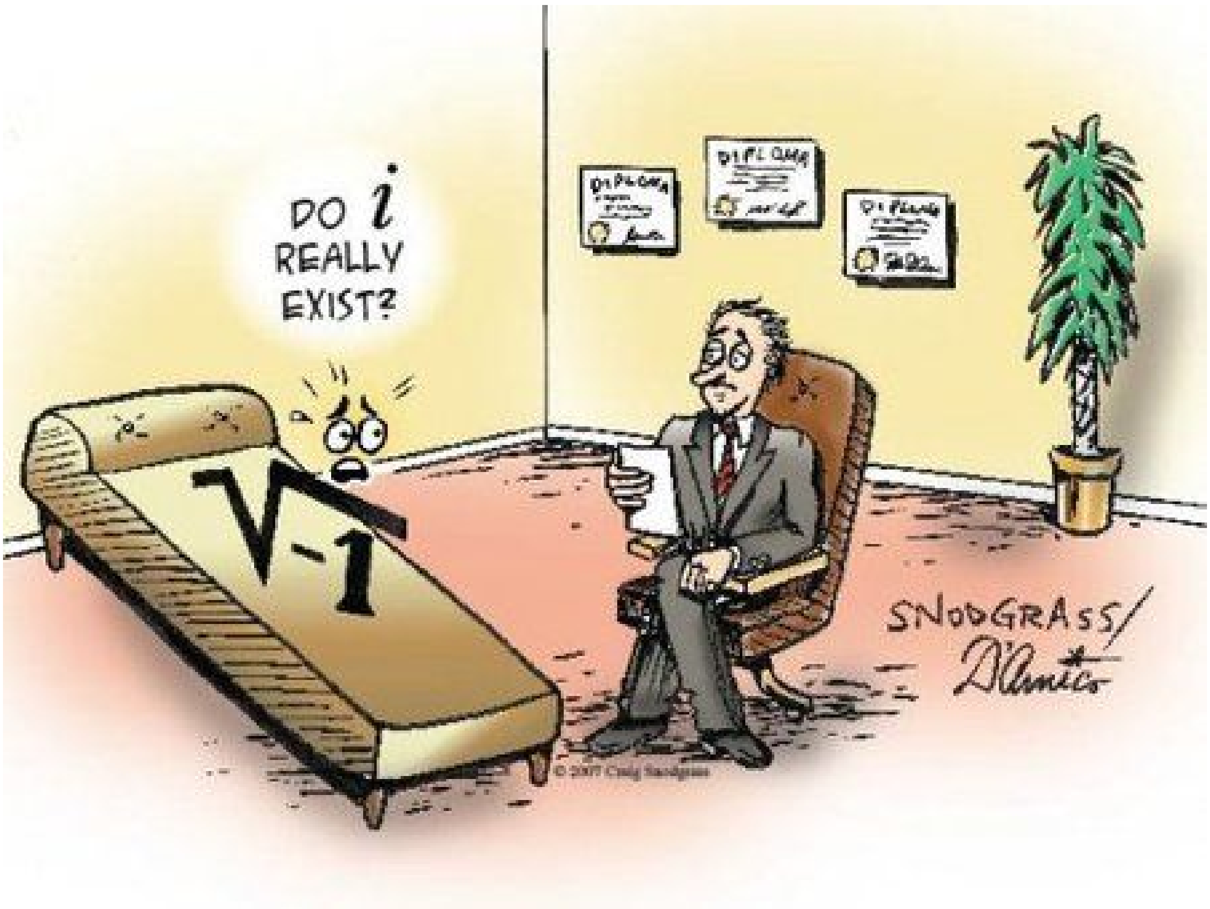
Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

COMPLEX NUMBERS II



Cartoon by Craig Snodgrass

COMPLEX NUMBERS II

Lesson 1

Further A-Level Pure Mathematics, Core 2
Complex Numbers II

1.1 Odd and Even Functions

Functions can be usefully classified as being one of three types,

- Odd
- Even
- Neither odd nor even

The terminology comes from the fact that $f(x) = x^n$ is an odd function if n is odd and an even function if n is even.

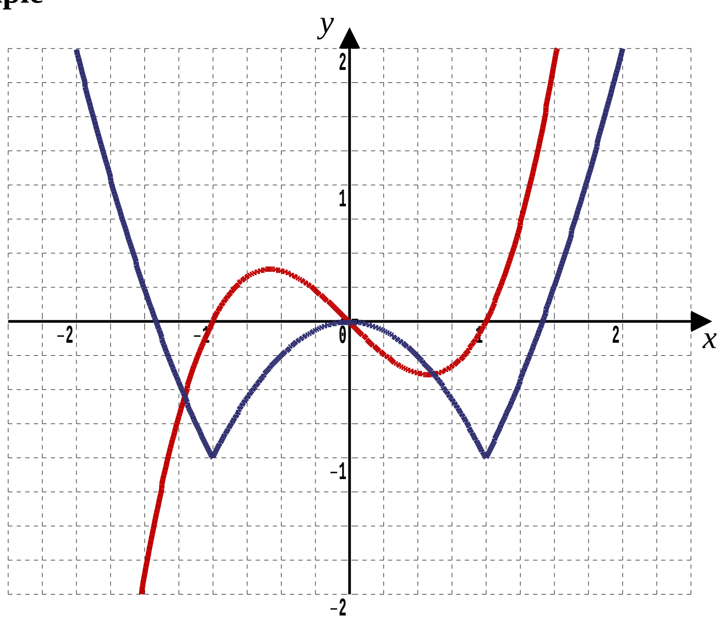
Odd Functions

If a function $f(x)$ has the property that $f(-x) = -f(x)$
then that function is described as being "odd".
Its graph has half-turn rotational symmetry about the origin.

Even Functions

If a function $f(x)$ has the property that $f(-x) = f(x)$
then that function is described as being "even".
Its graph has mirror symmetry in the y-axis.

1.2 Example



The odd function $f(x) = x^3 - x$ (like all odd functions) has a graph, shown red, that has half-turn rotational symmetry about the origin.

The even function $g(x) = \left|x^2 - 1\right| - 1$ (like all even functions) has a graph, shown blue, that has mirror symmetry in the y-axis.

1.3 Odd and Even Proofs

Prove that **(i)** $f(x) = x^3 - x$ is an odd function

(ii) $g(x) = |x^2 - 1| - 1$ is an even function

Teaching Video : http://www.NumberWonder.co.uk/v9099/1.mp4

[4 marks]

1.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Prove that $n(x) = \dfrac{3x^4 + 1}{x}$ is an odd function

[3 marks]

Question 2

$$h(x) = \sqrt{1 + x + x^2} + \sqrt{1 - x + x^2}$$

Prove that $h(x)$ is an even function

[4 marks]

Question 3

(i) What feature of the graph of $f(x) = sin\,x$ suggests that the sine function is an odd function ?

[1 mark]

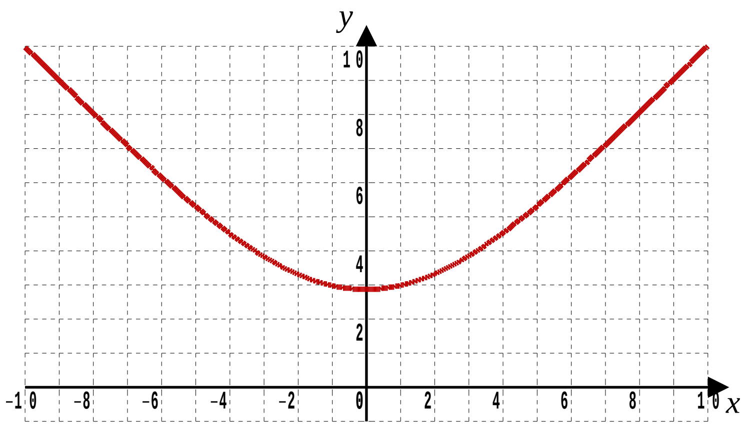
(ii) Prove that $f(x) = sin\,x$ is an odd function by making use of its Maclaurin series

[4 marks]

Question 4

$$v(x) = x \, \frac{2^x + 1}{2^x - 1}$$

The graph of $v(x)$ suggests that it may be an even function,



Prove that $v(x)$ is indeed an even function

[4 marks]

Question 5

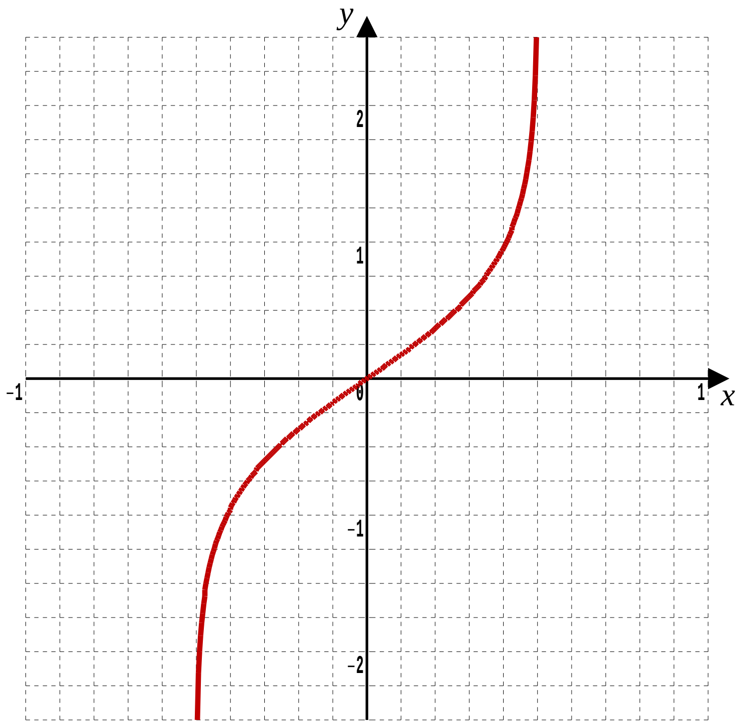
Prove that $s(x) = \dfrac{(1 + e^x)^2}{e^x}$ is an even function

[5 marks]

Question 6

$$L(x) = ln\left(\frac{0.5 + x}{0.5 - x}\right) \qquad x \in \mathbb{R}, \quad -0.5 < x < 0.5$$

The graph of $L(x)$ suggests that it may be an odd function,



Prove that $L(x)$ is indeed an odd function

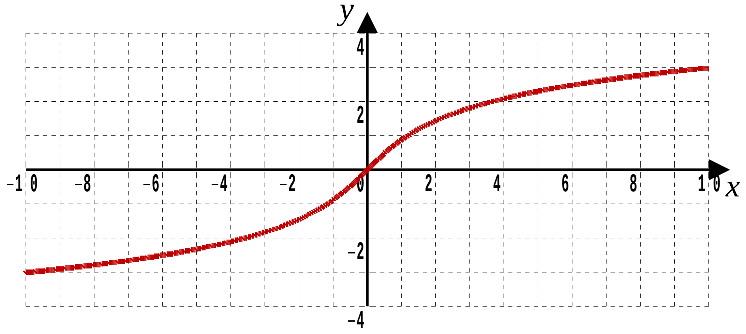
[5 marks]

Question 7

It can be shown† that,

- the derivative of an odd function is an even function
- the derivative of an even function is an odd function

The graph of $k(x) = ln\left(x + \sqrt{1 + x^2}\right)$ is given below and suggests that this may be an odd function



(i) Show that the derivative of $k(x)$ is $\dfrac{1}{\sqrt{1 + x^2}}$

[6 marks]

(ii) Hence prove that $k(x)$ is indeed an odd function

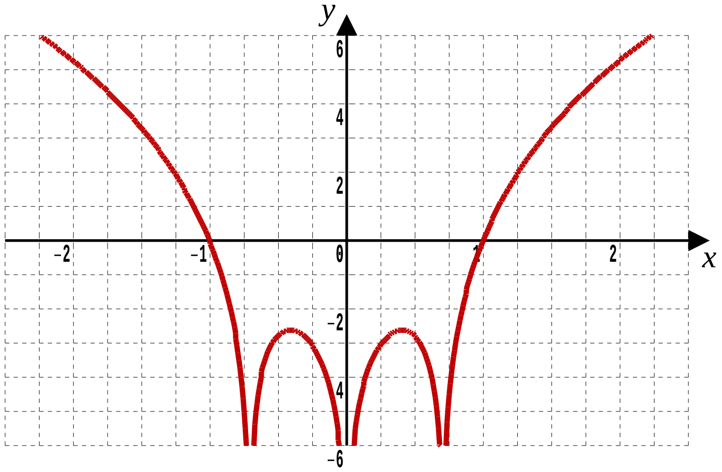
[2 marks]

† See Question 9

Question 8

$$p(x) = ln\left(2x^3 - x\right)^2 \qquad x \in \mathbb{R},\ x \neq 0,\ x \neq \pm\sqrt{\frac{1}{2}}$$

The graph of $p(x)$ suggests that it may be an even function,



Prove that $p(x)$ is indeed an even function by,

(i) First showing that $p'(x)$ is an odd function

[4 marks]

(ii) only algebra and no differentiation

[4 marks]

Question 9

(i) Study the following proof,

- The derivative of an odd function is an even function

Suppose that $f\,(x)$ is an odd function in which case $f\,(-x) = -f\,(x)$

$$\therefore \qquad -\ f\,(x) = f\,(-x)$$

differentiate both sides

$$-f'\,(x) = f'\,(-x) \times (-1) \text{ by the chain rule}$$

$$-f'\,(x) = -f'\,(-x)$$

$$\text{That is,} \qquad f'\,(x) = f'\,(-x)$$

Which is saying the derivative of f (which was an odd function) is an even function $\square$

(ii) Prove the following,

- The derivative of an even function is an odd function

[4 marks]

Question 10

Given a function $f(x)$ give a reason why $f(|x|)$ is guaranteed to be an even function but $|f(x)|$ is not.

[4 marks]

1.5 Answers

1.5.1 Solution to 1.3 Example (Teaching Video)

(i)

$$f(x) = x^3 - x$$
$$\therefore\ f(-x) = (-x)^3 - (-x)$$
$$= -x^3 + x$$
$$= -(x^3 - x)$$
$$= -f(x) \qquad \therefore\ f(x) \text{ is an odd function} \quad \square$$

[2 marks]

(ii)

$$g(x) = \left|x^2 - 1\right| - 1$$
$$\therefore\ g(-x) = \left|(-x)^2 - 1\right| - 1$$
$$= \left|x^2 - 1\right| - 1$$
$$= g(x) \qquad \therefore\ g(x) \text{ is an even function} \quad \square$$

[2 marks]

1.5.2 Solution to 1.4 Exercise

Answer 1

$$n(x) = \frac{3x^4 + 1}{x}$$
$$\Rightarrow \quad n(-x) = \frac{3(-x)^4 + 1}{(-x)}$$
$$= -\frac{3x^4 + 1}{x}$$
$$= -n(x)$$
$$\therefore\ n(x) \text{ is an odd function} \quad \square$$

[3 marks]

Answer 2

$$h(x) = \sqrt{1 + x + x^2} + \sqrt{1 - x + x^2}$$
$$\Rightarrow h(-x) = \sqrt{1 + (-x) + (-x)^2} + \sqrt{1 - (-x) + (-x)^2}$$
$$= \sqrt{1 - x + x^2} + \sqrt{1 + x + x^2}$$
$$= \sqrt{1 + x + x^2} + \sqrt{1 - x + x^2}$$
$$= h(x)$$
$$\therefore\ h(x) \text{ is an even function} \quad \square$$

[4 marks]

Answer 3

(i) The graph of $f(x) = sin\, x$ has half-turn rotational symmetry about the origin which is the hallmark of an odd function.

[1 mark]

(ii)

$$f(x) = sin\, x$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$
$$\Rightarrow \ f(-x) = sin(-x)$$
$$= (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \ldots$$
$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots$$
$$= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \right)$$
$$= -f(x)$$

$\therefore \ f(x)$ is an odd function ☐

[4 marks]

Answer 4

$$v(x) = x \, \frac{2^x + 1}{2^x - 1}$$
$$\Rightarrow \ v(-x) = (-x) \, \frac{2^{-x} + 1}{2^{-x} - 1}$$
$$= -x \, \frac{2^{-x} + 1}{2^{-x} - 1} \times \frac{2^x}{2^x}$$
$$= -x \, \frac{1 + 2^x}{1 - 2^x}$$
$$= x \, \frac{2^x + 1}{2^x - 1}$$
$$= v(x)$$

$\therefore \ v(x)$ is an even function ☐

[4 marks]

Answer 5

$$s(x) = \frac{(1 + e^{x})^2}{e^{x}}$$

$$\Rightarrow\ s(-x) = \frac{(1 + e^{-x})^2}{e^{-x}}$$

$$= \frac{1 + 2\,e^{-x} + e^{-2x}}{e^{-x}}$$

$$= \frac{1}{e^{-x}} + \frac{2\,e^{-x}}{e^{-x}} + \frac{e^{-2x}}{e^{-x}}$$

$$= e^{x} + 2 + e^{-x}$$

$$= \frac{e^{2x}}{e^{x}} + \frac{2\,e^{x}}{e^{x}} + \frac{1}{e^{x}}$$

$$= \frac{e^{2x} + 2\,e^{x} + 1}{e^{x}}$$

$$= \frac{(e^{x} + 1)^2}{e^{x}}$$

$$= \frac{(1 + e^{x})^2}{e^{x}}$$

$$= s(x)$$

$\therefore\ s(x)$ is an even function $\square$

[5 marks]

Answer 6

$$L(x) = ln\left(\frac{0.5 + x}{0.5 - x}\right)$$

$$\Rightarrow\ L(-x) = ln\left(\frac{0.5 + (-x)}{0.5 - (-x)}\right)$$

$$= ln\left(\frac{0.5 - x}{0.5 + x}\right)$$

$$= ln(0.5 - x) - ln(0.5 + x)$$

$$= -\left(-ln(0.5 - x) + ln(0.5 + x)\right)$$

$$= -\left(ln(0.5 + x) - ln(0.5 - x)\right)$$

$$= -ln\left(\frac{0.5 + x}{0.5 - x}\right)$$

$$= -L(x)$$

$\therefore\ L(x)$ is an odd function $\square$

[5 marks]

Answer 7

(i)

$$k(x) = ln\left(x + \sqrt{1 + x^2} \right)$$

$$= ln\left(x + \left(1 + x^2 \right)^{\frac{1}{2}} \right)$$

$$k'(x) = \frac{1}{\left(x + \sqrt{1 + x^2} \right)} \times \left(1 + \frac{1}{2}\left(1 + x^2 \right)^{-\frac{1}{2}} \times 2x \right)$$

$$= \frac{1}{\left(x + \sqrt{1 + x^2} \right)} \times \left(1 + \frac{x}{\sqrt{1 + x^2}} \right)$$

$$= \frac{1}{\left(x + \sqrt{1 + x^2} \right)} \times \left(\frac{\sqrt{1 + x^2}}{\sqrt{1 + x^2}} + \frac{x}{\sqrt{1 + x^2}} \right)$$

$$= \frac{1}{\left(x + \sqrt{1 + x^2} \right)} \times \frac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}}$$

$$= \frac{1}{\sqrt{1 + x^2}} \qquad \square$$

[6 marks]

(ii)

$$\Rightarrow k'(-x) = \frac{1}{\sqrt{1 + (-x)^2}}$$

$$= \frac{1}{\sqrt{1 + x^2}}$$

$$= k'(x)$$

$\therefore$ $k'(x)$ is an even function and so $k(x)$ is an odd function $\square$

[2 marks]

Answer 8
(i)

$$\begin{aligned} p(x) &= ln\left(2x^3 - x\right)^2 \\ &= 2\,ln\left(2x^3 - x\right) \\ p'(x) &= \frac{2}{\left(2x^3 - x\right)} \times \left(6x^2 - 1\right) \\ &= \frac{2\left(6x^2 - 1\right)}{\left(2x^3 - x\right)} \end{aligned}$$

$$\begin{aligned} \Rightarrow\ p'(-x) &= \frac{2\left(6\ (-x)^2 - 1\right)}{\left(2\ (-x)^3 - (-x)\ \right)} \\ &= \frac{2\ \left(6x^2 - 1\right)}{(-1)\left(2x^3 + x\right)} \\ &= -\frac{2\ \left(6x^2 - 1\right)}{\left(2x^3 - x\right)} \\ &= -p'(x) \end{aligned}$$

$\therefore\ p'(x)$ is an odd function and so $p(x)$ is an even function ☐

[4 marks]

(ii) Kevin's solution (2nd November 2022),

$$\begin{aligned} p(x) &= ln\left(2x^3 - x\right)^2 \\ &= ln\left(4x^6 - 4x^4 + x^2\right) \end{aligned}$$

$$\begin{aligned} \Rightarrow\ p(-x) &= ln\left(\ 4(-x)^6 - 4(-x)^4 + (-x)^2\ \right) \\ &= ln\left(4x^6 - 4x^4 + x^2\ \right) \\ &= p(x) \end{aligned}$$

$\therefore\ p(x)$ is an even function ☐

[4 marks]

Answer 9
(ii)

- The derivative of an even function is an odd function

Suppose that $f\,(x)$ is an even function in which case $f\,(-\,x) \;=\; f\,(x)$

$$\therefore \quad f\,(x) \;=\; f\,(-x)$$

differentiate both sides

$$f'\,(x) \;=\; f'\,(-\,x) \;\times\; (-\,1) \quad \text{by the chain rule}$$

$$f'\,(x) \;=\; -\,f'\,(-\,x)$$

Which is saying the derivative of f (which was an even function)
is an odd function ☐

[4 marks]

Answer 10

Let $g\,(x) \;=\; |\,x\,|$ and let $fg\,(x)$ be the function $h\,(x)$

By definition, $|\,(-\,x)\,| \;=\; |\,x\,|$ or, in other words, $g\,(-\,x) \;=\; g\,(x)$

$$\therefore \text{ As } g\,(-\,x) \;=\; g\,(x)$$

$$\text{it follows that } fg\,(-\,x) \;=\; fg\,(x)$$

$$\text{or } h\,(-\,x) \;=\; h\,(x)$$

which is the definition of an even function

but this is simply saying, $f\,(\,|\,(-\,x)\,|\,) \;=\; f\,(\,|\,x\,|\,)$

So, $f\,(\,|\,x\,|\,)$ is an even function

To show that $|\,f\,(x)\,|$ is not guaranteed to be an even function, simply find a a counter-example where it is an odd function or a counter-example where it is neither odd nor even.

[4 marks]

2.1 Tales Of The Infinite

For real x, the exponential function, e^x, can be written as the series expansion,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^r}{r!} + \ldots \qquad x \in \mathbb{R}$$

The utility of this function is enhanced by defining it to also apply when x is a complex number and exploring the consequences.

