\right) \\ &= e^{\frac{\theta}{2}\mathrm{i}}\,2\mathrm{i}\,sin\left(\frac{\theta}{2}\right) \\ &= -e^{\frac{\theta}{2}\mathrm{i}}\left(-2\mathrm{i}\,sin\left(\frac{\theta}{2}\right)\right) \\ &= -e^{\frac{\theta}{2}\mathrm{i}}\left(e^{-\frac{\theta}{2}\mathrm{i}} - e^{\frac{\theta}{2}\mathrm{i}}\right) \\ &= -e^{0\mathrm{i}} + e^{\theta\mathrm{i}} \\ &= e^{\theta\mathrm{i}} - 1 \\ &= \text{LHS} \qquad \square \end{aligned}$$

[4 marks]

Answer 2

(i)

$$C + \mathrm{i}S = 1 + \frac{1}{3}\left(\cos\theta + \mathrm{i}\sin\theta \right) + \frac{1}{9}\left(\cos 2\theta + \mathrm{i}\sin 2\theta \right) + \frac{1}{27}\left(\cos 3\theta + \mathrm{i}\sin 3\theta \right) + \ldots$$

$$= 1 + \frac{1}{3}e^{\mathrm{i}\theta} + \frac{1}{9}e^{\mathrm{i}2\theta} + \frac{1}{27}e^{\mathrm{i}3\theta} + \ldots$$

$$= \frac{1}{1 - \left(\frac{1}{3}e^{\mathrm{i}\theta} \right)} \times \frac{3}{3} \qquad \text{sum to infinity of a GP}$$

$$= \frac{3}{3 - e^{\mathrm{i}\theta}} \qquad \square$$

[4 marks]

(ii)

$$= \frac{3}{3 - \cos\theta - \mathrm{i}\sin\theta} \times \frac{3 - \cos\theta + \mathrm{i}\sin\theta}{3 - \cos\theta + \mathrm{i}\sin\theta}$$

$$= \frac{9 - 3\cos\theta + 3\mathrm{i}\sin\theta}{9 - 3\cos\theta - 3\mathrm{i}\sin\theta - 3\cos\theta + \cos^2\theta + \mathrm{i}\sin\theta\cos\theta + 3\mathrm{i}\sin\theta - \mathrm{i}\sin\theta\cos\theta - \mathrm{i}^2\sin^2\theta}$$

$$= \frac{9 - 3\cos\theta + 3\mathrm{i}\sin\theta}{9 - 3\cos\theta - 3\cos\theta + 1}$$

$$= \frac{9 - 3\cos\theta + 3\mathrm{i}\sin\theta}{10 - 6\cos\theta}$$

$$C + \mathrm{i}S = \frac{9 - 3\cos\theta + 3\mathrm{i}\sin\theta}{10 - 6\cos\theta}$$

$$= \frac{9 - 3\cos\theta}{10 - 6\cos\theta} + \mathrm{i}\,\frac{3\sin\theta}{10 - 6\cos\theta}$$

$$\therefore\ C = \frac{9 - 3\cos\theta}{10 - 6\cos\theta} \qquad \square$$

$$\text{and } S = \frac{3\sin\theta}{10 - 6\cos\theta}$$

[4 marks]

Answer 3
(i)

$$C + \mathrm{i}S = \cos\theta + \mathrm{i}\sin\theta$$
$$+ \frac{1}{2}\left(\cos 5\theta + \mathrm{i}\sin 5\theta\right)$$
$$+ \frac{1}{4}\left(\cos 9\theta + \mathrm{i}\sin 9\theta\right)$$
$$+ \frac{1}{8}\left(\cos 13\theta + \mathrm{i}\sin 13\theta\right)$$
$$+ \ldots$$
$$= e^{\mathrm{i}\theta} + \frac{1}{2}e^{\mathrm{i}5\theta} + \frac{1}{4}e^{\mathrm{i}9\theta} + \frac{1}{8}e^{\mathrm{i}13\theta} + \ldots$$
$$= \frac{e^{\mathrm{i}\theta}}{1 - \left(\frac{1}{2}e^{\mathrm{i}4\theta}\right)} \times \frac{2}{2} \qquad \text{sum to infinity of a GP}$$
$$= \frac{2e^{\mathrm{i}\theta}}{2 - e^{\mathrm{i}4\theta}} \qquad \square$$

[4 marks]

(ii)

$$\frac{2e^{\mathrm{i}\theta}}{2 - e^{\mathrm{i}4\theta}} = \frac{2\cos\theta + 2\mathrm{i}\sin\theta}{2 - \cos 4\theta - \mathrm{i}\sin 4\theta} \times \frac{2 - \cos 4\theta + \mathrm{i}\sin 4\theta}{2 - \cos 4\theta + \mathrm{i}\sin 4\theta}$$