Of particular interest is how the series expansion can be manipulated when x is the complex number, $x = \mathrm{i}\theta$ where θ is a real number constant.

$$\begin{aligned}
e^{\theta\,\mathrm{i}} &= 1 + \mathrm{i}\,\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \frac{(\mathrm{i}\theta)^5}{5!} + \frac{(\mathrm{i}\theta)^6}{6!} + \ldots \\
&= 1 + \theta\mathrm{i} + \frac{\mathrm{i}^2\theta^2}{2!} + \frac{\mathrm{i}^3\theta^3}{3!} + \frac{\mathrm{i}^4\theta^4}{4!} + \frac{\mathrm{i}^5\theta^5}{5!} + \frac{\mathrm{i}^6\theta^6}{6!} + \ldots \\
&= 1 + \theta\mathrm{i} - \frac{\theta^2}{2!} - \frac{\mathrm{i}\,\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\mathrm{i}\,\theta^5}{5!} - \frac{\theta^6}{6!} + \ldots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right) \\
&= \cos\theta + \mathrm{i}\sin\theta
\end{aligned}$$

Euler's Relation

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

2.2 Exponential Form

The everyday use of Euler's Relation is in writing complex numbers that are in

- rectangular form $z = a + \mathrm{i}b$ Also called Cartesian form

or
- polar form $z = r(\cos\theta + \mathrm{i}\sin\theta)$ Also called Trigonometric form

into the exponential form,

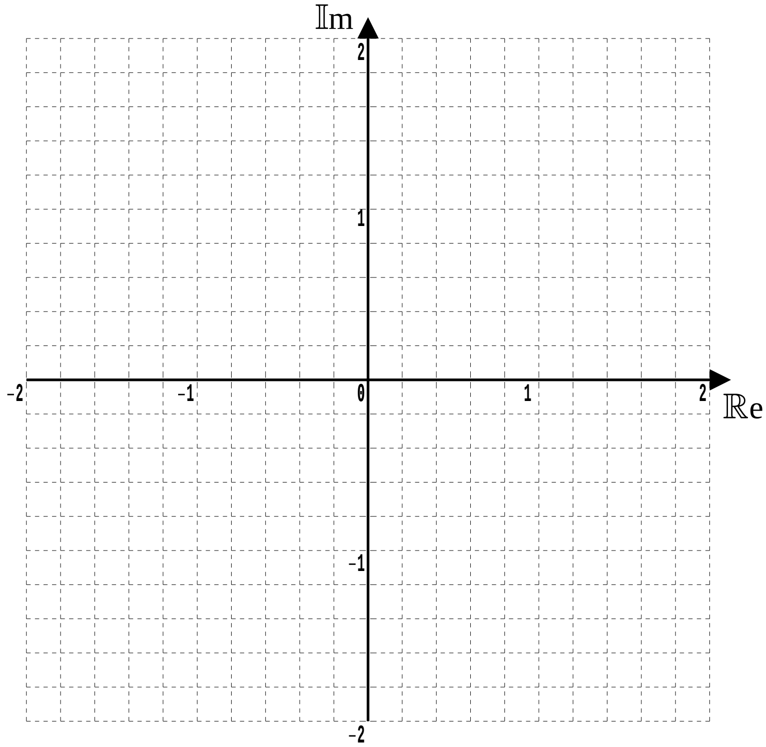
$$z = r\,e^{\mathrm{i}\theta} \text{ where } -\pi < \theta \leqslant \pi$$

Note : $r = |z|$ and $\theta = arg(z)$

2.3 Examples

Plot the following on an Argand diagram, then express them in exponential form,

(i) $\quad -1-\sqrt{3}\,\mathrm{i}$ $\qquad\qquad$ **(ii)** $\quad \sqrt{3}\left(cos\left(\frac{\pi}{6}\right) - \mathrm{i}\,sin\left(\frac{\pi}{6}\right)\right)$



Teaching Video : http://www.NumberWonder.co.uk/v9099/2a.mp4
http://www.NumberWonder.co.uk/v9099/2b.mp4

<= Part 1 $\qquad$ Part 2 =>

[3, 3 marks]

2.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 56

Question 1

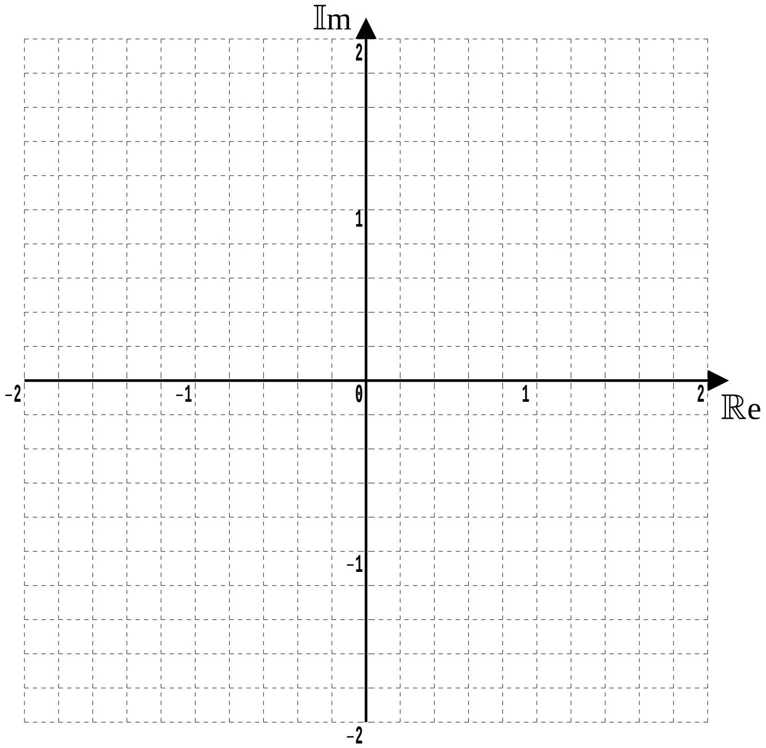
Plot the following on an Argand diagram, then express them in exponential form,

(i) $z_A = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right)$

(ii) $z_B = \frac{3}{2}\left(\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)$

(iii) $z_C = \sqrt{3}\left(\cos\left(\frac{5\pi}{6}\right) - \mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right)$

(iv) $z_D = 2\left(\cos\left(\frac{\pi}{5}\right) - \mathrm{i}\sin\left(\frac{\pi}{5}\right)\right)$



[3, 3, 3, 3 marks]

Question 2

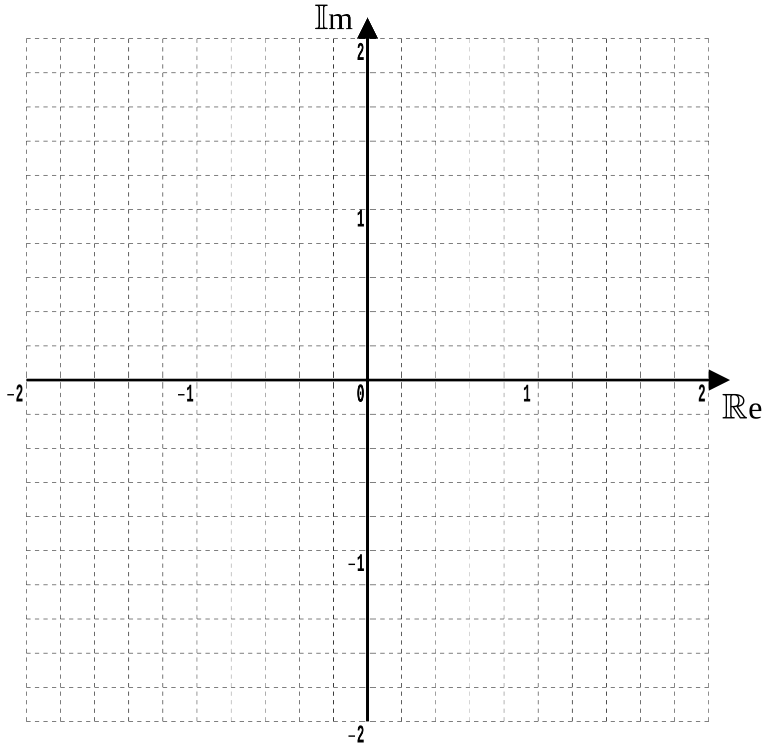
Plot the following on an Argand diagram, then express them in exponential form,

(i) $z_A = \dfrac{6}{5} + \dfrac{8}{5}\,\mathrm{i}$

(ii) $z_B = -\dfrac{3}{2} + \dfrac{1}{2}\,\mathrm{i}$

(iii) $z_C = -2\,\mathrm{i}$

(iv) $z_D = -1 - \dfrac{4}{3}\,\mathrm{i}$



[3, 3, 3, 3 marks]

Question 3

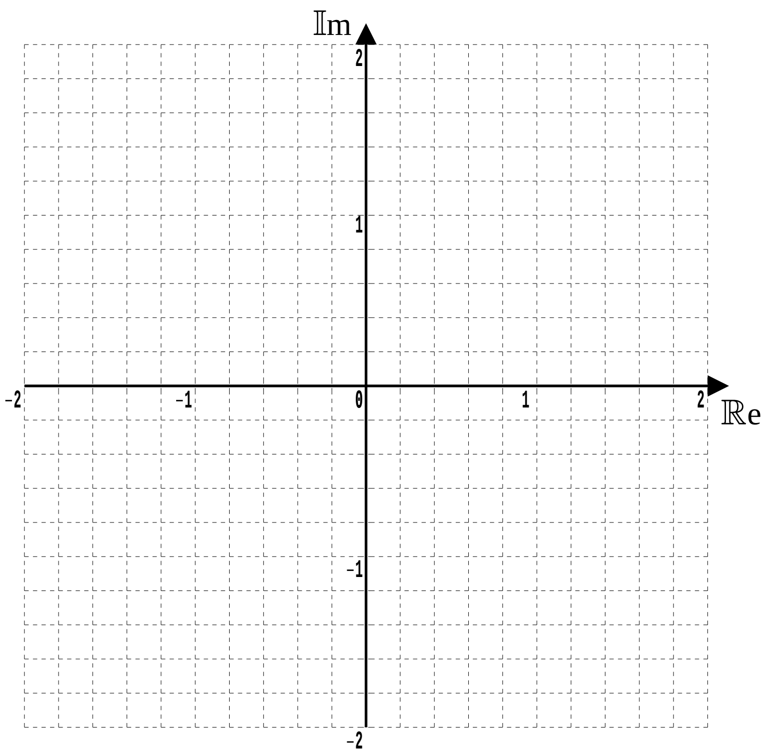
Express the following in the form $x + \mathrm{i}y$ where $x, y \in \mathbb{R}$ and also plot the points on an Argand diagram,

(i) $z_A = e^{\frac{\pi}{6}\mathrm{i}}$

(ii) $z_B = 2\,e^{\frac{3\pi}{2}\mathrm{i}}$

(iii) $z_C = \dfrac{\sqrt{2}}{2}\,e^{-\frac{\pi}{4}\mathrm{i}}$

(iv) $z_D = \dfrac{2}{3}\,e^{-\frac{4\pi}{5}\mathrm{i}}$



[3, 3, 3, 3 marks]

Question 4

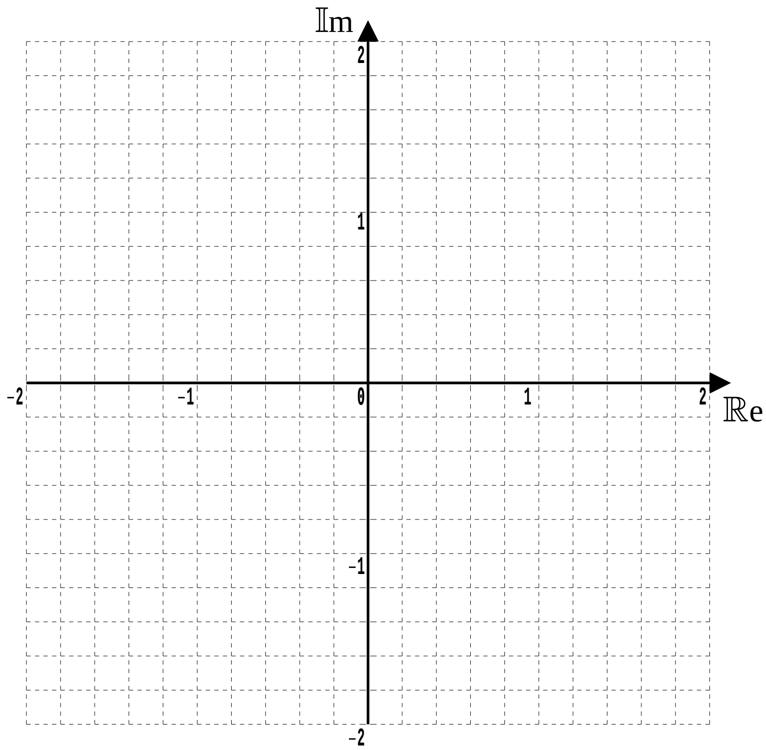
Express the following in the polar form $z = r\,(\,cos\,\theta + \mathrm{i}\,sin\,\theta\,)\,,\quad -\pi < \theta \leqslant \pi$ and plot the point on an Argand diagram,

(i) $z_A = \sqrt{3}\,e^{\frac{3\pi}{2}\mathrm{i}}$

(ii) $z_B = e^{\pi\,\mathrm{i}}$

(iii) $z_C = \sqrt{2}\,e^{-\frac{17\pi}{4}\mathrm{i}}$

(iv) $z_D = \dfrac{3}{2}\,e^{-\frac{4\pi}{3}\mathrm{i}}$



[3, 3, 3, 3 marks]

Question 5

(i) Use Euler's Relation to show that $\sin\theta = \frac{1}{2\mathrm{i}}\left(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}\right)$

[4 marks]

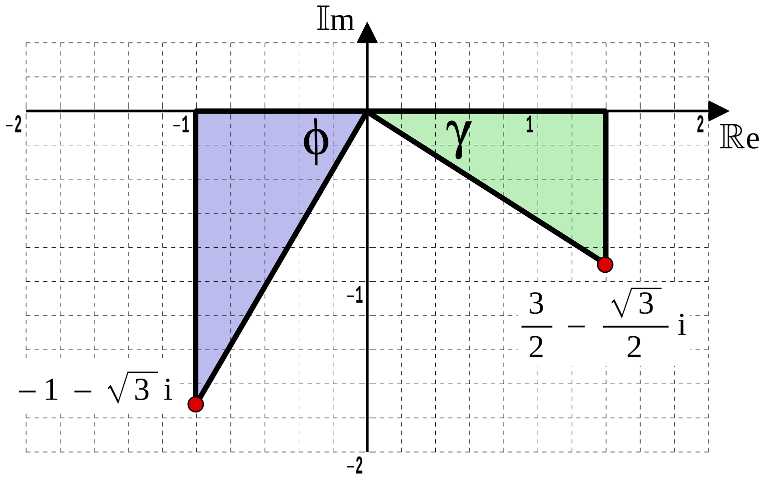
(ii) Suggest a similar result for $\cos\theta$ and demonstrate its validity, again by using Euler's Relation

[4 marks]

2.5 Answers

2.5.1 Solution to 2.3 Example (Teaching Video)

(i)



For the purple triangle with $z = -1 - \sqrt{3}\,\mathrm{i}$

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} \qquad \tan\phi = \frac{\sqrt{3}}{1}$$

$$= \sqrt{1+3} \qquad \phi = \arctan(\sqrt{3})$$

$$= 2 \qquad \phi = \frac{\pi}{3}$$

Angle required for exponential form is $-\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$

$$\therefore z = 2\,e^{-\frac{2\pi}{3}\mathrm{i}}$$

[3 marks]

For the green triangle with $z = \sqrt{3}\left(\cos\left(\frac{\pi}{6}\right) - \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right)$

As $\cos\theta$ is an even function, $\cos\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$

As $\sin\theta$ is an odd function, $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$

$$\therefore z = \sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$$

$$\therefore z = \sqrt{3}\,e^{-\frac{\pi}{6}\mathrm{i}}$$

[3 marks]

2.5.2 Solution to 2.4 Exercise

Answer 1

(i) $z_A = \sqrt{2}\left(cos\left(\frac{\pi}{4}\right) + \mathrm{i}\, sin\left(\frac{\pi}{4}\right)\right) = 1 + \mathrm{i}$

$z_A = \sqrt{2}\, e^{\frac{\pi}{4}\mathrm{i}}$

(ii) $z_B = \frac{3}{2}\left(cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\, sin\left(\frac{2\pi}{3}\right)\right) = -\frac{3}{4} + \frac{3\sqrt{3}}{4}\mathrm{i} = -0.75 + 1.30\,\mathrm{i}$

$z_B = \frac{3}{2}\, e^{\frac{2\pi}{3}\mathrm{i}}$

(iii) $z_C = \sqrt{3}\left(cos\left(\frac{5\pi}{6}\right) - \mathrm{i}\, sin\left(\frac{5\pi}{6}\right)\right) = -\frac{3}{2} - \frac{\sqrt{3}}{2}\mathrm{i} = -1.5 - 0.866\,\mathrm{i}$

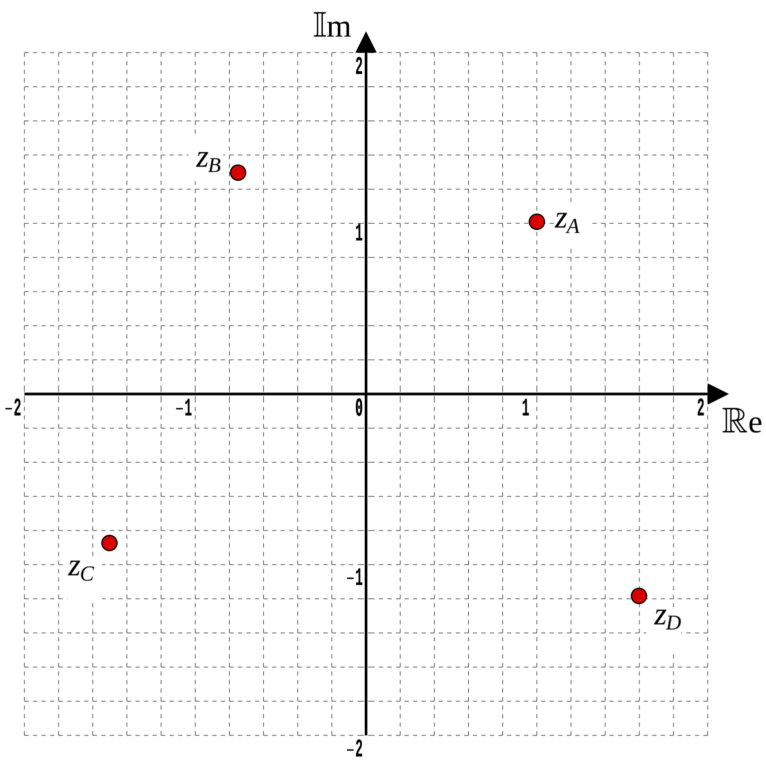
$= \sqrt{3}\left(cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\, sin\left(-\frac{5\pi}{6}\right)\right)$

$z_C = \sqrt{3}\, e^{-\frac{5\pi}{6}\mathrm{i}}$

(iv) $z_D = 2\left(cos\left(\frac{\pi}{5}\right) - \mathrm{i}\, sin\left(\frac{\pi}{5}\right)\right) = 1.62 - 1.18\,\mathrm{i}$

$= 2\left(cos\left(-\frac{\pi}{5}\right) + \mathrm{i}\, sin\left(-\frac{\pi}{5}\right)\right)$

$z_D = 2\, e^{-\frac{\pi}{5}\mathrm{i}}$



[3, 3, 3, 3 marks]

Answer 2

(i) $z_A = \dfrac{6}{5} + \dfrac{8}{5}\,\mathrm{i}$

$$|z| = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{8}{5}\right)^2} \qquad \tan\phi = \frac{8}{5} \div \frac{6}{5} = \frac{4}{3}$$

$$= \sqrt{\frac{36}{25} + \frac{64}{25}} = 2 \qquad \phi = \arctan\left(\frac{4}{3}\right) = 0.927$$

$$\theta = \phi \qquad \therefore z = 2\,e^{0.927\,\mathrm{i}}$$

(ii) $z_B = -\dfrac{3}{2} + \dfrac{1}{2}\,\mathrm{i}$

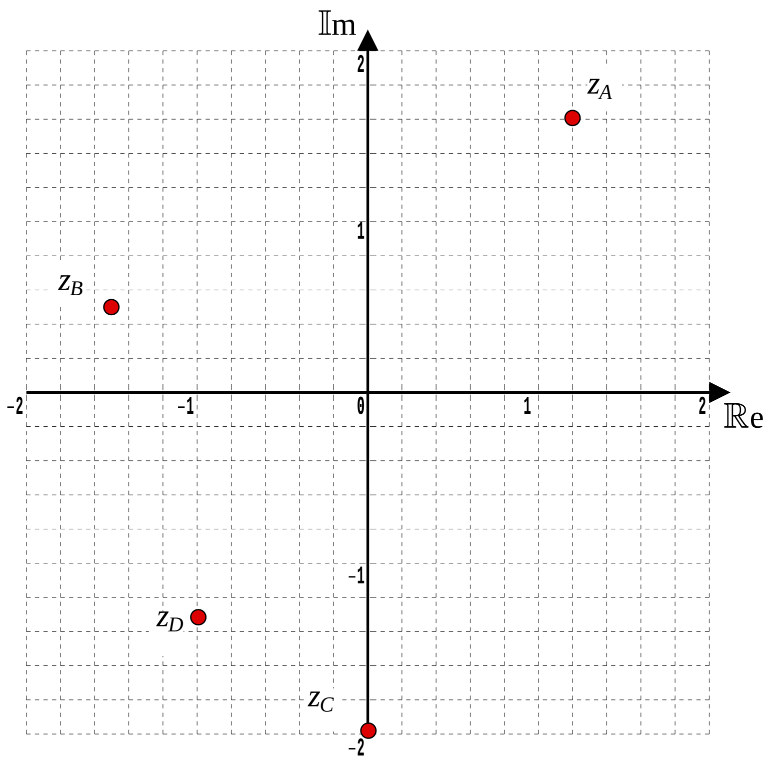
$$|z| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \qquad \tan\phi = \frac{1}{2} \div \frac{3}{2} = \frac{1}{3}$$

$$= \sqrt{\frac{9}{4} + \frac{1}{4}} = \frac{\sqrt{10}}{2} \qquad \phi = \arctan\left(\frac{1}{3}\right) = 0.321$$

$$\theta = \pi - \phi = 2.82 \qquad \therefore z = \frac{\sqrt{10}}{2}\,e^{2.82\,\mathrm{i}}$$

(iii) $z_C = -2\,\mathrm{i}$

$$|z| = 2 \qquad \theta = -\frac{\pi}{2} \qquad \therefore z = 2\,e^{-\frac{\pi}{2}\mathrm{i}}$$



(iv) $z_D = -1 - \frac{4}{3}\,\mathrm{i}$

$$|z| = \sqrt{\left(-\frac{3}{3}\right)^2 + \left(-\frac{4}{3}\right)^2} \qquad \tan\phi = \frac{4}{3} \div 1 = \frac{4}{3}$$

$$= \sqrt{\frac{9}{9} + \frac{16}{9}} = \frac{5}{3} \qquad \phi = \arctan\left(\frac{4}{3}\right) = 0.927$$

$$\theta = -\pi + \phi = -2.21 \qquad \therefore z = \frac{5}{3}\,e^{-2.21\,\mathrm{i}}$$

[3, 3, 3, 3 marks]

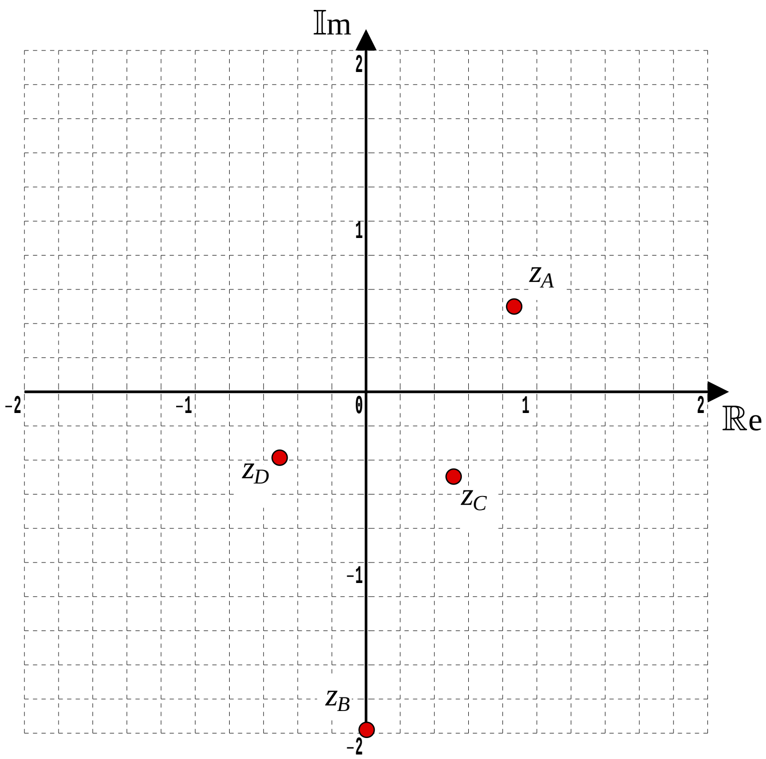
Answer 3

(i) $z_A = e^{\frac{\pi}{6}\mathrm{i}} = \cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\,\mathrm{i}$

(ii) $z_B = 2\,e^{\frac{3\pi}{2}\mathrm{i}} = 2\,e^{-\frac{\pi}{2}\mathrm{i}} = 2\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right) = 0 - 2\,\mathrm{i}$

(iii) $z_C = \frac{\sqrt{2}}{2}\,e^{-\frac{\pi}{4}\mathrm{i}} = \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right) = \frac{1}{2} - \frac{1}{2}\,\mathrm{i}$

(iv) $z_D = \frac{2}{3}\,e^{-\frac{4\pi}{5}\mathrm{i}} = \frac{2}{3}\left(\cos\left(-\frac{4\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{5}\right)\right) = -0.539 - 0.392\,\mathrm{i}$



[3, 3, 3, 3 marks]

Answer 4

(i) $z_A = \sqrt{3}\; e^{\frac{3\pi}{2}\,\mathrm{i}} = \sqrt{3}\; e^{-\frac{\pi}{2}\,\mathrm{i}}$

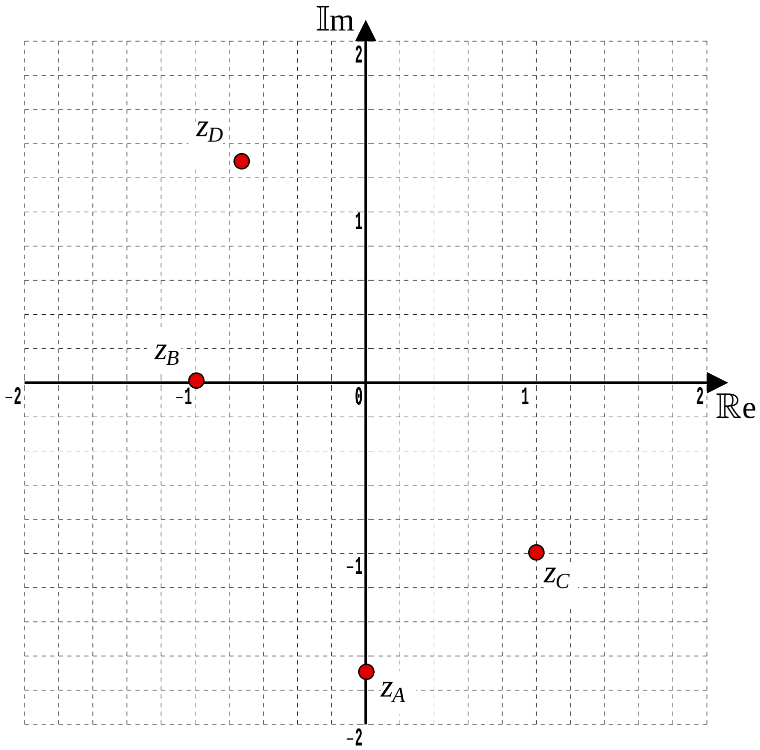
$$= \sqrt{3}\left(cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\,sin\left(-\frac{\pi}{2}\right)\right) = 0 - \sqrt{3}\,\mathrm{i}$$

(ii) $z_B = e^{\pi\,\mathrm{i}} = cos\,(\pi) - \mathrm{i}\,sin\,(\pi) = -1 + 0\,\mathrm{i}$

(iii) $z_C = \sqrt{2}\; e^{-\frac{17\pi}{4}\,\mathrm{i}} = \sqrt{2}\; e^{-\frac{\pi}{4}\,\mathrm{i}}$

$$= \sqrt{2}\left(cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\,sin\left(-\frac{\pi}{4}\right)\right) = 1 - \mathrm{i}$$

(iv) $z_D = \frac{3}{2}\, e^{-\frac{4\pi}{3}\,\mathrm{i}} = \frac{3}{2}\left(cos\left(-\frac{4\pi}{3}\right) + \mathrm{i}\,sin\left(-\frac{4\pi}{3}\right)\right) = -\frac{3}{4} + \frac{3\sqrt{3}}{4}\,\mathrm{i}$



[3, 3, 3, 3 marks]

Answer 5

(i)

$$\begin{aligned}
\text{RHS} &= \frac{1}{2\text{i}}\left(e^{\text{i}\theta} - e^{-\text{i}\theta}\right) \\
&= \frac{1}{2\text{i}}\left((\, cos\,\theta + \text{i}\, sin\,\theta\,) - (\, cos\,(-\theta) + \text{i}\, sin\,(-\theta)\,)\,\right) \\
&= \frac{1}{2\text{i}}\left(cos\,\theta + \text{i}\, sin\,\theta - (\, cos\,\theta - \text{i}\, sin\,\theta\,)\,\right) \\
&= \frac{1}{2\text{i}}\left(2\text{i}\, sin\,\theta \right) \\
&= sin\,\theta \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[4 marks]

(ii)

$$cos\,\theta = \frac{1}{2}\left(e^{\text{i}\theta} + e^{-\text{i}\theta}\right)$$

$$\begin{aligned}
\text{RHS} &= \frac{1}{2}\left(e^{\text{i}\theta} + e^{-\text{i}\theta}\right) \\
&= \frac{1}{2}\left((\, cos\,\theta + \text{i}\, sin\,\theta\,) + (\, cos\,(-\theta) + \text{i}\, sin\,(-\theta)\,)\,\right) \\
&= \frac{1}{2}\left(cos\,\theta + \text{i}\, sin\,\theta + (\, cos\,\theta - \text{i}\, sin\,\theta\,)\,\right) \\
&= \frac{1}{2}\left(2\, cos\,\theta \right) \\
&= cos\,\theta \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[4 marks]

3.1 Making Use of Exponential Form

Although it cannot be assumed that the laws of indices work in the same way with complex numbers as they do with real numbers, one of the merits of writing complex numbers in exponential form is the following key result,

---

The Multiply Divide Rule

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

- $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

3.2 Example

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then, by means of Euler's Relation, and without using the laws of indices, prove that $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Proof

$$\text{LHS} = \frac{z_1}{z_2}$$

$$= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \qquad \text{using Euler's Relation}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \times \frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 - i\cos\theta_2\sin\theta_2 + i\sin\theta_2\cos\theta_2 - i^2\sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \times \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \times \frac{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1}$$

$$= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \qquad \text{using Euler's Relation}$$

$$= \text{RHS} \qquad \square$$

[4 marks]

Teaching Video : http://www.NumberWonder.co.uk/v9099/3.mp4

The video will talk through the proof on the previous page

3.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then, by means of Euler's Relation. and without using the laws of indices, prove that $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

[5 marks]

Question 2

If $z = e^{i\theta}$ then, by means of Euler's Relation, and without using any laws of indices, prove that $\frac{1}{z} = e^{-i\theta}$

[5 marks]

Question 3

Use The Multiply Divide Rule to help calculate the following.

Give your simplified answers in the exponential form $z = r\,e^{i\theta}$ with $-\pi < \theta \leqslant \pi$

(i) $2\sqrt{3}\,e^{\frac{\pi}{2}i} \times 3\sqrt{3}\,e^{\frac{\pi}{3}i}$

[4 marks]

(ii) $\dfrac{\sqrt{8}\,e^{\frac{3\pi}{4}i}}{\sqrt{2}\,e^{-\frac{2\pi}{3}i}}$

[4 marks]

(iii) $\sqrt{6}\left(cos\left(-\frac{\pi}{12}\right) + i\,sin\left(-\frac{\pi}{12}\right)\right) \times \sqrt{3}\left(cos\left(\frac{\pi}{3}\right) + i\,sin\left(\frac{\pi}{3}\right)\right)$

[4 marks]

Question 4

Use the exponential form for a complex number to simplify,

$$\frac{(\cos 11\theta + \mathrm{i}\sin 11\theta)(\cos 5\theta + \mathrm{i}\sin 5\theta)}{\cos 7\theta + \mathrm{i}\sin 7\theta}$$

[3 marks]

Question 5

z and w are such that $z = -9 + 3\sqrt{3}\,\mathrm{i}$, $|w| = \sqrt{3}$ and $arg(w) = \frac{7\pi}{12}$

Express the following in the form $r\,e^{\mathrm{i}\theta}$ where $-\pi < \theta \leqslant \pi$

(i) z

(ii) w

(iii) zw

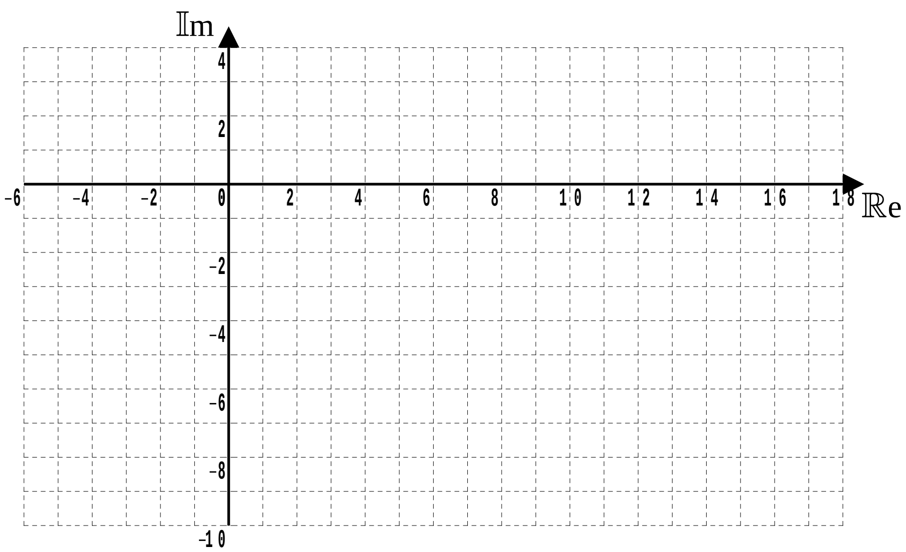
(iv) $\frac{z}{w}$

[3, 3, 3, 3 marks]

Question 6

Given that $z = \sqrt{2}\, e^{\frac{\pi}{4} i}$ write down, in exponential form z^n for n taking all integer values between 1 and 8 inclusive. Convert each power in the form $x + i\,y$ and plot each of the resulting points on an Argand diagram.
There is no need to cancel down any fractions, or restrict θ to the range $-\pi < \theta \leqslant \pi$

n	z^n (exponential form) $r\,e^{i\theta}$	z^n (Cartesian form) $x + i\,y$
1		
2		
3		
4		
5		
6		
7		
8		



What geometric interpretation can be attached to the complex number z ?

[7 marks]

Question 7

This question looks at a proof of Euler's Relation.

It begins by “pulling a rabbit out of a hat” in the form of defining this function;

$$f(\theta) = \frac{\cos\theta + \mathrm{i}\sin\theta}{e^{\mathrm{i}\theta}}$$

(i) Differentiate this function using the quotient rule and show that, for all values of θ, it is zero.

[4 marks]

(ii) If the derivative of a function is zero then the function must equal a constant. To work out what that constant must be, put zero into the original function. State the value of this constant.

[1 mark]

(iii) Explain how Euler's relation may now be deduced.

[1 mark]

Note :

There is an assumption in this proof that the derivative of the exponential function of a complex number behaves as you would hope !

3.4 Answers

Answer 1

Prove that, $z_1 z_2 = r_1 r_2 e^{i(\theta 1 + \theta 2)}$

$$
\begin{aligned}
\text{LHS} &= z_1 z_2 \\
&= r_1 e^{i\theta 1} \times r_2 e^{i\theta 2} \\
&= r_1 (cos\, \theta_1 + i\, sin\, \theta_1) r_2 (cos\, \theta_2 + i\, sin\, \theta_2) \\
&= r_1 r_2 \left(cos\, \theta_1\, cos\, \theta_2 + i\, cos\, \theta_1\, sin\, \theta_2 + i\, sin\, \theta_1\, cos\, \theta_2 + i^2\, sin\, \theta_1\, sin\, \theta_2 \right) \\
&= r_1 r_2 (cos\, \theta_1\, cos\, \theta_2 - sin\, \theta_1\, sin\, \theta_2 + i(sin\, \theta_1\, cos\, \theta_2 + cos\, \theta_1\, sin\, \theta_2)) \\
&= r_1 r_2 (cos (\theta_1 + \theta_2) + i\, sin (\theta_1 + \theta_2)) \\
&= r_1 r_2 e^{i(\theta 1 + \theta 2)} \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

Answer 2

Prove that $\dfrac{1}{z} = e^{-i\theta}$

$$
\begin{aligned}
\text{LHS} &= \frac{1}{z} \\
&= \frac{1}{e^{-i\theta}} \\
&= \frac{1}{cos\, \theta + i\, sin\, \theta} \\
&= \frac{1}{cos\, \theta + i\, sin\, \theta} \times \frac{cos\, \theta - i\, sin\, \theta}{cos\, \theta - i\, sin\, \theta} \\
&= \frac{cos\, \theta - i\, sin\, \theta}{cos^2\, \theta - i\, cos\, \theta\, sin\, \theta + i\, sin\, \theta\, cos\, \theta - i^2\, sin^2\, \theta} \\
&= \frac{cos\, \theta - i\, sin\, \theta}{cos^2\, \theta + sin^2\, \theta} \\
&= cos\, \theta - i\, sin\, \theta \\
&= cos\, (-\theta) + i\, sin\, (-\theta) \\
&= e^{-i\theta} \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

Answer 3

(i) $2\sqrt{3}\,e^{\frac{\pi}{2}\mathrm{i}} \times 3\sqrt{3}\,e^{\frac{\pi}{3}\mathrm{i}} = 18\,e^{\left(\frac{3\pi}{6}+\frac{2\pi}{6}\right)\mathrm{i}} = 18\,e^{\frac{5\pi}{6}\mathrm{i}}$

[4 marks]

(ii) $\dfrac{\sqrt{8}\,e^{\frac{3\pi}{4}\mathrm{i}}}{\sqrt{2}\,e^{-\frac{2\pi}{3}\mathrm{i}}} = 2\,e^{\left(\frac{3\pi}{4}+\frac{2\pi}{3}\right)\mathrm{i}} = 2\,e^{\frac{17\pi}{12}\mathrm{i}} = 2\,e^{-\frac{7\pi}{12}\mathrm{i}}$

[4 marks]

(iii) $\sqrt{6}\left(\cos\left(-\dfrac{\pi}{12}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{12}\right)\right) \times \sqrt{3}\left(\cos\left(\dfrac{\pi}{3}\right) + \mathrm{i}\sin\left(\dfrac{\pi}{3}\right)\right)$

$= \sqrt{6} \times \sqrt{3}\,e^{-\frac{\pi}{12}\mathrm{i}}e^{\frac{\pi}{3}\mathrm{i}} = 3\sqrt{2}\,e^{\frac{\pi}{4}\mathrm{i}}$

[4 marks]

Answer 4

$$\frac{(\cos 11\theta + \mathrm{i}\sin 11\theta)(\cos 5\theta + \mathrm{i}\sin 5\theta)}{\cos 7\theta + \mathrm{i}\sin 7\theta} = \frac{e^{11\theta\,\mathrm{i}} \times e^{5\theta\,\mathrm{i}}}{e^{7\theta\,\mathrm{i}}}$$

$$= e^{9\theta\,\mathrm{i}}$$

$$= \cos 9\theta + \mathrm{i}\sin 9\theta$$

[3 marks]

Answer 5

(i) $z = -9 + 3\sqrt{3}\,\mathrm{i}$

$|z| = \sqrt{(-9)^2 + \left(3\sqrt{3}\right)^2}$ $\qquad$ $\tan\phi = \dfrac{3\sqrt{3}}{9}$

$= \sqrt{81 + 27} = 6\sqrt{3}$ $\qquad$ $\phi = \arctan\left(\dfrac{\sqrt{3}}{3}\right) = \dfrac{\pi}{6}$

$\theta = \pi - \phi \qquad \therefore z = 6\sqrt{3}\,e^{\frac{5\pi}{6}\mathrm{i}}$

[3 marks]

(ii) $w = \sqrt{3}\,e^{\frac{7\pi}{12}\mathrm{i}}$

[3 marks]

(iii) $zw = 6\sqrt{3}\,e^{\frac{5\pi}{6}\mathrm{i}} \times \sqrt{3}\,e^{\frac{7\pi}{12}\mathrm{i}} = 18\,e^{\frac{17\pi}{12}\mathrm{i}} = 18\,e^{-\frac{7\pi}{12}\mathrm{i}}$

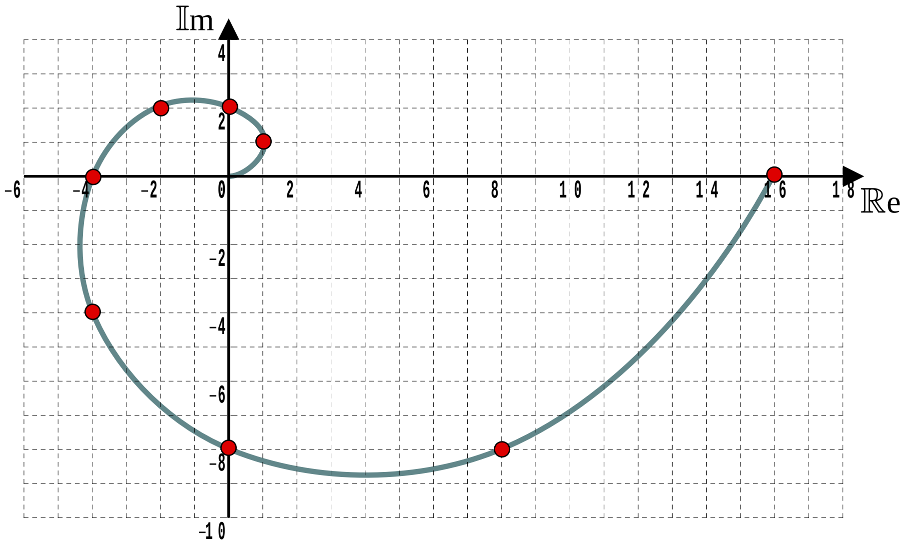
[3 marks]

(iv) $\dfrac{z}{w} = \dfrac{6\sqrt{3}\,e^{\frac{5\pi}{6}\mathrm{i}}}{\sqrt{3}\,e^{\frac{7\pi}{12}\mathrm{i}}} = 6\,e^{\frac{\pi}{4}\mathrm{i}}$

[3 marks]

Answer 6

n	z^n (exponential form) $r\,e^{i\theta}$	z^n (Cartesian form) $x + i\,y$
1	$\sqrt{2}\,e^{\frac{\pi}{4}i}$	$1 + \ i$
2	$2\,e^{\frac{2\pi}{4}i}$	$0 + 2\,i$
3	$2\sqrt{2}\,e^{\frac{3\pi}{4}i}$	$-2 + 2\,i$
4	$4\,e^{\frac{4\pi}{4}i}$	$-4 + 0\,i$
5	$4\sqrt{2}\,e^{\frac{5\pi}{4}i}$	$-4 - 4\,i$
6	$8\,e^{\frac{6\pi}{4}i}$	$0 - 8\,i$
7	$8\sqrt{2}\,e^{\frac{7\pi}{4}i}$	$8 - 8\,i$
8	$16\,e^{\frac{8\pi}{4}i}$	$16 + 0\,i$



Each successive multiplication by $\sqrt{2}\,e^{\frac{\pi}{4}i}$ is rotating the point by $\dfrac{\pi}{4}$ radians

and increasing its distance from the origin by a length scale factor of $\sqrt{2}$

[7 marks]

Answer 7

(i)

$$f\,(\theta) \;=\; \frac{\cos\theta \;+\; \mathrm{i}\sin\theta}{e^{\mathrm{i}\theta}}$$

$$f'\,(\theta) \;=\; \frac{e^{\mathrm{i}\theta}\;(-\sin\theta \;+\; \mathrm{i}\cos\theta\,) \;-\; \mathrm{i}\,e^{\mathrm{i}\theta}\,(\,\cos\theta \;+\; \mathrm{i}\sin\theta\,)}{\left(\,e^{\mathrm{i}\theta}\,\right)^2}$$

$$=\; \frac{e^{\mathrm{i}\theta}\,(-\sin\theta \;+\; \mathrm{i}\cos\theta \;-\; \mathrm{i}\cos\theta \;-\; \mathrm{i}^2\sin\theta\,)}{\left(\,e^{\mathrm{i}\theta}\,\right)^2}$$

$$=\; \frac{e^{\mathrm{i}\theta}\,(-\sin\theta \;+\; \mathrm{i}\cos\theta \;-\; \mathrm{i}\cos\theta \;+\; \sin\theta\,)}{\left(\,e^{\mathrm{i}\theta}\,\right)^2}$$

$$=\; \frac{e^{\mathrm{i}\theta}\;(0)}{\left(\,e^{\mathrm{i}\theta}\,\right)^2}$$

$$=\; 0$$

[4 marks]

(ii)

$$f\,(0) \;=\; \frac{\cos\;(0) \;+\; \mathrm{i}\sin\;(0)}{e^{\mathrm{i}0}}$$

$$=\; \frac{1 \;+\; 0}{1}$$

$$=\; 1$$

[1 mark]

(iii)

$$f\,(x) \;=\; 1$$

$$\frac{\cos\theta \;+\; \mathrm{i}\sin\theta}{e^{\mathrm{i}\theta}} \;=\; 1$$

$$\therefore\;\; e^{\mathrm{i}\theta} \;=\; \cos\theta \;+\; \mathrm{i}\sin\theta \qquad\qquad \square$$

[1 mark]

4.1 The Merge of Trigonometry and Complex Numbers

Euler's Relation is important because it firmly links together the subjects of trigonometry and complex numbers in a fundamental way. This merging of two topics, each important in their own right, is further strengthened by a formula found by Abraham de Moivre in 1707,

---

De Moivre's Theorem

For any positive integer, n,

$$\left(r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right)^n = r^n\left(\cos(n\theta) + \mathrm{i}\sin(n\theta)\right)$$

The formula will also work with negative integers although not, as it stands, with fractional values of n. Here it will be proven for positive integers.