The “clutter” can be reduced by noting that all of the imaginary terms in the denominator cancel each other out when complex conjugates multiply.
Also, as only the imaginary part of the answer is sought, there is no need to work out any of the real terms in the numerator.

$$\mathrm{i}S = \frac{2\mathrm{i}\cos\theta\sin 4\theta + 4\mathrm{i}\sin\theta - 2\mathrm{i}\sin\theta\cos 4\theta}{4 - 2\cos 4\theta - 2\cos 4\theta + \cos^2 4\theta + \sin^2 4\theta}$$

Carefully shuffling the pieces around yields,

$$S = \frac{4\sin\theta + 2\left(\sin 4\theta\cos\theta - \cos 4\theta\sin\theta\right)}{4 - 2\cos 4\theta - 2\cos 4\theta + 1}$$
$$= \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta} \qquad \square$$

[4 marks]

Answer 4

(i) $$e^{\mathrm{i}n\theta} = \cos n\theta + \mathrm{i} \sin n\theta$$

[1 mark]

(ii) $$e^{-\mathrm{i}n\theta} = \cos n\theta - \mathrm{i} \sin n\theta$$

[1 mark]

(iii)

$$e^{\mathrm{i}n\theta} = \cos n\theta + \mathrm{i} \sin n\theta$$

$$\text{ADD} \quad e^{-\mathrm{i}n\theta} = \cos n\theta - \mathrm{i} \sin n\theta$$

$$e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta} = 2 \cos n\theta$$

$$\therefore \quad \cos n\theta = \frac{1}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta} \right) \qquad \square$$

[1 mark]

(iv)

$$e^{\mathrm{i}n\theta} = \cos n\theta + \mathrm{i} \sin n\theta$$

$$\text{SUBTRACT} \quad e^{-\mathrm{i}n\theta} = \cos n\theta - \mathrm{i} \sin n\theta$$

$$e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta} = 2\mathrm{i} \sin n\theta$$

$$\therefore \quad \sin n\theta = \frac{1}{2}\left(e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta} \right) \qquad \square$$

[1 mark]

Answer 5

$$\sum_{k=0}^{7} (1 + \mathrm{i})^k = \sum_{k=0}^{7} \left(\sqrt{2}\, e^{\frac{\pi}{4}\mathrm{i}} \right)^k$$

$$= 1 + \sqrt{2}\, e^{\frac{\pi}{4}\mathrm{i}} + \left(\sqrt{2}\, e^{\frac{\pi}{4}\mathrm{i}} \right)^2 + \ldots + \left(\sqrt{2}\, e^{\frac{\pi}{4}\mathrm{i}} \right)^7$$

$$= \frac{1\left(1 - \left(\sqrt{2}\, e^{\frac{\pi}{4}\mathrm{i}} \right)^8 \right)}{\left(1 - \left(\sqrt{2}\, e^{\frac{\pi}{4}\mathrm{i}} \right) \right)}$$

$$= \frac{1 - 16\, e^{\frac{8\pi}{4}\mathrm{i}}}{1 - \sqrt{2} \cos\left(\frac{\pi}{4}\right) - \sqrt{2}\, \mathrm{i} \sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{1 - 16 \cos(2\pi) - 16\, \mathrm{i} \sin(2\pi)}{1 - 1 - \mathrm{i}} \times \frac{\mathrm{i}}{\mathrm{i}}$$

$$= \mathrm{i} - 16\mathrm{i}$$

$$= -15\mathrm{i} \qquad \square$$

[4 marks]

Answer 6

(i)

$$\begin{aligned}
\text{LHS} &= (2 + e^{i\theta})(2 + e^{-i\theta}) \\
&= 4 + 2e^{i\theta} + 2e^{-i\theta} + e^{0i} \\
&= 4 + 2\cos\theta + 2i\sin\theta + 2\cos(-\theta) + 2i\sin(-\theta) + 1 \\
&= 5 + 2\cos\theta + 2i\sin\theta + 2\cos\theta - 2i\sin\theta \qquad \text{even and odd functions} \\
&= 5 + 4\cos\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned}
C - iS &= 1 \\
&\quad - \frac{1}{2}(\cos\theta + i\sin\theta) \\
&\quad + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) \\
&\quad - \frac{1}{8}(\cos 3\theta + i\sin 3\theta) \\
&\quad + \ldots \\
&= 1 - \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{i2\theta} - \frac{1}{8}e^{i3\theta} + \ldots \\
&= \frac{1}{1 + \frac{1}{2}e^{i\theta}} \times \frac{2}{2} \\
&= \frac{2}{2 + e^{i\theta}} \\
&= \frac{2}{2 + \cos\theta + i\sin\theta} \times \frac{2 + \cos\theta - i\sin\theta}{2 + \cos\theta - i\sin\theta} \\
&= \frac{4 + 2\cos\theta - 2i\sin\theta}{4 + 2\cos\theta + 2\cos\theta + \cos^2\theta + \sin^2\theta} \\
&= \frac{4 + 2\cos\theta - 2i\sin\theta}{5 + 4\cos\theta}
\end{aligned}$$

$$\therefore C = \frac{4 + 2\cos\theta}{5 + 4\cos\theta} \quad \text{and} \quad S = \frac{\sin\theta}{5 + 4\cos\theta}$$