Proof (by induction)

Basis step :

For $n = 1$ LHS $= \left(r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right)^1 = r\left(\cos\theta + \mathrm{i}\sin\theta\right)$

RHS $= r^1\left(\cos(1\theta) + \mathrm{i}\sin(1\theta)\right) = r\left(\cos\theta + \mathrm{i}\sin\theta\right)$

As LHS = RHS, de Moivre's theorem is true for $n = 1$

Assumption step :

Assume that de Moivre's theorem is true for $n = k,\ k \in \mathbb{Z}^+$ in which case,

$$\left(r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right)^k = r^k\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)$$

Inductive step :

When $n = k + 1$, the LHS of the theorem becomes,

$$\left(r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right)^{k+1}$$

$$= \left(r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right)^k\left(r\left(\cos\theta + \mathrm{i}\sin\theta\right)\right)^1$$

$$= r^k\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)r\left(\cos\theta + \mathrm{i}\sin\theta\right)$$

$$= r^{k+1}\left(\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + \mathrm{i}\left(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta\right)\right)$$

$$= r^{k+1}\left(\cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta)\right)$$

$$= r^{k+1}\left(\cos((k + 1)\theta) + \mathrm{i}\sin((k + 1)\theta)\right)$$

which is the RHS of the theorem with $n = k + 1$

Conclusion step:

Thus, if the theorem is true for $n = k$, then it is true for $n = k + 1$

As the theorem is true for $n = 1$, it is true for all positive integers, by induction □

Teaching Video : http://www.NumberWonder.co.uk/v9099/4.mp4

The video talks through the proof of de Moivre's theorem on the previous page

Abraham de Moivre
(1667-1754)
A French mathematician known for de Moivre's theorem, a formula that links complex numbers and trigonometry, and for his statistics work with the normal distribution.
He wrote a book on probability theory, “The Doctrine of Chances”, much prized by gamblers.
He moved to England to escape religious persecution, there becoming a friend of Isaac Newton, Edmond Halley and James Sterling.
He was the first to postulate the Central Limit Theorem, a cornerstone of probability theory.

4.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

$$z = \sqrt{2}\left(cos\left(\frac{\pi}{6}\right) + \mathrm{i}\,sin\left(\frac{\pi}{6}\right)\right)$$

Find the exact value of z^5, giving your answer in the form $a + \mathrm{i}\,b$ where $a,\ b \in \mathbb{R}$

[3 marks]

Question 2

$$w = 2\left(cos\left(\frac{\pi}{3}\right) + \mathrm{i}\,sin\left(\frac{\pi}{3}\right)\right)$$

Find the exact value of w^4, giving your answer in the form $a + \mathrm{i}\,b$ where $a,\ b \in \mathbb{R}$

[3 marks]

Question 3

$$v = \sqrt{3}\left(cos\left(\frac{3\pi}{4}\right) - \mathrm{i}\,sin\left(\frac{3\pi}{4}\right)\right)$$

Find the exact value of v^6, giving your answer in the form $a + \mathrm{i}\,b$ where $a,\ b \in \mathbb{R}$

[3 marks]

Question 4

$$z = -2 + \left(2\sqrt{3}\right)\mathrm{i}$$

(i) Express z in the polar form $r\left(\cos\theta + \mathrm{i}\sin\theta\right)$ where r and θ have values that you have determined.

[3 marks]

Using de Moivre's theorem,

(ii) show that z^6 is a real number, and state its numerical value

[2 marks]

Question 5

Express $\left(3 + \sqrt{3}\,\mathrm{i}\right)^5$ in the form $a + b\sqrt{3}\,\mathrm{i}$ where a and b are integers

[4 marks]

Question 6

De Moivre's theorem can be used to establish to following useful result,

---

Exponential Form, Powered

$$\left(r\,e^{\mathrm{i}\theta}\right)^n = r^n\,e^{\mathrm{i}n\theta}$$

Prove this result without using any laws of indices, but using de Moivre's theorem

[4 marks]

Question 7

Using de Moivre's theorem or any of the laws of indices previously proven,

prove that $\dfrac{\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\,\sin\left(\frac{3\pi}{4}\right)\right)^6}{\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\,\sin\left(\frac{\pi}{4}\right)\right)^{11}} = \cos\left(\dfrac{\pi}{4}\right) - \mathrm{i}\,\sin\left(\dfrac{\pi}{4}\right)$

[6 marks]

Question 8

(i) Express $\dfrac{1 + \sqrt{3}\,\mathrm{i}}{1 - \sqrt{3}\,\mathrm{i}}$ in the form $r\,e^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$

[4 marks]

(ii) Hence find the smallest positive integer value of n for which

$$\left(\frac{1 + \sqrt{3}\,\mathrm{i}}{1 - \sqrt{3}\,\mathrm{i}}\right)^n$$

is real and positive

[3 marks]

Question 9

Determine, in as simple a form as possible, the value of

$$\frac{18(1-i)^{39}}{(1+i)^{41}}$$

[5 marks]

4.3 Answers

Answer 1

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

$$\therefore\; z^5 = (\sqrt{2})^5\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)^5$$

$$= 4\sqrt{2}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$

$$= -2\sqrt{6} + 2\sqrt{2}\,i$$

[3 marks]

Answer 2

$$w = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$$

$$\therefore\; w^4 = 2^4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)^4$$

$$= 16\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$$

$$= 16\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right) \qquad \text{can skip this step}$$

$$= 16\left(\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right) \qquad \text{can skip this step}$$

$$= -8 - 8\sqrt{3}\,i$$

[3 marks]

Answer 3

$$v = \sqrt{3}\left(\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right)$$

$$= \sqrt{3}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) \qquad \text{being pedantic !}$$

$$\therefore\; v^6 = (\sqrt{3})^6\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)^6$$

$$= 27\left(\cos\left(\frac{18\pi}{4}\right) + i\sin\left(-\frac{18\pi}{4}\right)\right)$$

$$= 27\left(\cos\left(\frac{9\pi}{2}\right) + i\sin\left(-\frac{9\pi}{2}\right)\right)$$

$$= 27\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$$= 0 - 27\,i$$

[3 marks]

Answer 4

(i)

$$z = -2 + \left(2\sqrt{3}\right) i$$

$$|z| = \sqrt{(-2)^2 + \left(2\sqrt{3}\right)^2} \qquad \tan\phi = \frac{2\sqrt{3}}{2}$$

$$= \sqrt{4 + 12} \qquad \phi = \arctan\left(\sqrt{3}\right)$$

$$= 4 \qquad \phi = \frac{\pi}{3}$$

Angle required for polar form is $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$$\therefore z = 4\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

[3 marks]

(ii)

$$z^6 = 4^6\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)^6$$

$$= 4096\left(\cos\left(4\pi\right) + i\sin\left(4\pi\right)\right)$$

$$= 4096$$

[2 marks]

Answer 5

Let $z = 3 + \sqrt{3}\, i$

$$|z| = \sqrt{(3)^2 + \left(\sqrt{3}\right)^2} \qquad \tan\phi = \frac{\sqrt{3}}{3}$$

$$= \sqrt{9 + 3} \qquad \phi = \arctan\left(\frac{\sqrt{3}}{3}\right)$$

$$= 2\sqrt{3} \qquad \phi = \frac{\pi}{6}$$

Angle required for polar form is unchanged $\frac{\pi}{6}$

$$z = 2\sqrt{3}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

$$z^5 = \left(2\sqrt{3}\right)^5\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)^5$$

$$= 32 \times 9\sqrt{3}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$

$$= 288\sqrt{3}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$

$$= -432 + 144\sqrt{3}\, i$$

[4 marks]

Answer 6

$$\left(r\,e^{i\theta}\right)^n = r^n\,e^{in\theta}$$

$$\begin{aligned}
\text{LHS} &= \left(r\,e^{i\theta}\right)^n \\
&= r^n\left(e^{i\theta}\right)^n \\
&= r^n\left(\cos\theta + i\sin\theta\right)^n \\
&= r^n\left(\cos(n\theta) + i\sin(n\theta)\right) \\
&= r^n\,e^{in\theta} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 7

$$\frac{\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)^6}{\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)^{11}} = \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)$$

$$\begin{aligned}
\text{LHS} &= \frac{\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)^6}{\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)^{11}} \\
&= \left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)^6\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)^{-11} \\
&= \left(\cos\left(\frac{18\pi}{4}\right) + i\sin\left(\frac{18\pi}{4}\right)\right)\left(\cos\left(-\frac{11\pi}{4}\right) + i\sin\left(-\frac{11\pi}{4}\right)\right) \\
&= e^{\frac{18\pi}{4}i} \times e^{-\frac{11\pi}{4}i} \\
&= e^{\frac{7\pi}{4}i} \\
&= e^{-\frac{\pi}{4}i} \\
&= \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[6 marks]

Answer 8

(i)

$$\frac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i} = \frac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i} \times \frac{1+\sqrt{3}\,i}{1+\sqrt{3}\,i}$$

$$= \frac{1+\sqrt{3}\,i+\sqrt{3}\,i+3i^2}{1+\sqrt{3}\,i-\sqrt{3}\,i-3i^2}$$

$$= \frac{-2+2\sqrt{3}\,i}{4}$$

$$= -\frac{1}{2}+\frac{\sqrt{3}}{2}\,i$$

$$|z| = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \qquad \tan\phi = \frac{\sqrt{3}}{1}$$

$$= \sqrt{\frac{1}{4}+\frac{3}{4}} \qquad \phi = \arctan\left(\sqrt{3}\right)$$

$$= 1 \qquad \phi = \frac{\pi}{3}$$

Angle required for polar form is $\pi - \phi = \dfrac{2\pi}{3}$

$$\therefore \quad \frac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i} = e^{\frac{2\pi}{3}i}$$

[4 marks]

(ii)

$$\left(\frac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i}\right)^n = \left(e^{\frac{2\pi}{3}i}\right)^n$$

$$= e^{\frac{2n\pi}{3}i}$$

$$= \cos\left(\frac{2n\pi}{3}\right) + i\sin\left(\frac{2n\pi}{3}\right)$$

For a real positive solution, $\sin\left(\dfrac{2n\pi}{3}\right) = 0$

$$\frac{2n\pi}{3} = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots$$

$$2n\pi = 0,\ 3\pi,\ 6\pi,\ 9\pi,\ \ldots$$

$$n = 0,\ \frac{3}{2},\ 3,\ \frac{9}{2},\ \ldots$$

$\therefore$ Smallest positive integer value of n is 3 as in that case

$$\left(\frac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i}\right)^3 = \cos(2\pi) + i\sin(2\pi) = 1$$

[3 marks]

Answer 9

$$\frac{18\,(1 - \mathrm{i})^{39}}{(1 + \mathrm{i})^{41}} = 18 \times \frac{\left(\sqrt{2}\, e^{-\frac{\pi}{4}\mathrm{i}}\right)^{39}}{\left(\sqrt{2}\, e^{\frac{\pi}{4}\mathrm{i}}\right)^{41}}$$

$$= 18 \times \left(\frac{1}{\sqrt{2}}\right)^{2} \times e^{-\frac{39\pi}{4}\mathrm{i}} \times e^{-\frac{41\pi}{4}\mathrm{i}}$$

$$= 9 \times e^{-\frac{80\pi}{4}\mathrm{i}}$$

$$= 9\, e^{-20\pi\mathrm{i}}$$

$$= 9\, e^{0}$$

$$= 9$$

[5 marks]

5.1 Trigonometric Identities, Type 1

Thanks to Euler's Relation and de Moivre's theorem, the world of complex numbers is a natural setting in which to generate trigonometric identities.

One fruitful approach is to consider an expression of the form $(\cos\theta + i \sin\theta)^n$ for some integer value of *n*. By then expanding the brackets in two different ways, and setting the resulting expressions equal to each other, many useful trigonometric identities are obtained.

5.2 Example

By expanding the brackets of $(\cos\theta + \mathrm{i} \sin\theta)^2$ in two different ways, derive two well known trigonometric identities.

Teaching Video : http://NumberWonder.co.uk/v9099/5.mp4

[6 marks]

5.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

By expanding the brackets of $(\,cos\,\theta\;+\;\mathrm{i}\,sin\,\theta\,)^3$ in two different ways, derive the following two well known trigonometric identities, frequently obtained "the long way" in the single A-Level examination papers,

$$cos\,3\theta \;=\; 4\,cos^3\,\theta \;-\; 3\,cos\,\theta \quad \text{and} \quad sin\,3\theta \;=\; 3\,sin\,\theta \;-\; 4\,sin^3\,\theta$$

[6 marks]

Question 2

By means of the substitution $x = \cos\theta$, and making use of the trigonometric identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, to find all three solutions to the equation,

$$8x^3 - 6x - 1 = 0$$

Give your answers to 3 decimal places

[5 marks]

Question 3

This question is about finding all the solution to the equation

$$\cos 5\theta + 5\cos 3\theta = 0 \qquad 0 \leqslant \theta < \pi$$

You are given that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

and also that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Give your answers to 3 decimal places

[6 marks]

Question 4

Further A-Level Examination Question from June 2014, FP2(R), Q7 (Edexcel)

(a) Use de Moivre's theorem to show that,

$$\sin 5\theta \ = \ 16 \sin^5 \theta \ - \ 20 \sin^3 \theta \ + \ 5 \sin \theta$$

[5 marks]

(b) Hence find the five distinct solutions of the equation

$$16x^5 - 20x^3 + 5x + \frac{1}{2} = 0$$

giving your answers to 3 decimal places where necessary.

[5 marks]

(c) Use the identity given in (a) to find

$$\int_0^{\frac{\pi}{4}} \left(4\sin^5\theta - 5\sin^3\theta \right) d\theta$$

expressing your answer in the form $a\sqrt{2} + b$ where a and b are rational numbers.

[4 marks]

Question 5

(i) Use de Moivre's theorem to show that

$$sin\ 4\theta \ = \ 4\,cos^3\,\theta\ sin\ \theta\ -\ 4\,cos\ \theta\ sin^3\,\theta$$

[4 marks]

(ii) Hence, or otherwise, show that,

$$tan\ 4\theta \ = \ \frac{4\,tan\ \theta\ -\ 4\,tan^3\,\theta}{1\ -\ 6\,tan^2\,\theta\ +\ tan^4\,\theta}$$

[4 marks]

(iii) Use your answer to part (b) to find, to 2 decimal places, the four solutions of the equation,

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

[5 marks]

Question 6

Further A-Level Examination Question from June 2018, FP2, Q7(a) (Edexcel)

Use de Moivre's theorem to show that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

[6 marks]

5.4 Answers

Further A-Level Pure Mathematics, Core 2
Complex Numbers II

5.4.1 Solutions to 5.2 Example (Teaching Video)

$$(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$$ by de Moivre's theorem

$$\begin{aligned}(\cos\theta + i\sin\theta)^2 &= 1 \times (\cos\theta)^2 \times (i\sin\theta)^0 \\ &\quad + 2 \times (\cos\theta)^1 \times (i\sin\theta)^1 \\ &\quad + 1 \times (\cos\theta)^0 \times (i\sin\theta)^2 \\ &= \cos^2\theta + i\,2\cos\theta\sin\theta - \sin^2\theta\end{aligned}$$ by the Binomial theorem

Equating the two expansions gives,

$$\cos(2\theta) + i\sin(2\theta) = \cos^2\theta + i\,2\cos\theta\sin\theta - \sin^2\theta$$

Matching the real part on the LHS with the real part on the RHS yields,

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Matching the imaginary part on the LHS with the imaginary part on the RHS yields,

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

[6 marks]

5.4.2 Solution to 5.3 Exercise

Answer 1

$$\left(\cos\theta + \mathrm{i}\sin\theta \right)^3 = \cos(3\theta) + \mathrm{i}\sin(3\theta) \qquad \text{by de Moivre's theorem}$$

$$\begin{aligned}
\left(\cos\theta + \mathrm{i}\sin\theta \right)^3 &= 1 \times (\cos\theta)^3 \times \left(\mathrm{i}\sin\theta\right)^0 \qquad \text{by the Binomial theorem} \\
&+ 3 \times (\cos\theta)^2 \times \left(\mathrm{i}\sin\theta\right)^1 \\
&+ 3 \times (\cos\theta)^1 \times \left(\mathrm{i}\sin\theta\right)^2 \\
&+ 1 \times (\cos\theta)^0 \times \left(\mathrm{i}\sin\theta\right)^3 \\
&= \cos^3\theta + \mathrm{i}\,3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta \\
&= \cos^3\theta - 3\cos\theta\sin^2\theta + \mathrm{i}\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)
\end{aligned}$$

Equating the two expansions gives,

$$\cos(3\theta) + \mathrm{i}\sin(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta + \mathrm{i}\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$$

Matching the real part on the LHS with the real part on the RHS yields,

$$\begin{aligned}
\cos(3\theta) &= \cos^3\theta - 3\cos\theta\sin^2\theta \\
&= \cos^3\theta - 3\cos\theta\left(1 - \cos^2\theta\right) \\
&= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta \qquad \square
\end{aligned}$$

Matching the imaginary part on the LHS with the imaginary part on the RHS yields,

$$\begin{aligned}
\sin(3\theta) &= 3\cos^2\theta\sin\theta - \sin^3\theta \\
&= 3\left(1 - \sin^2\theta\right)\sin\theta - \sin^3\theta \\
&= 3\sin\theta - 3\sin^2\theta - \sin^3\theta \\
&= 3\sin\theta - 4\sin^2\theta \qquad \square
\end{aligned}$$

[6 marks]

Answer 2

$$8x^3 - 6x - 1 = 0$$

let $x = \cos\theta$ to obtain,

$$8\cos^3\theta - 6\cos\theta - 1 = 0$$

$$2\left(4\cos^3\theta - 3\cos\theta\right) = 1$$

$$4\cos^3\theta - 3\cos\theta = \frac{1}{2}$$

$$\cos 3\theta = \frac{1}{2}$$

$$3\theta = \frac{\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3},\ \ldots$$

$$\theta = \frac{\pi}{9},\ \frac{5\pi}{9},\ \frac{7\pi}{9},\ \ldots$$

$$\therefore x = 0.940,\ -0.174,\ -0.766$$

[5 marks]

Answer 3

$$\cos 5\theta + 5\cos 3\theta = 0$$

$$\left(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\right) + 5\left(4\cos^3\theta - 3\cos\theta\right) = 0$$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta + 20\cos^3\theta - 15\cos\theta = 0$$

$$16\cos^5\theta - 10\cos\theta = 0$$

$$2\cos\theta\left(8\cos^4\theta - 5\right) = 0$$

Either $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} = 1.571$

Or $\cos\theta = \pm\sqrt[4]{\frac{5}{8}} \Rightarrow \theta = 0.475,\ 2.666$

That is, $\theta \in \{0.475,\ 1.571,\ 2.666\}$

[6 marks]

Answer 4
(a)

$$\left(cos\,\theta + \mathrm{i}\,sin\,\theta \right)^5 = cos(5\theta) + \mathrm{i}\,sin(5\theta)$$ by de Moivre's theorem

$$\begin{aligned}\left(cos\,\theta + \mathrm{i}\,sin\,\theta \right)^5 =\ & 1 \times (cos\,\theta)^5 \times \left(\mathrm{i}\,sin\,\theta \right)^0 \\ & + 5 \times (cos\,\theta)^4 \times \left(\mathrm{i}\,sin\,\theta \right)^1 \\ & + 10 \times (cos\,\theta)^3 \times \left(\mathrm{i}\,sin\,\theta \right)^2 \\ & + 10 \times (cos\,\theta)^2 \times \left(\mathrm{i}\,sin\,\theta \right)^3 \\ & + 5 \times (cos\,\theta)^1 \times \left(\mathrm{i}\,sin\,\theta \right)^4 \\ & + 1 \times (cos\,\theta)^0 \times \left(\mathrm{i}\,sin\,\theta \right)^5 \\ =\ & cos^5\,\theta + \mathrm{i}\,5\,cos^4\,\theta\,sin\,\theta - 10\,cos^3\,\theta\,sin^2\,\theta \\ & - \mathrm{i}\,10\,cos^2\,\theta\,sin^3\,\theta + 5\,cos\,\theta\,sin^4\,\theta + \mathrm{i}\,sin^5\,\theta\end{aligned}$$ by the Binomial theorem

Equating the two expansions gives,

$$\begin{aligned}cos(5\theta) + \mathrm{i}\,sin(5\theta) =\ & cos^5\,\theta - 10\,cos^3\,\theta\,sin^2\,\theta + 5\,cos\,\theta\,sin^4\,\theta \\ & + \mathrm{i}\left(5\,cos^4\,\theta\,sin\,\theta - 10\,cos^2\,\theta\,sin^3\,\theta + sin^5\,\theta \right)\end{aligned}$$

Matching the imaginary part on the LHS with the imaginary part on the RHS yields,

$$\begin{aligned}sin(5\theta) &= 5\,cos^4\,\theta\,sin\,\theta - 10\,cos^2\,\theta\,sin^3\,\theta + sin^5\,\theta \\ &= 5\left(1 - sin^2\,\theta \right)^2 sin\,\theta - 10\,(1 - sin^2\,\theta)\,sin^3\,\theta + sin^5\,\theta \\ &= 5\left(1 - 2\,sin^2\,\theta + sin^4\,\theta \right) sin\,\theta - 10\,sin^3\,\theta + 10\,sin^5\,\theta + sin^5\,\theta \\ &= 5\,sin\,\theta - 10\,sin^3\,\theta + 5\,sin^5\,\theta - 10\,sin^3\,\theta + 10\,sin^5\,\theta + sin^5\,\theta \\ &= 5\,sin\,\theta - 20\,sin^3\,\theta + 16\,sin^5\,\theta \\ &= 16\,sin^5\,\theta - 20\,sin^3\,\theta + 5\,sin\,\theta \qquad \square\end{aligned}$$

[5 marks]

(b)

$$16x^5 - 20x^3 + 5x + \frac{1}{2} = 0$$

let $x = \sin\theta$ to obtain,

$$16\sin^5\theta - 20\sin^3\theta + 5\sin\theta + \frac{1}{2} = 0$$

$$\sin 5\theta + \frac{1}{2} = 0$$

$$\sin 5\theta = -\frac{1}{2}$$ (OK to do the next part in degrees)

$$5\theta = -2\pi - \frac{\pi}{6}, -\pi + \frac{\pi}{6}, -\frac{\pi}{6}, \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$

$$5\theta = -\frac{13\pi}{6}, -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\theta = -\frac{13\pi}{30}, -\frac{5\pi}{30}, -\frac{\pi}{30}, \frac{7\pi}{30}, \frac{11\pi}{30}$$

$$x = -0.978, -0.5, -0.105, 0.669, 0.914$$

[5 marks]

(c)

$$\int_0^{\frac{\pi}{4}} \left(4\sin^5\theta - 5\sin^3\theta\right) d\theta = \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(\sin 5\theta - 5\sin\theta\right) d\theta$$

$$= \frac{1}{4}\left[-\frac{\cos 5\theta}{5} + 5\cos\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[\frac{\sqrt{2}}{10} + \frac{5\sqrt{2}}{2}\right] - \frac{1}{4}\left[-\frac{1}{5} + 5\right]$$

$$= \frac{1}{4}\left[\frac{26\sqrt{2}}{10}\right] - \frac{1}{4}\left[\frac{24}{5}\right]$$

$$= \frac{13\sqrt{2}}{20} - \frac{6}{5}$$

[4 marks]

Answer 5

(i)

$\left(cos\,\theta \,+\, \mathrm{i}\, sin\,\theta \right)^4 \,=\, cos\,(4\theta) \,+\, \mathrm{i}\, sin\,(4\theta)$ by de Moivre's theorem

$$\begin{aligned}\left(cos\,\theta \,+\, \mathrm{i}\, sin\,\theta \right)^4 &= 1 \times (\,cos\,\theta\,)^4 \times \left(\,\mathrm{i}\, sin\,\theta\,\right)^0 \quad \text{by the Binomial theorem}\\ &\quad + 4 \times (\,cos\,\theta\,)^3 \times \left(\,\mathrm{i}\, sin\,\theta\,\right)^1\\ &\quad + 6 \times (\,cos\,\theta\,)^2 \times \left(\,\mathrm{i}\, sin\,\theta\,\right)^2\\ &\quad + 4 \times (\,cos\,\theta\,)^1 \times \left(\,\mathrm{i}\, sin\,\theta\,\right)^3\\ &\quad + 1 \times (\,cos\,\theta\,)^0 \times \left(\,\mathrm{i}\, sin\,\theta\,\right)^4\\ &= cos^4\,\theta \,+\, \mathrm{i}\,4\,cos^3\,\theta\,sin\,\theta \,-\, 6\,cos^2\,\theta\,sin^2\,\theta\\ &\quad -\, \mathrm{i}\,4\,cos\,\theta\,sin^3\,\theta \,+\, sin^4\,\theta\end{aligned}$$

Equating the two expansions gives,

$$\begin{aligned}&cos\,(4\theta) \,+\, \mathrm{i}\, sin\,(4\theta)\\ &\quad = cos^4\,\theta \,+\, \mathrm{i}\,4\,cos^3\,\theta\,sin\,\theta \,-\, 6\,cos^2\,\theta\,sin^2\,\theta \,-\, \mathrm{i}\,4\,cos\,\theta\,sin^3\,\theta \,+\, sin^4\,\theta\end{aligned}$$

Matching the imaginary part on the LHS with the imaginary part on the RHS yields,

$$sin\,(4\theta) \,=\, 4\,cos^3\,\theta\,sin\,\theta \,-\, 4\,cos\,\theta\,sin^3\,\theta \qquad \square$$

[4 marks]

(ii)

Matching the real part on the LHS with the real part on the RHS yields,

$$cos\,(4\theta) \,=\, cos^4\,\theta \,-\, 6\,cos^2\,\theta\,sin^2\,\theta \,+\, sin^4\,\theta$$

$$\begin{aligned}\text{LHS} &= tan\,4\theta\\ &= \frac{sin\,4\theta}{cos\,4\theta}\\ &= \frac{4\,cos^3\,\theta\,sin\,\theta \,-\, 4\,cos\,\theta\,sin^3\,\theta}{cos^4\,\theta \,-\, 6\,cos^2\,\theta\,sin^2\,\theta \,+\, sin^4\,\theta} \times \frac{\frac{1}{cos^4\,\theta}}{\frac{1}{cos^4\,\theta}}\\ &= \frac{4\,tan\,\theta \,-\, 4\,tan^3\,\theta}{1 \,-\, 6\,tan^2\,\theta \,+\, tan^4\,\theta}\\ &= \text{RHS} \qquad \square\end{aligned}$$

[4 marks]

(iii)

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

Let $x = tan\,\theta$ in which case,

$$tan^4\theta + 4\,tan^3\theta - 6\,tan^2\theta - 4\,tan\,\theta + 1 = 0$$

$$tan^4\theta - 6\,tan^2\theta + 1 = 4\,tan\,\theta - 4\,tan^3\theta$$

$$\frac{4\,tan\,\theta - 4\,tan^3\theta}{1 - 6\,tan^2\theta + tan^4\theta} = 1$$

$$tan\,4\theta = 1$$

$$4\theta = \frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{9\pi}{4},\ \frac{13\pi}{4},\ \ldots$$

$$\theta = \frac{\pi}{16},\ \frac{5\pi}{16},\ \frac{9\pi}{16},\ \frac{13\pi}{16},\ \ldots$$

$$x = 0.20,\ 1.50,\ -5.03,\ -0.67$$

[5 marks]

Answer 6

$\left(\cos\theta + \mathrm{i}\sin\theta \right)^7 = \cos(7\theta) + \mathrm{i}\sin(7\theta)$ by de Moivre's theorem

$$\begin{aligned}
\left(\cos\theta + \mathrm{i}\sin\theta \right)^7 = \; & 1 \times (\cos\theta)^7 \times \left(\mathrm{i}\sin\theta\right)^0 \quad \text{by the Binomial theorem} \\
& + 7 \times (\cos\theta)^6 \times \left(\mathrm{i}\sin\theta\right)^1 \\
& + 21 \times (\cos\theta)^5 \times \left(\mathrm{i}\sin\theta\right)^2 \\
& + 35 \times (\cos\theta)^4 \times \left(\mathrm{i}\sin\theta\right)^3 \\
& + 35 \times (\cos\theta)^3 \times \left(\mathrm{i}\sin\theta\right)^4 \\
& + 21 \times (\cos\theta)^2 \times \left(\mathrm{i}\sin\theta\right)^5 \\
& + 7 \times (\cos\theta)^1 \times \left(\mathrm{i}\sin\theta\right)^6 \\
& + 1 \times (\cos\theta)^0 \times \left(\mathrm{i}\sin\theta\right)^7 \\
= \; & \cos^7\theta + \mathrm{i}\,7\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - \mathrm{i}\,35\cos^4\theta\sin^3\theta \\
& + 35\cos^3\theta\sin^4\theta + \mathrm{i}\,21\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - \sin^7\theta
\end{aligned}$$

Equating the two expansions gives,

$$\begin{aligned}
& \cos(7\theta) + \mathrm{i}\sin(7\theta) \\
& = \cos^7\theta + \mathrm{i}\,7\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - \mathrm{i}\,35\cos^4\theta\sin^3\theta \\
& + 35\cos^3\theta\sin^4\theta + \mathrm{i}\,21\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - \mathrm{i}\sin^7\theta
\end{aligned}$$

Matching the real part on the LHS with the real part on the RHS yields,

$$\begin{aligned}
& \cos(7\theta) \\
& = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta \\
& = \cos^7\theta - 21\cos^5\theta\left(1 - \cos^2\theta\right) + 35\cos^3\theta\left(1 - \cos^2\theta\right)^2 - 7\cos\theta\left(1 - \cos^2\theta\right)^3 \\
& = \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta\left(1 - 2\cos^2\theta + \cos^4\theta \right) \\
& \qquad - 7\cos\theta\left(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta \right) \\
& = \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta \\
& \qquad - 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta \\
& = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta \qquad \square
\end{aligned}$$

[6 marks]

6.1 Trigonometric Identities, Type 2

Previously, identities were derived where a trigonometric function was applied to a multiple of the angle θ and it was equated to an expression containing powers of trigonometric functions in θ.

For example,

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta \qquad \text{(Type 1)}$$

This can be done the other way around, by which is meant a power of a trigonometric function in θ is equated to a power free expression of trigonometric functions of multiples of the angle θ.

For example,

$$\cos^4\theta = \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right) \qquad \text{(Type 2)}$$

6.2 A Pair of Essentials

In finding such Type 2 identities the following pair of results is invaluable,

---

The Type 2 Key

- $z^n + \dfrac{1}{z^n} = 2\cos(n\theta)$
- $z^n - \dfrac{1}{z^n} = 2\,\mathrm{i}\sin(n\theta)$

Proof

$$\begin{aligned}
\text{LHS} &= z^n + \frac{1}{z^n} \\
&= z^n + z^{-n} \\
&= (\cos\theta + \mathrm{i}\sin\theta)^n + (\cos\theta + \mathrm{i}\sin\theta)^{-n} \\
&= \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(-n\theta) + \mathrm{i}\sin(-n\theta) && \text{by de Moivre's theorem} \\
&= \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(n\theta) - \mathrm{i}\sin(n\theta) && \text{even and odd functions} \\
&= 2\cos(n\theta) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

A proof of the other result is very similar and is left as an exercise for the reader.

6.3 Example

Use the appropriate Type 2 Key result to prove that,

$$cos^4 \theta = \frac{1}{8}\left(cos\, 4\theta + 4\, cos\, 2\theta + 3 \right)$$

Teaching Video : http://www.NumberWonder.co.uk/v9099/6.mp4

[4 marks]

6.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

The binomial theorem gives that $\left(z + \frac{1}{z}\right)^2 = \left(z^2 + \frac{1}{z^2}\right) + 2$

(i) Make use of the “The Type 2 Key” to obtain a useful formula for $cos^2\,\theta$

[2 marks]

(ii) Hence determine the exact value of $\int_0^{\frac{\pi}{2}} cos^2\,\theta\,d\theta$

[2 marks]

(iii) In a similar manner, consider $\left(z - \frac{1}{z}\right)^2$ and use “The Type 2 Key” to determine the exact value of $\int_0^{\frac{\pi}{6}} sin^2\,\theta\,d\theta$

[3 marks]

Question 2

(i) Beginning with an expression for $\left(z - \frac{1}{z}\right)^3$ find the constants p and q in the identity $sin^3\,\theta \;=\; p\,sin\,3\theta \;+\; q\,sin\theta$

[4 marks]

(ii) Prove $sin\left(\frac{\pi}{2} - \theta\right) = cos\,\theta$ and $sin\left(\frac{3\pi}{2} - 3\theta\right) = -\,cos\,3\theta$

[2 marks]

(iii) Combine your part (i) and (ii) results to find the constants r and s in the identity $cos^3\,\theta \;=\; r\,cos\,3\theta \;+\; s\,cos\,\theta$

[2 marks]

Question 3

(i) Express $\left(z^2 + \dfrac{1}{z^2}\right)^3$ in terms of $\cos 6\theta$ and $\cos 2\theta$

[3 marks]

(ii) Hence find constants a and b such that $\cos^3 2\theta = a \cos 6\theta + b \cos 2\theta$

[3 marks]

(iii) Hence show that $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = k\sqrt{3}$ where k is a rational constant

[4 marks]

Question 4

Further A-Level Examination Question from June 2016, FP2, Q5 (Edexcel)

(a) Use de Moivre's theorem to show that

$$sin^5\,\theta \;\equiv\; a\,sin\,5\theta \;+\; b\,sin\,3\theta \;+\; c\,sin\,\theta$$

where a, b and c are constants to be found

[5 marks]

(b) Hence show that $\int_0^{\frac{\pi}{3}} sin^5\,\theta\,d\theta \;=\; \frac{53}{480}$

[5 marks]

Question 5

Further A-Level Examination Question from June 2007, FP3, Q4(b)(c) (Edexcel)

(i) Express $32\, cos^6\, \theta$ in the form $p\, cos\, 6\theta \,+\, q\, cos\, 4\theta \,+\, r\, cos\, 2\theta \,+\, s$ where p, q, r and s are integers

[5 marks]

(ii) Hence find the exact value of $\int_0^{\frac{\pi}{3}} cos^6\, \theta\, d\theta$

[4 marks]

Question 6

Further A-Level Examination Question from June 2006, FP2, Q2(a)(ii) (MEI)

By considering $\left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^2$ find A, B, C and D such that

$$sin^4\,\theta\;cos^2\,\theta \;=\; A\,cos\,6\theta \;+\; B\,cos\,4\theta \;+\; C\,cos\,2\theta \;+\; D$$

[6 marks]

6.5 Answers

6.5.1 Solution to 6.3 Example (Teaching Video)

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

Apply the binomial theorem to $\left(z + \frac{1}{z}\right)^4$

$$\begin{aligned}
\left(z + \frac{1}{z}\right)^4 &= 1 \times z^4 \times \left(\frac{1}{z}\right)^0 \\
&+ 4 \times z^3 \times \left(\frac{1}{z}\right)^1 \\
&+ 6 \times z^2 \times \left(\frac{1}{z}\right)^2 \\
&+ 4 \times z^1 \times \left(\frac{1}{z}\right)^3 \\
&+ 1 \times z^0 \times \left(\frac{1}{z}\right)^4 \\
\therefore \left(z + \frac{1}{z}\right)^4 &= z^4 + 4z^2 + 6 + 4\left(\frac{1}{z^2}\right) + \left(\frac{1}{z^4}\right) \\
(2\cos\theta)^4 &= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6 \\
16\cos^4\theta &= 2\cos 4\theta + 8\cos 2\theta + 6 \\
8\cos^4\theta &= \cos 4\theta + 4\cos 2\theta + 3 \\
\cos^4\theta &= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3) \qquad \square
\end{aligned}$$

[4 marks]

6.5.2 Solutions to 6.4 Exercise

Answer 1

(i)

$$\left(z + \frac{1}{z}\right)^2 = \left(z^2 + \frac{1}{z^2}\right) + 2$$

$$(2\cos\theta)^2 = 2\cos 2\theta + 2$$

$$4\cos^2\theta = 2\cos 2\theta + 2$$

$$\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$$

[2 marks]

(ii)

$$\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2\theta + 1 \, d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[0 + \frac{\pi}{2} - 0 - 0\right]$$

$$= \frac{\pi}{4}$$

[2 marks]

(iii)

$$\left(z - \frac{1}{z}\right)^2 = \left(z^2 + \frac{1}{z^2}\right) - 2$$

$$(2\mathrm{i}\sin\theta)^2 = 2\cos 2\theta - 2$$

$$-4\sin^2\theta = 2\cos 2\theta - 2$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

$$\int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{6}} 1 - \cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{1}{2} \times \frac{\sqrt{3}}{2} - 0 + 0\right]$$

$$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

$$= \frac{2\pi - 3\sqrt{3}}{24}$$

[3 marks]

Answer 2

(i)

$$\left(z - \frac{1}{z} \right)^3 = z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$$

$$= \left(z^3 - \frac{1}{z^3} \right) - 3\left(z - \frac{1}{z} \right)$$

$$(\ 2\mathrm{i}\,sin\,\theta\ )^3 = 2\mathrm{i}\,sin\,3\theta - 3 \times 2\mathrm{i}\,sin\,\theta$$

$$-8\mathrm{i}\,sin^3\,\theta = 2\mathrm{i}\,sin\,3\theta - 6\mathrm{i}\,sin\,\theta$$

$$sin^3\,\theta = -\frac{1}{4}\,sin\,3\theta + \frac{3}{4}\,sin\,\theta$$

That is, $p = -\frac{1}{4}$ and $q = \frac{3}{4}$

[4 marks]

(ii)

$$\text{LHS} = sin\left(\frac{\pi}{2} - \theta \right)$$

$$= sin\left(\frac{\pi}{2}\right)cos\,\theta - cos\left(\frac{\pi}{2}\right)sin\,\theta$$

$$= cos\,\theta$$

$$= \text{RHS} \qquad \square$$

$$\text{LHS} = sin\left(\frac{3\pi}{2} - 3\theta \right)$$

$$= sin\left(\frac{3\pi}{2}\right)cos\,3\theta - cos\left(\frac{3\pi}{2}\right)sin\,3\theta$$

$$= -cos\,3\theta$$

$$= \text{RHS} \qquad \square$$

[2 marks]

(iii) Substitute in $\left(\frac{\pi}{2} - \theta \right)$ in place of θ in

$$sin^3\,\theta = -\frac{1}{4}\,sin\,3\theta + \frac{3}{4}\,sin\,\theta$$

to get $$sin^3\left(\frac{\pi}{2} - \theta \right) = -\frac{1}{4}\,sin\left(\frac{3\pi}{2} - 3\theta \right) + \frac{3}{4}\,sin\left(\frac{\pi}{2} + \theta \right)$$

$$\therefore\ cos^3\,\theta = \frac{1}{4}\,cos\,3\theta + \frac{3}{4}\,cos\,\theta$$

i.e. $r = \frac{1}{4}$ and $s = \frac{3}{4}$

[2 marks]

Answer 3

(i)

$$\left(z^2 + \frac{1}{z^2} \right)^3 = z^6 + 3z^2 + \frac{3}{z^2} + \frac{1}{z^6}$$

$$= \left(z^6 + \frac{1}{z^6} \right) + 3\left(z^2 + \frac{1}{z^2} \right)$$

$$= 2\cos 6\theta + 6\cos 2\theta$$

[3 marks]

(ii)

$$\therefore \ (2\cos 2\theta)^3 = 2\cos 6\theta + 6\cos 2\theta$$

$$\cos^3 2\theta = \frac{1}{4}\cos 6\theta + \frac{3}{4}\cos 2\theta$$

$$\text{i.e. } a = \frac{1}{4} \text{ and } b = \frac{3}{4}$$

[3 marks]

(iii)

$$\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \frac{1}{4}\int_0^{\frac{\pi}{6}} \cos 6\theta + 3\cos 2\theta \, d\theta$$

$$= \frac{1}{4}\left[\frac{1}{6}\sin 6\theta + \frac{3}{2}\sin 2\theta \right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{24} \times 0 + \frac{3}{8} \times \frac{\sqrt{3}}{2} - 0 - 0$$

$$= \frac{3\sqrt{3}}{16}$$

$$\text{i.e. } k = \frac{3}{16} \text{ (A rational constant)} \qquad \square$$

[4 marks]

Answer 4

(a)

$$\left(z - \frac{1}{z} \right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

$$(2i\sin\theta)^5 = \left(z^5 - \frac{1}{z^5} \right) - \left(z^3 - \frac{1}{z^3} \right) + 10\left(z - \frac{1}{z} \right)$$

$$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

$$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$$

$$\text{i.e. } a = \frac{1}{16},\ b = -\frac{5}{16} \text{ and } c = \frac{5}{8}$$

[5 marks]

(b)

$$\int_0^{\frac{\pi}{3}} sin^5\,\theta\; d\theta \;=\; \int_0^{\frac{\pi}{3}} \frac{1}{16}\, sin\, 5\theta \;-\; \frac{5}{16}\, sin\, 3\theta \;+\; \frac{5}{8}\, sin\, \theta\; d\theta$$

$$= \left[-\frac{1}{80}\, cos\, 5\theta \;+\; \frac{5}{48}\, cos\, 3\theta \;-\; \frac{5}{8}\, cos\, \theta \right]_0^{\frac{\pi}{3}}$$

$$= \left[-\frac{1}{160} \;-\; \frac{5}{48} \;-\; \frac{5}{16} \right] - \left[-\frac{1}{80} \;+\; \frac{5}{48} \;-\; \frac{5}{8} \right]$$

$$= \left[-\frac{203}{480} \right] - \left[-\frac{8}{15} \times \frac{32}{32} \right]$$

$$= \frac{53}{480} \qquad \square$$

[5 marks]

Answer 5

(i)

$$\left(z \;+\; \frac{1}{z} \right)^6 \;=\; z^6 \;+\; 6\, z^4 \;+\; 15\, z^2 \;+\; 20 \;+\; \frac{15}{z^2} \;+\; \frac{6}{z^4} \;+\; \frac{1}{z^6}$$

$$(\, 2\, cos\, \theta\,)^6 \;=\; \left(z^6 \;+\; \frac{1}{z^6} \right) \;+\; 6 \left(z^4 \;+\; \frac{1}{z^4} \right) \;+\; 15 \left(z^2 \;+\; \frac{1}{z^2} \right) \;+\; 20$$

$$64\, cos^6\, \theta \;=\; 2\, cos\, 6\theta \;+\; 12\, cos\, 4\theta \;+\; 30\, cos\, 2\theta \;+\; 20$$

$$32\, cos^6\, \theta \;=\; cos\, 6\theta \;+\; 6\, cos\, 4\theta \;+\; 15\, cos\, 2\theta \;+\; 10$$

i.e. $p \;=\; 1,\;\; q \;=\; 6,\;\; r \;=\; 15$ and $s \;=\; 10$

[5 marks]

(ii)

$$\int_0^{\frac{\pi}{3}} cos^6\, \theta\; d\theta \;=\; \frac{1}{32} \int_0^{\frac{\pi}{3}} cos\, 6\theta \;+\; 6\, cos\, 4\theta \;+\; 15\, cos\, 2\theta \;+\; 10\; d\theta$$

$$= \frac{1}{32} \left[\frac{1}{6}\, sin\, 6\theta \;+\; \frac{3}{2}\, sin\, 4\theta \;+\; \frac{15}{2}\, sin\, 2\theta \;+\; 10\theta \right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{32} \left[0 \;-\; \frac{3\sqrt{3}}{4} \;+\; \frac{15\sqrt{3}}{4} \;+\; \frac{10\pi}{3} \;-\; 0 \right]$$

$$= \frac{3\sqrt{3}}{32} \;+\; \frac{5\pi}{48}$$

[4 marks]

Answer 6

$$\left(z - \frac{1}{z}\right)^4\left(z + \frac{1}{z}\right)^2 = \left(z - \frac{1}{z}\right)^2\left(\left(z - \frac{1}{z}\right)\left(z + \frac{1}{z}\right)\right)^2$$

$$= \left(z - \frac{1}{z}\right)^2\left(z^2 - \frac{1}{z^2}\right)^2$$

$$= \left(\left(z - \frac{1}{z}\right)\left(z^2 - \frac{1}{z^2}\right)\right)^2$$

$$= \left(z^3 - \frac{1}{z} - z + \frac{1}{z^3}\right)^2$$

$$= \left(\left(z^3 + \frac{1}{z^3}\right) - \left(z + \frac{1}{z}\right)\right)^2$$

$$= \left(z^3 + \frac{1}{z^3}\right)^2 - 2\left(z^3 + \frac{1}{z^3}\right)\left(z + \frac{1}{z}\right) + \left(z + \frac{1}{z}\right)^2$$

$$= z^6 + 2 + \frac{1}{z^6} - 2\left(z^4 + z^2 + \frac{1}{z^2} + \frac{1}{z^4}\right) + z^2 + 2 + \frac{1}{z^2}$$

$$= z^6 + \frac{1}{z^6} + 2 - 2z^4 - \frac{2}{z^4} - 2z^2 - \frac{2}{z^2} + z^2 + 2 + \frac{1}{z^2}$$

$$= z^6 + \frac{1}{z^6} - 2z^4 - \frac{2}{z^4} - z^2 - \frac{1}{z^2} + 4$$

$$(2\mathrm{i}\,sin\,\theta)^4\;(2\,cos\,\theta)^2 = 2\,cos\,6\theta - 2 \times 2\,cos\,4\theta - 2\,cos\,2\theta + 4$$

$$64\,sin^4\theta\,cos^2\theta = 2\,cos\,6\theta - 4\,cos\,4\theta - 2\,cos\,2\theta + 4$$

$$sin^4\theta\,cos^2\theta = \frac{1}{32}\,cos\,6\theta - \frac{1}{16}\,cos\,4\theta - \frac{1}{32}\,cos\,2\theta + \frac{1}{16}$$

That is, $A = \frac{1}{32}$, $B = -\frac{1}{16}$, $C = -\frac{1}{32}$ and $D = \frac{1}{16}$

[6 marks]

7.1 Complex Number Polygons

The focus of this lesson is upon solving equations of the form $z^n = w$ where z and w are complex numbers and n is a positive integer.