Notice that the minus sign was already in the expression $C - iS$

[4 marks]

Answer 7
(i)

$$\text{LHS} = 1 + e^{2\theta \mathrm{i}}$$
$$= 1 + \cos 2\theta + \mathrm{i} \sin 2\theta$$
$$= \cos^2\theta + \sin^2\theta + \cos^2\theta - \sin^2\theta + \mathrm{i}\, 2 \sin\theta \cos\theta$$
$$= 2\cos^2\theta + 2\mathrm{i} \sin\theta \cos\theta$$
$$= 2\cos\theta\ (\cos\theta + \mathrm{i}\sin\theta)$$
$$= \text{RHS} \qquad \square$$

[2 marks]

(ii)

$$C + \mathrm{i}S = 1$$
$$+ \binom{n}{1}(\cos 2\theta + \sin 2\theta)$$
$$+ \binom{n}{2}(\cos 4\theta + \sin 4\theta)$$
$$+ \ldots$$
$$+ \binom{n}{n}(\cos 2n\theta + \mathrm{i}\sin 2n\theta)$$
$$= \binom{n}{0} e^{0\mathrm{i}} + \binom{n}{1} e^{2\theta\mathrm{i}} + \binom{n}{2} e^{4\theta\mathrm{i}} + \ldots + \binom{n}{n} e^{2n\theta\mathrm{i}}$$
$$= \binom{n}{0}\left(e^{2\theta\mathrm{i}}\right)^0 + \binom{n}{1}\left(e^{2\theta\mathrm{i}}\right)^1 + \binom{n}{2}\left(e^{2\theta\mathrm{i}}\right)^2 + \ldots + \binom{n}{n}\left(e^{2\theta\mathrm{i}}\right)^n$$
$$= \left(1 + e^{2\theta\mathrm{i}}\right)^n \qquad \text{Binomial Theorem “backwards”}$$
$$= (2\cos\theta\ (\cos\theta + \mathrm{i}\sin\theta))^n \qquad \text{using the part (i) result}$$
$$= 2^n \cos^n\theta\ (\cos\theta + \mathrm{i}\sin\theta)^n$$
$$= 2^n \cos^n\theta\ (\cos n\theta + \mathrm{i}\sin n\theta) \qquad \text{by de Moivre's Theorem}$$
$$\therefore\ C = 2^n \cos^n\theta \cos n\theta \qquad \square$$
$$\text{and } S = 2^n \sin^n\theta \sin n\theta$$

[7 marks]

Answer 8

(i)

$$\sum_{k=0}^{3} \mathrm{i}^k = \mathrm{i}^0 + \mathrm{i}^1 + \mathrm{i}^2 + \mathrm{i}^3$$

$$= 1 + \mathrm{i} - 1 - \mathrm{i}$$

$$= 0$$

[1 mark]

(ii)

$$\sum_{k=0}^{15} \mathrm{i}^k = \sum_{k=0}^{3} \mathrm{i}^k - \mathrm{i}^0 + \sum_{k=4}^{7} \mathrm{i}^k + \sum_{k=8}^{11} \mathrm{i}^k + \sum_{k=12}^{15} \mathrm{i}^k$$

$$= 0 - 1 + 0 + 0 + 0$$

$$= -1$$

[2 marks]

(iii)

$$e^{\frac{\pi}{2}\mathrm{i}} = \mathrm{i} \qquad \therefore \sum_{k=0}^{12} z^k = \sum_{k=0}^{12} \mathrm{i}^k$$

$$= \sum_{k=0}^{3} \mathrm{i}^k + \sum_{k=4}^{7} \mathrm{i}^k + \sum_{k=8}^{11} \mathrm{i}^k + \mathrm{i}^{12}$$

$$= \sum_{m=0}^{2} \left(\sum_{k=4m}^{4m+3} \mathrm{i}^k \right) + \mathrm{i}^{12}$$

$$= \sum_{m=0}^{2} (0) + 1$$

$$= 1$$

[2 marks]

(iv)

$$e^{-\frac{\pi}{2}\mathrm{i}} = -\mathrm{i} \qquad \therefore \sum_{k=0}^{121} z^k = \sum_{k=0}^{121} (-\mathrm{i})^k$$

$$= \sum_{m=0}^{29} \left(\sum_{k=4m}^{4m+3} \mathrm{i}^k \right) + (-\mathrm{i})^{120} + (-\mathrm{i})^{121}$$

$$= \sum_{m=0}^{29} (0) + 1 - \mathrm{i}$$

$$= 1 - \mathrm{i}$$

[2 marks]

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