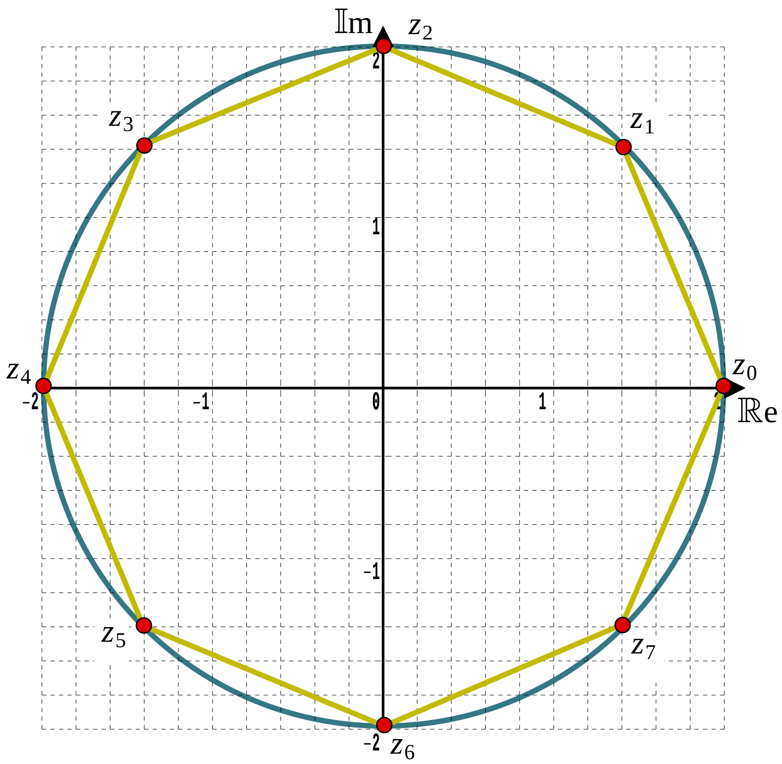
The Fundamental Theorem of Algebra states that over the complex numbers a polynomial of degree n will have n roots. For any given n and w the task is to then find all n roots.

When the equation $z^8 = 256$ is solved over the real numbers, two answers are are obtained, $z = \pm 2$, because they are the only two real numbers that make the equation true.

Over the complex numbers, however, the same equation has these eight roots;

$$z \in \{\pm 2, \pm 2\text{i}, \pm(\sqrt{2} + \sqrt{2}\,\text{i}), \pm(\sqrt{2} - \sqrt{2}\,\text{i})\}$$

On an Argand diagram this collection of roots lie at the vertices of a regular octagon with its centre at the origin.



Thus there is thus a natural relationship between the roots of equations of the form $z^n = w$ and a regular n-gon. The radius of the circle that circumscribes the n-gon and the angle by which the n-gon is rotated about the origin depend upon w.

7.2 Root Finding

Complex Roots Theorem

For a positive integer n, $w = r(\cos\theta + \mathrm{i}\sin\theta)$ has exactly n distinct n^{th} roots given by

$$z_k = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + \mathrm{i}\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

where $k = 0, 1, 2, 3, ..., n - 1$

On an Argand diagram the roots lie at the vertices of a regular n-gon with its centre at the origin.

Example

To solve the equation $z^8 = 256$ with the answers in Cartesian form...

Begin by observing that $r = 256$, $\theta = 0$ and so $w = 256$ is better expressed in trigonometric form as $w = 256(\cos 0 + \mathrm{i}\sin 0)$

From the Complex Roots Theorem, the roots will be given by,

$$z_k = \sqrt[8]{256}\left(\cos\left(\frac{0 + 2\pi k}{8}\right) + \mathrm{i}\sin\left(\frac{0 + 2\pi k}{8}\right)\right)$$

$$= 2\left(\cos\left(\frac{\pi k}{4}\right) + \mathrm{i}\sin\left(\frac{\pi k}{4}\right)\right) \text{ for } k = 0, 1, 2, ..., n - 1$$

For $k = 0 : z_0 = 2(\cos 0 + \mathrm{i}\sin 0) = 2$

For $k = 1 : z_1 = 2\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right) = \sqrt{2} + \sqrt{2}\,\mathrm{i}$

For $k = 2 : z_2 = 2\left(\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)\right) = 2\,\mathrm{i}$

For $k = 3 : z_3 = 2\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right) = -\sqrt{2} + \sqrt{2}\,\mathrm{i}$

For $k = 4 : z_4 = 2(\cos(\pi) + \mathrm{i}\sin(\pi)) = -2$

For $k = 5 : z_5 = 2\left(\cos\left(\frac{5\pi}{4}\right) + \mathrm{i}\sin\left(\frac{5\pi}{4}\right)\right) = -\sqrt{2} - \sqrt{2}\,\mathrm{i}$

For $k = 6 : z_6 = 2\left(\cos\left(\frac{3\pi}{2}\right) + \mathrm{i}\sin\left(\frac{3\pi}{2}\right)\right) = -2\,\mathrm{i}$

For $k = 7 : z_7 = 2\left(\cos\left(\frac{7\pi}{4}\right) + \mathrm{i}\sin\left(\frac{7\pi}{4}\right)\right) = \sqrt{2} - \sqrt{2}\,\mathrm{i}$

These are the roots shown on the diagram on the previous page

7.3 A Special Case

Considerable importance is attached to the special case of solving the equation $z^n = w$ when $w = 1$. The resulting roots are termed the n^{th} roots of unity. Geometrically, on an Argand diagram the roots will be equally spaced around a unit circle, with one root always at $1 + 0\text{i}$

The n^{th} Roots of Unity

The n^{th} roots of unity are the roots of the equation $z^n = 1$

If those roots are denoted as $1,\ \omega,\ \omega^2,\ \ldots,\ \omega^{n-1}$ then those roots sum to zero

$$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$$

Proof #1

The n^{th} roots of unity are the solutions to the equation $z^n - 1 = 0$.

In this equation the coefficient of z^n is 1 and the coefficient of z^{n-1} is zero.

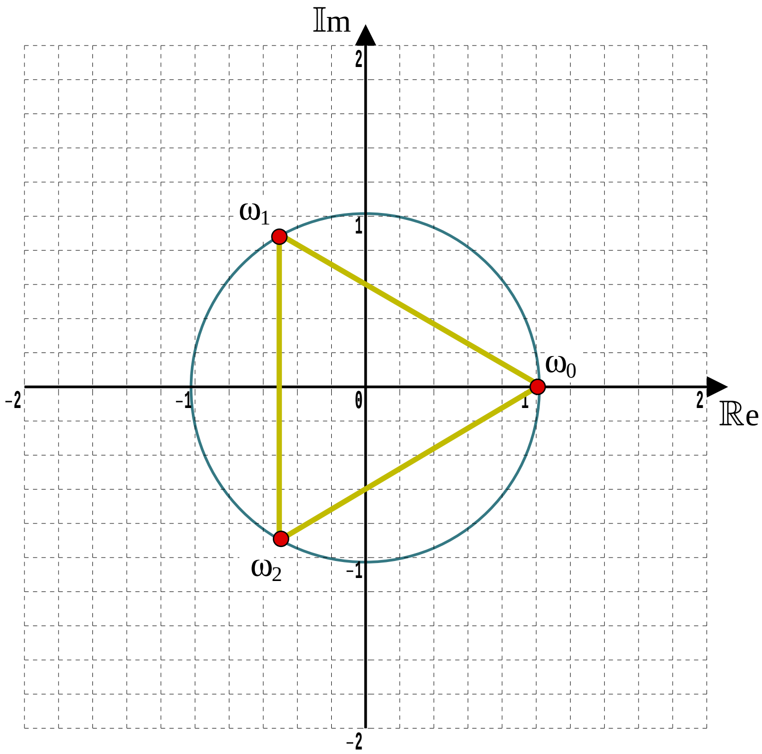
The sum of the roots of a polynomial is given by the coefficient of z^{n-1} □

Proof #2

Geometrically, the n^{th} roots of unity are equally spaced vectors around a unit circle and so their sum is the centre of the circle which is $0 + 0\text{i}$ □

Example

The cube roots of unity are $1,\ -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}\text{i}$



7.4 Exercise

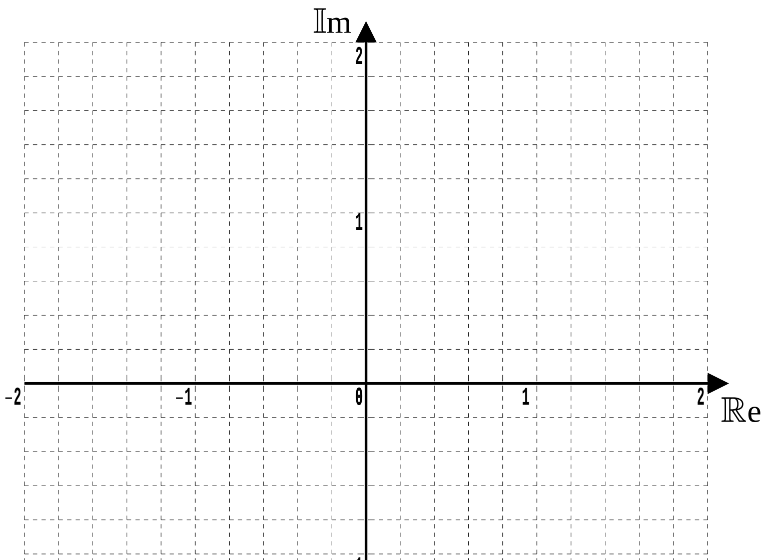
Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

(i) Solve the equation $z^3 = \frac{27}{8}\text{i}$ presenting the exact answers in Cartesian form.

[6 marks]

(ii) Plot the part (i) answers on an Argand diagram. By considering them the vertices of an equilateral triangle, determine the exact area of the triangle.



[5 marks]

Question 2

Further A-Level Examination Question from June 2009, FP2, Q2 (Edexcel)

Solve the equation

$$z^3 = 4\sqrt{2} - 4\sqrt{2}\,\mathrm{i}$$

giving your answers in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$

[6 marks]

Question 3
Further A-Level Examination Question from June 2017, FP2, Q3 (Edexcel)
Solve the equation,

$$z^3 + 32 + 32\mathrm{i}\sqrt{3} = 0$$

giving your answers in the form $r\,e^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$

[6 marks]

Question 4

(i) Find the five roots of the equation $z^5 - 1 = 0$

Give your answers in the form $r\,(\,cos\,\theta + \mathrm{i}\,sin\,\theta\,)$ where $-\pi < \theta \leqslant \pi$

[6 marks]

(ii) Hence, or otherwise, show that,

$$cos\left(\frac{2\pi}{5}\right) + cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$$

[3 marks]

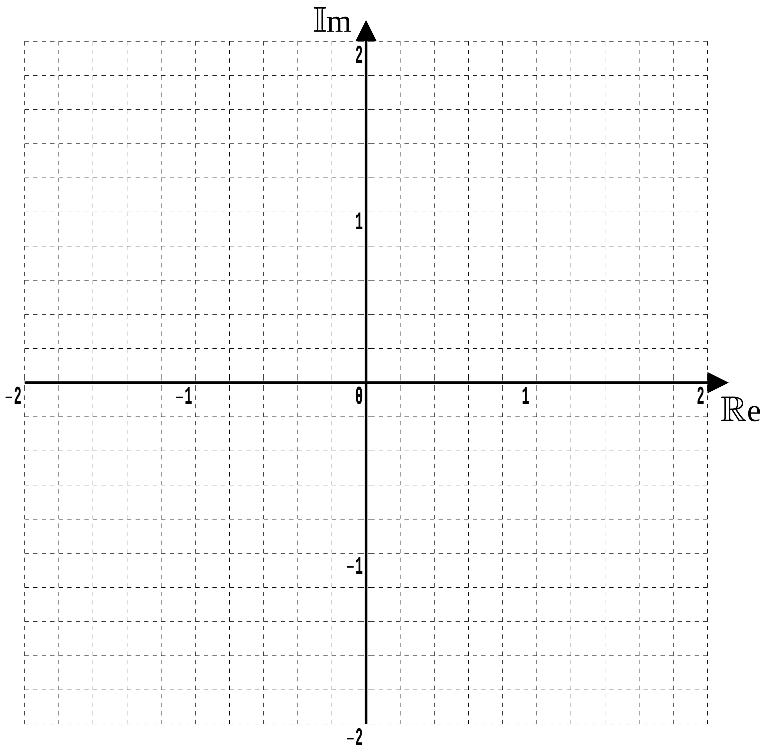
Question 5

(i) Find the three roots of the equation $(z+1)^3 = -1$

Give your answers in the form $x + iy$, where $x, y \in \mathbb{R}$

[6 marks]

(ii) Plot the points representing these three roots on an Argand diagram.



[2 marks]

(iii) Given that these three points lie on a circle, state its centre and radius

[1 mark]

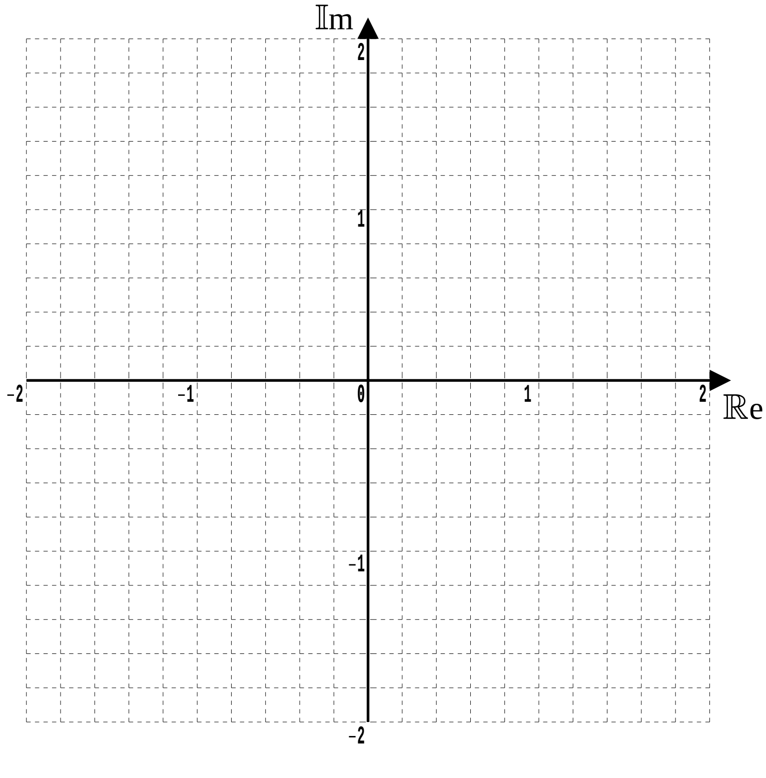
Question 6

Further A-Level Examination Question from June 2016, FP2, Q3 (Edexcel)

(a) Find the four roots of the equation $z^4 = 8\left(\sqrt{3} + \mathrm{i}\right)$ in the form $z = re^{\mathrm{i}\theta}$

[5 marks]

(b) Show these roots on an Argand diagram



[2 marks]

7.5 Answers

Answer 1

(i)
$$z^3 = \frac{27}{8}\,\mathrm{i}$$

Begin by observing that $r = \frac{27}{8}$, $\theta = \frac{\pi}{2}$ so $w = \frac{27}{8}\,\mathrm{i}$ is better expressed as,

$$w = \frac{27}{8}\left(\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)\right)$$

From the Complex Roots Theorem, the roots will be given by,

$$z_k = \sqrt[3]{\frac{27}{8}}\left(\cos\left(\frac{\frac{\pi}{2} + 2\pi k}{3}\right) + \mathrm{i}\sin\left(\frac{\frac{\pi}{2} + 2\pi k}{3}\right)\right)$$
$$= \frac{3}{2}\left(\cos\left(\frac{\pi + 4\pi k}{6}\right) + \mathrm{i}\sin\left(\frac{\pi + 4\pi k}{6}\right)\right) \text{ for } k = 0,\ 1,\ 2$$

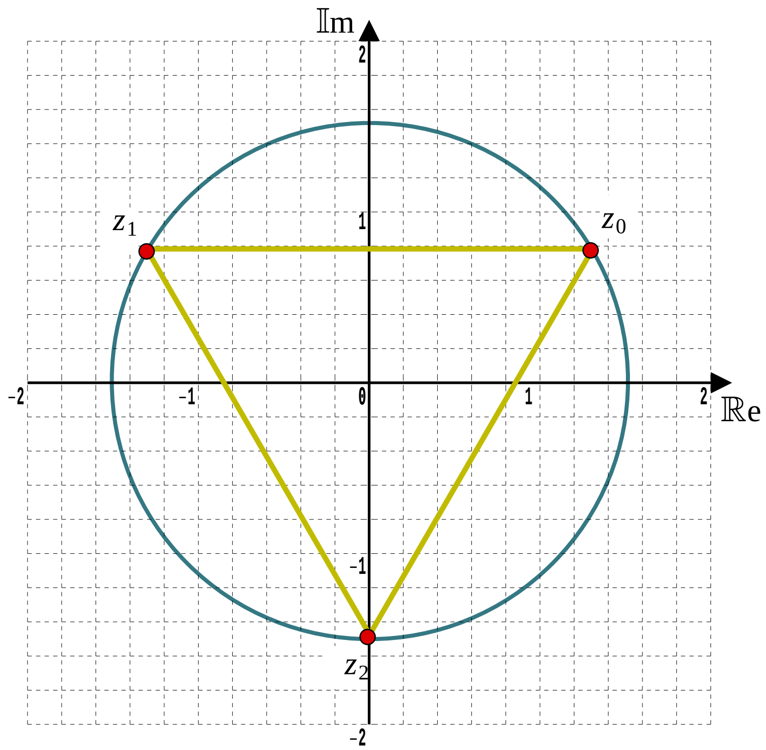
For $k = 0 : z_0 = \frac{3}{2}\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right) = \frac{3\sqrt{3}}{4} + \frac{3}{4}\mathrm{i}$

For $k = 1 : z_1 = \frac{3}{2}\left(\cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right) = -\frac{3\sqrt{3}}{4} + \frac{3}{4}\mathrm{i}$

For $k = 2 : z_2 = \frac{3}{2}\left(\cos\left(\frac{3\pi}{2}\right) + \mathrm{i}\sin\left(\frac{3\pi}{2}\right)\right) = -\frac{3}{2}\mathrm{i}$

[6 marks]

(ii)



$$Area\,\Delta = \frac{1}{2} \times \frac{6\sqrt{3}}{4} \times \frac{9}{4} = \frac{27\sqrt{3}}{16} \qquad \text{(About 2.923)}$$

[5 marks]

Answer 2

$$z^3 = 4\sqrt{2} - 4\sqrt{2}\,i = 4\sqrt{2}\,(1 - i)$$

Observe that $r = 8$ and $\theta = -\dfrac{\pi}{4}$ so $w = 4\sqrt{2} - 4\sqrt{2}\,i$ becomes,

$$w = 8\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

From the Complex Roots Theorem, the roots will be given by,

$$z_k = \sqrt[3]{8}\left(\cos\left(\frac{-\frac{\pi}{4} + 2\pi k}{3}\right) + i\sin\left(\frac{-\frac{\pi}{4} + 2\pi k}{3}\right)\right)$$

$$= 2\left(\cos\left(\frac{-\pi + 8\pi k}{12}\right) + i\sin\left(\frac{-\pi + 8\pi k}{12}\right)\right) \text{ for } k = 0,\ 1,\ 2$$

For $k = 0 : z_0 = 2\left(\cos\left(\dfrac{-\pi}{12}\right) + i\sin\left(\dfrac{-\pi}{12}\right)\right)$

For $k = 1 : z_1 = 2\left(\cos\left(\dfrac{7\pi}{12}\right) + i\sin\left(\dfrac{7\pi}{12}\right)\right)$

For $k = 2 : z_2 = 2\left(\cos\left(\dfrac{5\pi}{4}\right) + i\sin\left(\dfrac{5\pi}{4}\right)\right)$

$$= 2\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$

[6 marks]

Answer 3

$$z^3 = -32 - 32i\sqrt{3} = 32\left(-1 - \sqrt{3}\,i\right)$$

Observe that $r = 64$ and $\theta = -\dfrac{2\pi}{3}$ so $w = -32 - 32i\sqrt{3}$ becomes,

$$w = 64\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

From the Complex Roots Theorem, the roots will be given by,

$$z_k = \sqrt[3]{64}\left(\cos\left(\frac{-\frac{2\pi}{3} + 2\pi k}{3}\right) + i\sin\left(\frac{-\frac{2\pi}{3} + 2\pi k}{3}\right)\right)$$

$$= 4\left(\cos\left(\frac{-2\pi + 6\pi k}{9}\right) + i\sin\left(\frac{-2\pi + 6\pi k}{9}\right)\right) \text{ for } k = 0,\ 1,\ 2$$

For $k = 0 : z_0 = 4\left(\cos\left(\dfrac{-2\pi}{9}\right) + i\sin\left(\dfrac{-2\pi}{9}\right)\right) = 4e^{-\frac{2\pi}{9}i}$

For $k = 1 : z_1 = 4\left(\cos\left(\dfrac{4\pi}{9}\right) + i\sin\left(\dfrac{4\pi}{9}\right)\right) = 4e^{\frac{4\pi}{9}i}$

For $k = 2 : z_2 = 4\left(\cos\left(\dfrac{10\pi}{9}\right) + i\sin\left(\dfrac{10\pi}{9}\right)\right)$

$$= 4\left(\cos\left(-\frac{8\pi}{9}\right) + i\sin\left(-\frac{8\pi}{9}\right)\right) = 4e^{-\frac{8\pi}{9}i}$$

[6 marks]

Answer 4

(i)

$$z^5 - 1 = 0 \Leftrightarrow z^5 = 1$$

Observe that $r = 1$ and $\theta = 0$ so $w = 1$ becomes,

$$w = 1\left(cos\left(0 \right) + \mathrm{i}\, sin\left(0 \right) \right)$$

From the Complex Roots Theorem, the roots will be given by,

$$\begin{aligned} z_k &= \sqrt[5]{1}\left(cos\left(\frac{0 + 2\pi k}{5} \right) + \mathrm{i}\, sin\left(\frac{0 + 2\pi k}{5} \right) \right) \\ &= \left(cos\left(\frac{2\pi k}{5} \right) + \mathrm{i}\, sin\left(\frac{2\pi k}{5} \right) \right) \text{ for } k = 0,\ 1,\ 2,\ 3,\ 4 \end{aligned}$$

For $k = 0 : z_0 = \left(cos\left(0 \right) + \mathrm{i}\, sin\left(0 \right) \right)$

For $k = 1 : z_1 = \left(cos\left(\frac{2\pi}{5} \right) + \mathrm{i}\, sin\left(\frac{2\pi}{5} \right) \right)$

For $k = 2 : z_2 = \left(cos\left(\frac{4\pi}{5} \right) + \mathrm{i}\, sin\left(\frac{4\pi}{5} \right) \right)$

For $k = 3 : z_2 = \left(cos\left(\frac{6\pi}{5} \right) + \mathrm{i}\, sin\left(\frac{6\pi}{5} \right) \right)$

$$= \left(cos\left(-\frac{4\pi}{5} \right) + \mathrm{i}\, sin\left(-\frac{4\pi}{5} \right) \right)$$

For $k = 4 : z_2 = \left(cos\left(\frac{8\pi}{5} \right) + \mathrm{i}\, sin\left(\frac{8\pi}{5} \right) \right)$

$$= \left(cos\left(-\frac{2\pi}{5} \right) + \mathrm{i}\, sin\left(-\frac{2\pi}{5} \right) \right)$$

[6 marks]

(ii) The sum of the 5 roots is zero.
This means the real part of the sum must be zero and the imaginary part of the sum must also be zero.
Focussing on the real part,

$$cos\left(0 \right) + cos\left(\frac{2\pi}{5} \right) + cos\left(\frac{4\pi}{5} \right) + cos\left(-\frac{4\pi}{5} \right) + cos\left(-\frac{2\pi}{5} \right) = 0$$

$$1 + cos\left(\frac{2\pi}{5} \right) + cos\left(\frac{4\pi}{5} \right) + cos\left(\frac{4\pi}{5} \right) + cos\left(\frac{2\pi}{5} \right) = 0$$

...the last line is because $cos\ \theta$ is an even function...

$$2\, cos\left(\frac{2\pi}{5} \right) + 2\, cos\left(\frac{4\pi}{5} \right) = -1$$

$$cos\left(\frac{2\pi}{5} \right) + cos\left(\frac{4\pi}{5} \right) = -\frac{1}{2} \qquad \square$$

[3 marks]

Answer 5

(i) Let $v = z + 1$ in which case the equation becomes,

$$v^3 = -1$$

Observe that $r = 1$ and $\theta = \pi$ so $w = -1$ becomes,

$$w = 1\left(\cos(\pi) + \mathrm{i}\sin(\pi)\right)$$

From the Complex Roots Theorem, the roots will be given by,

$$v_k = \sqrt[3]{1}\left(\cos\left(\frac{-\pi + 2\pi k}{3}\right) + \mathrm{i}\sin\left(\frac{-\pi + 2\pi k}{3}\right)\right) \quad \text{for } k = 0,\ 1,\ 2$$

For $k = 0 : v_0 = \left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

$$\therefore\ z_0 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$$

For $k = 1 : v_1 = \left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$

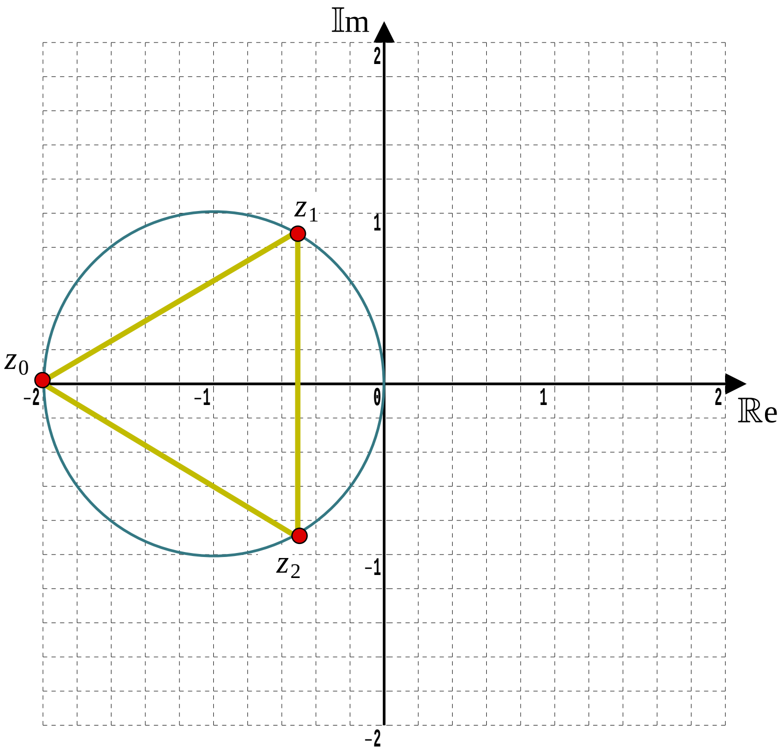
$$\therefore\ z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

For $k = 2 : v_2 = \left(\cos(\pi) + \mathrm{i}\sin(\pi)\right) = -1$

$$z_2 = -2$$

[6 marks]

(ii)



[2 marks]

(iii) Centre $(-1,\ 0)$ That is, $-1 + 0\mathrm{i}$ and radius 1

[1 mark]

Answer 6

(a)

$$z^4 = 8\left(\sqrt{3} + \mathrm{i}\right)$$

Observe that $r = 16$ and $\theta = \dfrac{\pi}{6}$ so $w = -8\left(\sqrt{3} + \mathrm{i}\right)$ becomes,

$$w = 16\left(cos\left(\frac{\pi}{6}\right) + \mathrm{i}\,sin\left(\frac{\pi}{6}\right)\right)$$

From the Complex Roots Theorem, the roots will be given by,

$$z_k = \sqrt[4]{16}\left(cos\left(\frac{\frac{\pi}{6} + 2\pi k}{4}\right) + \mathrm{i}\,sin\left(\frac{\frac{\pi}{6} + 2\pi k}{4}\right)\right)$$

$$= 2\left(cos\left(\frac{\pi + 12\pi k}{24}\right) + \mathrm{i}\,sin\left(\frac{\pi + 12\pi k}{24}\right)\right) \text{ for } k = 0,\ 1,\ 2,\ 3$$

For $k = 0 : z_0 = 2\left(cos\left(\dfrac{\pi}{24}\right) + \mathrm{i}\,sin\left(\dfrac{\pi}{24}\right)\right) = 2e^{\frac{\pi}{24}\mathrm{i}}$

$$\therefore\ z_0 = 1.98 + 0.26\mathrm{i}$$

For $k = 1 : z_1 = 2\left(cos\left(\dfrac{13\pi}{24}\right) + \mathrm{i}\,sin\left(\dfrac{13\pi}{24}\right)\right) = 2e^{\frac{13\pi}{24}\mathrm{i}}$

$$\therefore\ z_1 = -0.26 + 1.98\mathrm{i}$$

For $k = 2 : z_2 = 2\left(cos\left(\dfrac{25\pi}{24}\right) + \mathrm{i}\,sin\left(\dfrac{25\pi}{24}\right)\right)$

$$= 2\left(cos\left(-\frac{23\pi}{24}\right) + \mathrm{i}\,sin\left(-\frac{23\pi}{24}\right)\right) = 2e^{-\frac{23\pi}{24}\mathrm{i}}$$

$$\therefore\ z_2 = -1.98 - 0.26\mathrm{i}$$

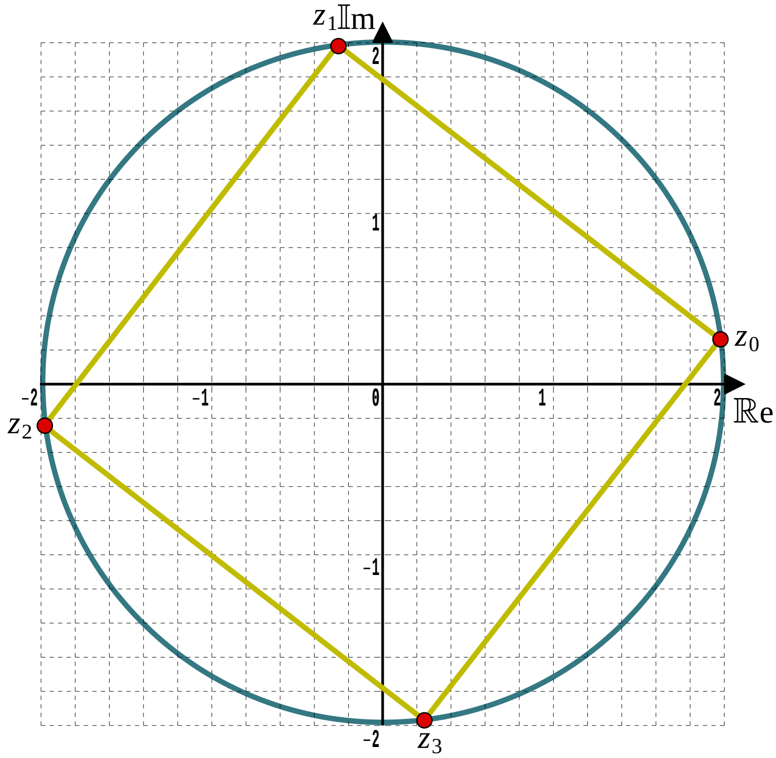
For $k = 3 : z_3 = 2\left(cos\left(\dfrac{37\pi}{24}\right) + \mathrm{i}\,sin\left(\dfrac{37\pi}{24}\right)\right)$

$$= 2\left(cos\left(-\frac{11\pi}{24}\right) + \mathrm{i}\,sin\left(-\frac{11\pi}{24}\right)\right) = 2e^{-\frac{11\pi}{24}\mathrm{i}}$$

$$\therefore\ z_3 = 0.26 - 1.98\mathrm{i}$$

[5 marks]

(b)



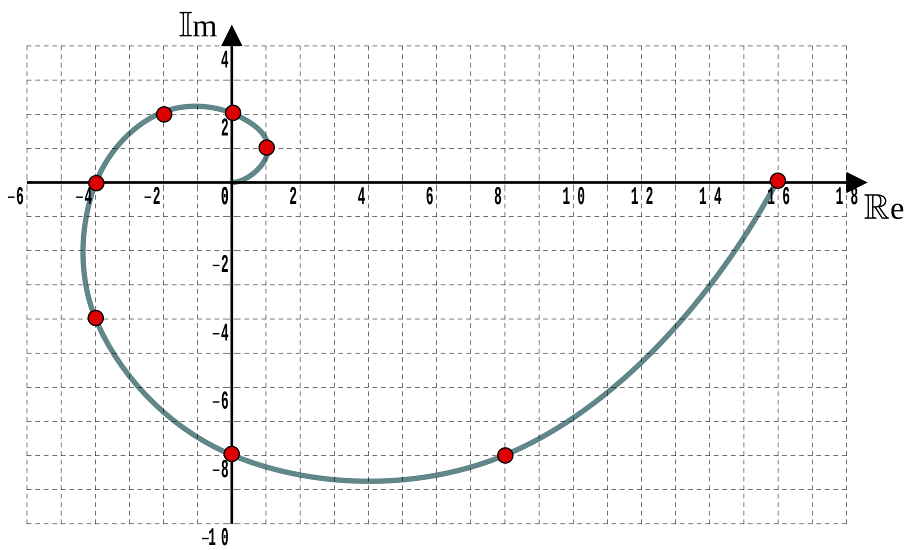
[2 marks]

8.1 The Spiral Similarity

In lesson 3† the effect of repeatedly multiplying the point $(1 + \mathrm{i})$ by the complex number $(1 + \mathrm{i})$ was investigated. Although the calculation can be performed with the point and the multiplying number both in Cartesian form, the use of exponential form made both the calculation easier, and clarified the underlying structure of the process, especially by not simplifying fractions or insisting that θ be in the range $-\pi < \theta \leqslant \pi$

n	z^n (exponential form) $r\,e^{\mathrm{i}\theta}$	z^n (Cartesian form) $x + \mathrm{i}\,y$
1	$\sqrt{2}\,e^{\frac{\pi}{4}\mathrm{i}}$	$1 + \mathrm{i}$
2	$2\,e^{\frac{2\pi}{4}\mathrm{i}}$	$0 + 2\,\mathrm{i}$
3	$2\sqrt{2}\,e^{\frac{3\pi}{4}\mathrm{i}}$	$-2 + 2\,\mathrm{i}$
4	$4\,e^{\frac{4\pi}{4}\mathrm{i}}$	$-4 + 0\,\mathrm{i}$
5	$4\sqrt{2}\,e^{\frac{5\pi}{4}\mathrm{i}}$	$-4 - 4\,\mathrm{i}$
6	$8\,e^{\frac{6\pi}{4}\mathrm{i}}$	$0 - 8\,\mathrm{i}$
7	$8\sqrt{2}\,e^{\frac{7\pi}{4}\mathrm{i}}$	$8 - 8\,\mathrm{i}$
8	$16\,e^{\frac{8\pi}{4}\mathrm{i}}$	$16 + 0\,\mathrm{i}$

Plotting the point after each multiplication revealed an attractive spiral lurking in the background;



† Lesson 3, Exercise 3.3, Question 6

Each successive multiplication by $\sqrt{2}\, e^{\frac{\pi}{4}\mathrm{i}}$ rotated the point by $\dfrac{\pi}{4}$ radians and increasing its distance from the origin by a length scale factor of $\sqrt{2}$

This type of transformation, a rotation in combination with an enlargement is termed a spiral similarity.

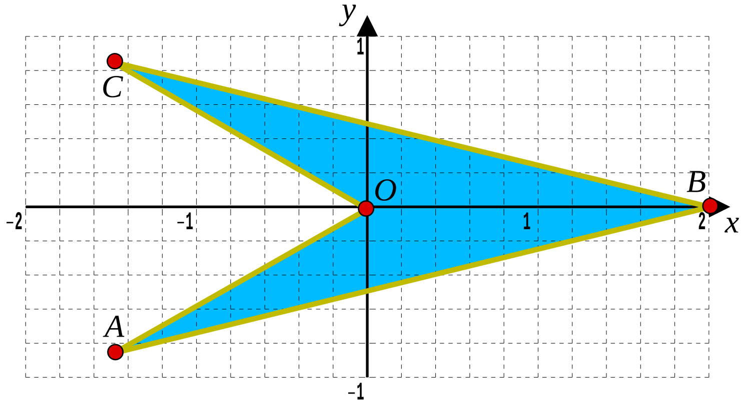
The Spiral Similarity

Multiplication of a point, or a set of points, on an Argand diagram by $r\, e^{\theta \mathrm{i}}$ gives rise to a spiral similarity where all points are rotated anticlockwise about the origin by an angle θ, and their distances from the origin alter by the factor r.

When $r = 1$ the transformation reduces to a rotation by θ about the origin.

8.2 Example

A dart has vertices O, A, B and C. B is the point (2, 0) and A and C are spiral similarities of B rotated by $\pm\,\dfrac{5\pi}{6}$ radians about the origin and scale factor $\dfrac{\sqrt{3}}{2}$.



Find the exact Cartesian coordinates of A and C and the exact area of the dart.

Teaching Video : http://www.NumberWonder.co.uk/v9099/8.mp4

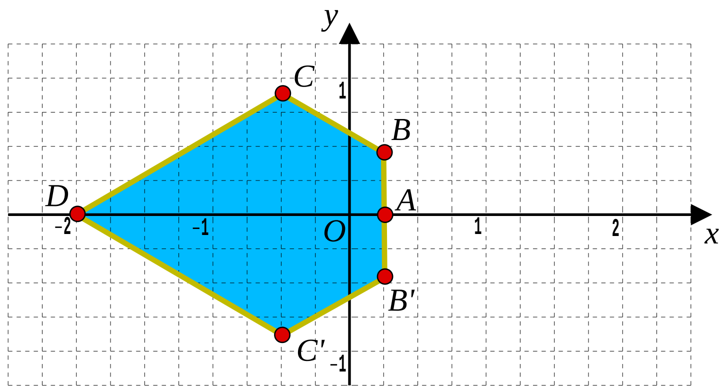
[6 marks]

8.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

An irregular pentagon has vertices B, C, D, C' and B' where C' and B' are reflections in the x-axis of C and B respectively, as shown in the diagram.



A is the midpoint of the vertical line BB' and has coordinates $\left(\frac{1}{4},\ 0\right)$.

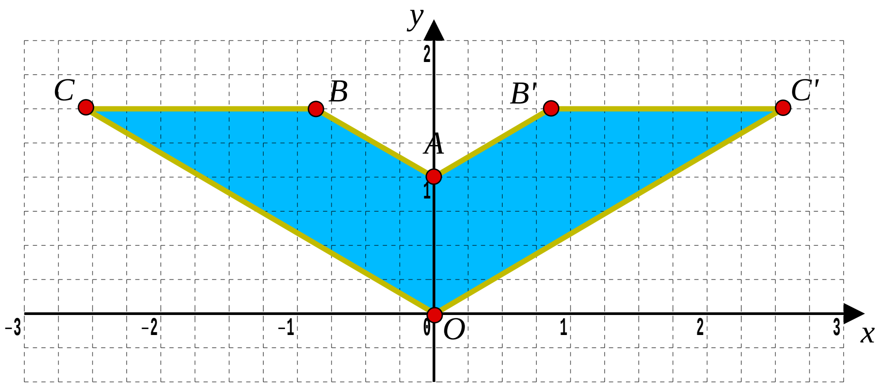
B is a spiral similarity of A rotated by $\frac{\pi}{3}$ radians about O and scale factor 2.

The same spiral similarity applied to B gives C, and applied to C gives D.
Find the exact Cartesian coordinates of B, C and D and the exact area of the pentagon.

[7 marks]

Question 2

An irregular hexagon has vertices *A*, *B*, *C*, *O*, *C′* and *B′* where *C′* and *B′* are reflections in the *y*-axis of *C* and *B* respectively, as shown in the diagram.



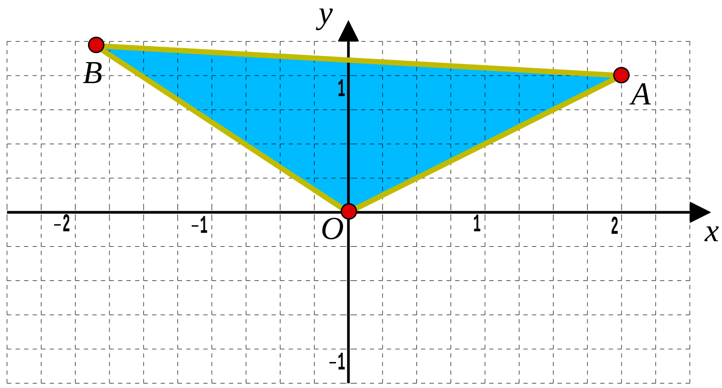
A has coordinates $(0,\ 1)$ and B is a spiral similarity of A rotated by $\dfrac{\pi}{6}$ radians about O with scale factor $\sqrt{3}$. The same spiral similarity applied to B gives C. Find the exact coordinates of B and C and the exact area of the hexagon.

[7 marks]

Question 3

Triangle *OAB* has vertex *O* at the origin and vertex *A* at coordinates (2, 1).

Vertex *B* is a rotation of vertex *A* by $\frac{2\pi}{3}$ radians about the origin.



(i) Given that the exact coordinates of vertex *B* are sought, identify the issue that will arise if vertex *A* is written in exponential form.

[2 marks]

(ii) By writing the rotation in the Cartesian form $x + \mathrm{i}y$ find the exact coordinates of vertex *B*.

[5 marks]

(iii) Find the exact perimeter of triangle *OAB*

[2 marks]

Question 4

Further A-Level Examination Question from June 2019, Core 2, Q6 (Edexcel)

In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin.

The point A represents the complex number $6 + 2\text{i}$

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + \text{i}y$, where x and y are real and exact.

[6 marks]

The points D, E and F are the midpoints of the sides of triangle ABC

(b) Find the exact area of triangle DEF

[3 marks]

Question 5

Further A-Level Examination Question from June 2018, FP2, Q2(b) (MEI)

The vertices of a square with sides of length 1 unit lie on the axes on an Argand diagram. The vertices represent the complex numbers z_1, z_2, z_3 and z_4 and the midpoints of the sides of the square represent complex numbers z_5, z_6, z_7 and z_8

(i) Express z_5, z_6, z_7 and z_8 in modulus-argument form, and hence determine a polynomial equation of degree 4, with integer coefficients, whose roots are z_5, z_6, z_7 and z_8

[4 marks]

Let $P(z)$ be a polynomial equation of degree 8, with integer coefficients, whose roots are z_1, z_2, z_3, z_4, z_5, z_6, z_7 and z_8

(ii) Explain why $P(z)$ cannot be of the form $a z^8 + b$ where a and b are integers

[1 mark]

(iii) Find $P(z)$

[4 marks]

Question 6

Further A-Level Examination Question from January 2013, FP2, Q2(b) (MEI)

(i) Express $e^{\frac{2\pi}{3}\mathrm{i}}$ in the form $x + \mathrm{i}y$ where the real numbers x and y should be given exactly

[1 mark]

An equilateral triangle in the Argand diagram has its centre at the origin.
One vertex of the triangle is at the point representing $2 + 4\mathrm{i}$

(ii) Obtain the complex numbers representing the other two vertices, giving your answers in the form $x + \mathrm{i}y$ where the real numbers x and y should be given exactly.

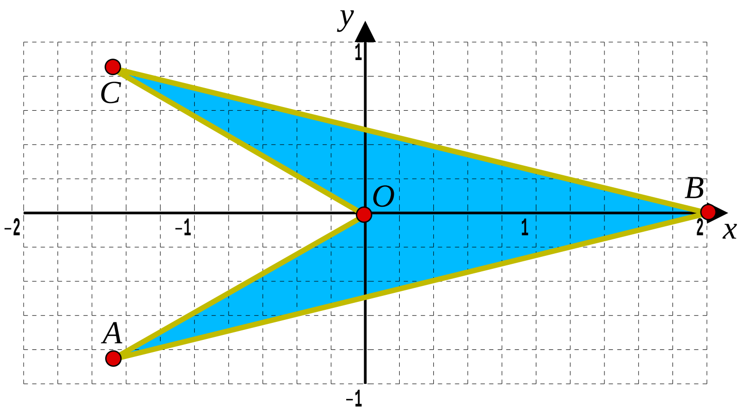
[6 marks]

(iii) Show that the length of a side is $2\sqrt{15}$

[2 marks]

8.5 Answers

8.5.1 Solutions to 8.3 Example (Teaching Video)



The dart can be thought of as lying, not in the Cartesean plane but in the complex plane; the point $B(\,2,\ \ 0\,)$ can then be written in exponential form as $2e^{0\mathrm{i}}$

The spiral similarity can be written in exponential form as $\dfrac{\sqrt{3}}{2}\ e^{\pm\frac{5\pi}{6}\mathrm{i}}$

Points A and C are thus found by the following multiplication,

$$\frac{\sqrt{3}}{2}\ e^{\pm\frac{5\pi}{6}} \times 2e^{0\mathrm{i}} = \sqrt{3}\ e^{\pm\frac{5\pi}{6}\mathrm{i}}$$

$$= \sqrt{3}\left(cos\left(\pm\frac{5\pi}{6}\right) + \mathrm{i}\ sin\left(\pm\frac{5\pi}{6}\right)\right)$$

as *cos* is an even function,

and *sin* is an odd function this can be written as

$$= \sqrt{3}\left(cos\left(\frac{5\pi}{6}\right) \pm \mathrm{i}\ sin\left(\frac{5\pi}{6}\right)\right)$$

$$= -\frac{3}{2} \pm \frac{\sqrt{3}}{2}\ \mathrm{i}$$

Although we've been working in the complex plane, the question was originally set in the Cartesean plane; our answer should be given in Cartesean coordinates.

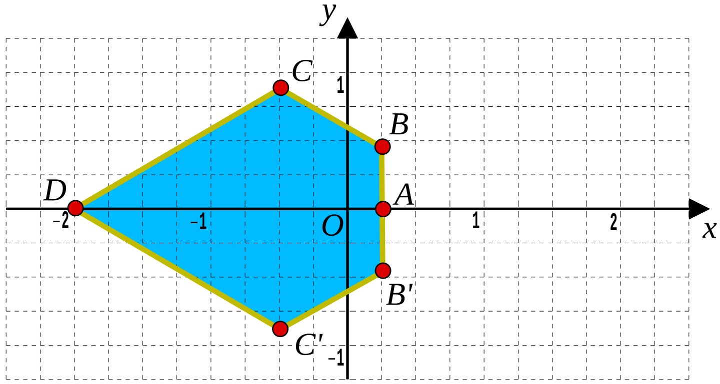
Thus, with the help of the diagram, $A\left(-\dfrac{3}{2}.\ -\ \dfrac{\sqrt{3}}{2}\right)$ and $C\left(-\dfrac{3}{2},\ \ \dfrac{\sqrt{3}}{2}\right)$

The area of the dart is given by $2 \times \dfrac{1}{2} \times 2 \times \dfrac{\sqrt{3}}{2} = \sqrt{3}$ units^2

[6 marks]

8.5.2 Solutions to 8.4 Exercise

Answer 1



The spiral similarity can be written as $2\,e^{\frac{\pi}{3}i}$ and this is applied first to A which, viewed as a complex number, can be written in exponential form as $\frac{1}{4}\,e^{0i}$

B has coordinates given by,

$$\begin{aligned} 2\,e^{\frac{\pi}{3}i} \times \frac{1}{4}\,e^{0i} &= \frac{1}{2}\,e^{\frac{\pi}{3}i} \\ &= \frac{1}{2}\left(cos\left(\frac{\pi}{3}\right) + i\,sin\left(\frac{\pi}{3}\right)\right) \\ &= \frac{1}{4} + \frac{\sqrt{3}}{4}i \qquad \Leftrightarrow \qquad B\left(\frac{1}{4},\ \frac{\sqrt{3}}{4}\right) \end{aligned}$$

C has coordinates given by,

$$\begin{aligned} 2\,e^{\frac{\pi}{3}i} \times \frac{1}{2}\,e^{\frac{\pi}{3}i} &= e^{\frac{2\pi}{3}i} \\ &= cos\left(\frac{2\pi}{3}\right) + i\,sin\left(\frac{2\pi}{3}\right) \\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i \qquad \Leftrightarrow \qquad C\left(-\frac{1}{2},\ \frac{\sqrt{3}}{2}\right) \end{aligned}$$

D has coordinates given by,

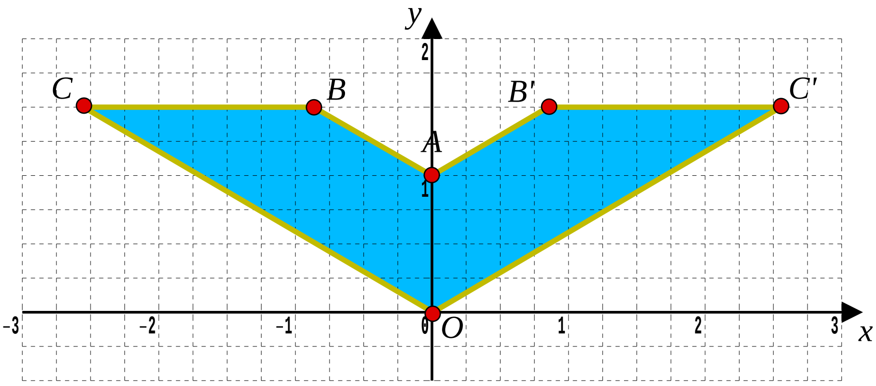
$$\begin{aligned} 2\,e^{\frac{\pi}{3}i} \times e^{\frac{2\pi}{3}i} &= 2\,e^{\pi i} \\ &= -2 \qquad \Leftrightarrow \qquad D(-2,\ 0) \end{aligned}$$

The area can be found in various ways, I cut it vertically CC' into a triangle plus a trapezium;

$$\begin{aligned} Area &= \frac{1}{2} \times \sqrt{3} \times \frac{3}{2} + \frac{1}{2}\left(\sqrt{3} + \frac{\sqrt{3}}{2}\right) \times \frac{3}{4} \\ &= \frac{3\sqrt{3}}{4} + \frac{9\sqrt{3}}{16} \\ &= \frac{21\sqrt{3}}{16} \end{aligned}$$

[7 marks]

Answer 2



The spiral similarity can be written as $\sqrt{3}\ e^{\frac{\pi}{6}\mathrm{i}}$ and this is applied first to A which can be thought of as a complex number written in exponential form as $e^{\frac{\pi}{2}\mathrm{i}}$

B has coordinates given by,

$$\begin{aligned}\sqrt{3}\ e^{\frac{\pi}{6}\mathrm{i}} \times e^{\frac{\pi}{2}\mathrm{i}} &= \sqrt{3}\ e^{\frac{2\pi}{3}\mathrm{i}}\\ &= \sqrt{3}\left(\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)\\ &= -\frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i} \qquad \Leftrightarrow \qquad B\left(-\frac{\sqrt{3}}{2},\ \frac{3}{2}\right)\end{aligned}$$

C has coordinates given by,

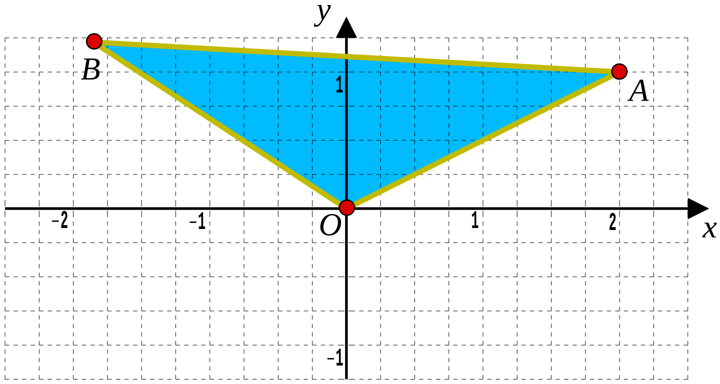
$$\begin{aligned}\sqrt{3}\ e^{\frac{\pi}{6}\mathrm{i}} \times \sqrt{3}\ e^{\frac{2\pi}{3}\mathrm{i}} &= 3\,e^{\frac{5\pi}{6}\mathrm{i}}\\ &= 3\left(\cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right)\\ &= -\frac{3\sqrt{3}}{2} + \frac{3}{2}\mathrm{i} \qquad \Leftrightarrow \qquad C\left(-\frac{3\sqrt{3}}{2},\ \frac{3}{2}\right)\end{aligned}$$

The area can be found in various ways, I thought of it as the large triangle OCC' minus the small triangle ABB',

$$\begin{aligned}Area &= \frac{1}{2} \times 2\left(-\frac{3\sqrt{3}}{2}\right) \times \frac{3}{2} - \frac{1}{2} \times 2\left(\frac{\sqrt{3}}{2}\right) \times \frac{1}{2}\\ &= \frac{9\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\\ &= 2\sqrt{3}\ \text{units}^2\end{aligned}$$

[7 marks]

Answer 3



(i) Trying to write the point A in exponential form gives $\sqrt{5}\, e^{arctan\left(\frac{1}{2}\right)}$ and the issue now is that $arctan\left(\frac{1}{2}\right)$ is now obviously expressible as an exact quantity. The result is that there is not an obvious way to obtain the exact coordinates of vertex B by this method (even although it worked so well in the previous two questions).

[2 marks]

(ii) Working on the rotation, in exponential form it would be $e^{\frac{2\pi}{3}\mathrm{i}}$ which can be turned into Cartesian form via Euler's relation;

$$e^{\frac{2\pi}{3}\mathrm{i}} = cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\, sin\left(\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i} = \frac{1}{2}\left(-1 + \sqrt{3}\,\mathrm{i}\right)$$

Now the multiplication on A can be performed,

$$\frac{1}{2}\left(-1 + \sqrt{3}\,\mathrm{i}\right)\left(2 + \mathrm{i}\right) = \frac{1}{2}\left(-2 - \mathrm{i} + 2\sqrt{3}\,\mathrm{i} + \sqrt{3}\,\mathrm{i}^2\right)$$

$$= \frac{-2 - \sqrt{3}}{2} + \frac{-1 + 2\sqrt{3}}{2}\mathrm{i}$$

So the exact coordinates of B are $\left(\frac{-2 - \sqrt{3}}{2}, \frac{-1 + 2\sqrt{3}}{2}\right)$

[5 marks]

(iii) Easiest method is to use the cosine rule to get the length of o

$$o^2 = a^2 + b^2 - 2ab\, cos\, O$$

$$= \left(\sqrt{5}\right)^2 + \left(\sqrt{5}\right)^2 - 2\left(\sqrt{5}\right)\left(\sqrt{5}\right) cos\left(\frac{2\pi}{3}\right)$$

$$= 5 + 5 - 10 \times \left(-\frac{1}{2}\right)$$

$$= 15 \Rightarrow o = \sqrt{15}$$

The exact perimeter is $2\sqrt{5} + \sqrt{15}$ or $\sqrt{5}\left(2 + \sqrt{3}\right)$

[2 marks]

Answer 4

(a) The get point B, rotate A by 120°, that is, $\dfrac{2\pi}{3}$ radians

To do this multiply by $e^{\frac{2\pi}{3}\mathrm{i}}$

$$= \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

$$= \frac{1}{2}\left(-1 + \sqrt{3}\,\mathrm{i}\right)$$

$$(6 + 2\mathrm{i})\frac{1}{2}\left(-1 + \sqrt{3}\,\mathrm{i}\right) = (3 + \mathrm{i})\left(-1 + \sqrt{3}\,\mathrm{i}\right)$$

$$= -3 + 3\sqrt{3}\,\mathrm{i} - \mathrm{i} + \sqrt{3}\,\mathrm{i}^2$$

$$= \left(-3 - \sqrt{3}\right) + \mathrm{i}\left(-1 + 3\sqrt{3}\right) \qquad \text{for } B$$

To get point C, rotate A by – 120°, that is, $-\dfrac{2\pi}{3}$ radians

To do this multiply by $e^{-\frac{2\pi}{3}\mathrm{i}}$

$$= \cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$$

$$= \frac{1}{2}\left(-1 - \sqrt{3}\,\mathrm{i}\right)$$

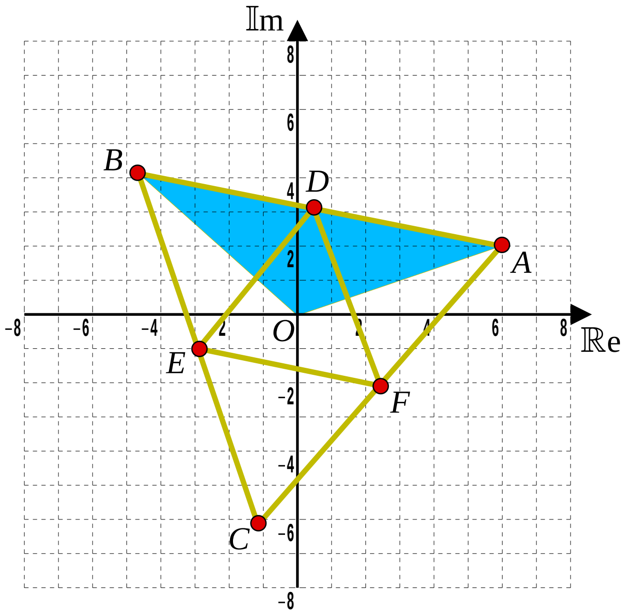
$$(6 + 2\mathrm{i})\frac{1}{2}\left(-1 - \sqrt{3}\,\mathrm{i}\right) = (3 + \mathrm{i})\left(-1 - \sqrt{3}\,\mathrm{i}\right)$$

$$= -3 - 3\sqrt{3}\,\mathrm{i} - \mathrm{i} - \sqrt{3}\,\mathrm{i}^2$$

$$= \left(-3 + \sqrt{3}\right) + \mathrm{i}\left(-1 - 3\sqrt{3}\right) \qquad \text{for } C$$

[6 marks]

(b)



There are several ways of answering this question.
I'm going to focus on the blue isosceles triangle, three copies of which will have the same area as the large equilateral triangle *ABC*

$$Area\ \Delta ABC = 3 \times \frac{1}{2} \times \sqrt{6^2 + 2^2} \times \sqrt{6^2 + 2^2} \times sin\left(\frac{2\pi}{3}\right)$$

$$= 30\sqrt{3}$$

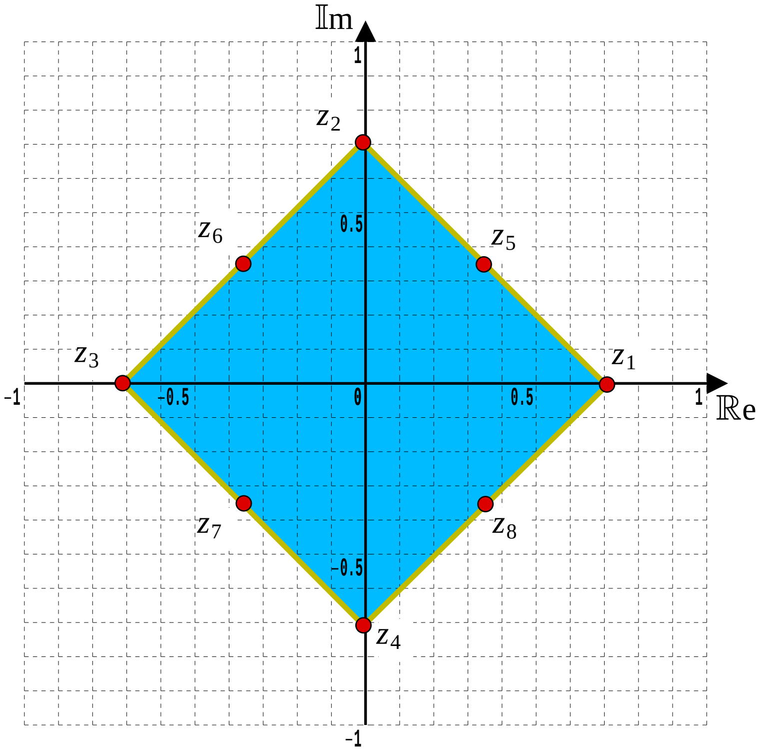
Equilateral triangle *DEF* has an area that is clearly one quarter of the area of the larger equilateral triangle *ABC*

$$Area\ \Delta DEF = \frac{1}{4} \times 30\sqrt{3}$$

$$= \frac{15\sqrt{3}}{2}$$

[3 marks]

Answer 5



(i) With reference to the diagram,

$$z_5 = \frac{1}{2}e^{\frac{\pi}{4}i} \qquad z_6 = \frac{1}{2}e^{\frac{3\pi}{4}i} \qquad z_7 = \frac{1}{2}e^{-\frac{3\pi}{4}i} \qquad z_8 = \frac{1}{2}e^{-\frac{\pi}{4}i}$$

As all four roots lie of a circle centre origin and radius 0.5 the equation will be of the form $z^4 = k$, for some constant k

By substituting in one of the points, z_5

$$\left(\frac{1}{2}e^{\frac{\pi}{4}i}\right)^4 = k$$

$$\frac{1}{16}e^{\pi i} = k$$

$$\therefore \ k = -\frac{1}{16}$$

The equation is thus, $z^4 = -\dfrac{1}{16}$

$$16z^4 + 1 = 0$$

[4 marks]

(ii) The equation cannot be of the form $a\,z^8 + b = 0$ because the roots of such an equation would all lie on the circumference of a circle centred on the origin which these eight roots do not.

[1 mark]

(iii) The factors associated with the original square are,

$$\left(z - \frac{1}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}}\right)\left(z - \frac{1}{\sqrt{2}}i\right)\left(z + \frac{1}{\sqrt{2}}i\right) = 0$$

$$\left(z^2 - \frac{1}{2}\right)\left(z^2 + \frac{1}{2}\right) = 0$$

$$z^4 - \frac{1}{4} = 0$$

$$4z^4 - 1 = 0$$

$\therefore$ For $P(z) = 0$

$$P(z) = \left(16z^4 + 1\right)\left(4z^4 - 1\right)$$

$$P(z) = 64z^8 + 20z^4 - 1$$

Any multiple of this will, of course, have the same roots

[4 marks]

Answer 6

(i)

$$e^{\frac{2\pi}{3}\mathrm{i}} = cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\,sin\left(\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i}$$

[1 mark]

(ii)

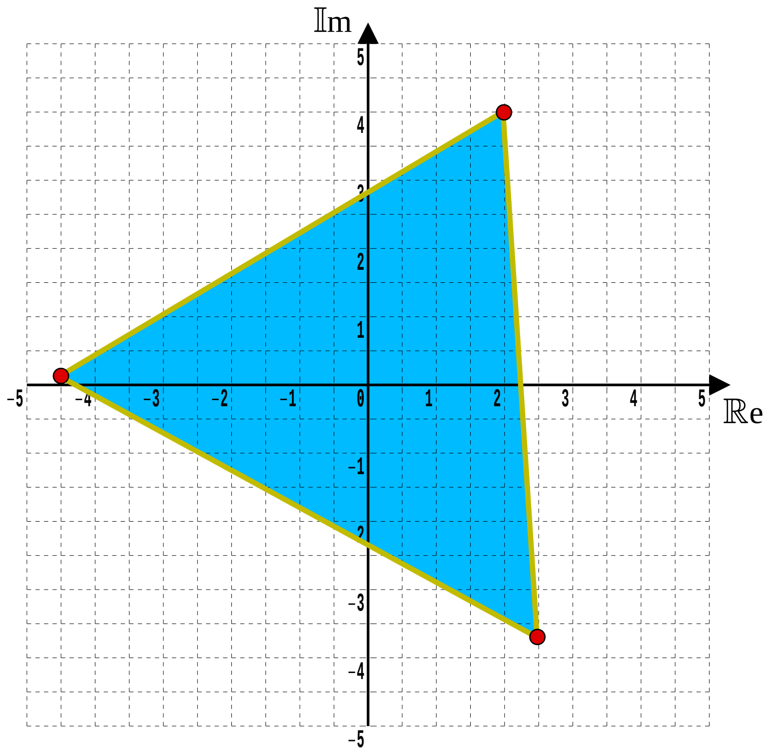
$$\frac{1}{2}\left(-1 + \sqrt{3}\,\mathrm{i}\right)\left(2 + 4\mathrm{i}\right) = \left(-1 + \sqrt{3}\,\mathrm{i}\right)\left(1 + 2\mathrm{i}\right)$$

$$= \left(-1 - 2\sqrt{3}\right) + \mathrm{i}\left(-2 + \sqrt{3}\right)$$

$$\frac{1}{2}\left(-1 - \sqrt{3}\,\mathrm{i}\right)\left(2 + 4\mathrm{i}\right) = \left(-1 - \sqrt{3}\,\mathrm{i}\right)\left(1 + 2\mathrm{i}\right)$$

$$= \left(-1 + 2\sqrt{3}\right) + \mathrm{i}\left(-2 - \sqrt{3}\right)$$

[6 marks]



(iii)

$$\sqrt{\left(\left(-1 + 2\sqrt{3}\right) - \left(-1 - 2\sqrt{3}\right)\right)^2 + \left(\left(-2 - \sqrt{3}\right) - \left(-2 + \sqrt{3}\right)\right)^2}$$

$$= \sqrt{\left(-1 + 2\sqrt{3} + 1 + 2\sqrt{3}\right)^2 + \left(-2 - \sqrt{3} + 2 - \sqrt{3}\right)^2}$$

$$= \sqrt{\left(4\sqrt{3}\right)^2 + \left(-2\sqrt{3}\right)^2}$$

$$= \sqrt{16 \times 3 + 4 \times 3} = \sqrt{60} = \sqrt{4 \times 15} = 2\sqrt{15} \qquad \square$$

[2 marks]

9.1 Complex Geometric Progressions

It has been established over the previous eight lessons that many results from the world of real numbers have corresponding results in the complex numbers. For example, several of the rules of indices carry across, albeit with very different reasoning behind why they still work. One very useful real number result that more directly carries across is that for summing a Geometric Progression.

The Sum of a Geometric Progression

- $\displaystyle\sum_{k=0}^{n-1} a\,r^k = a + ar + a\,r^2 + \ldots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}$
- $\displaystyle\sum_{k=0}^{\infty} a\,r^k = a + ar + a\,r^2 + \ldots = \frac{a}{1 - r},\ |r| < 1$

where $a,\ r \in \mathbb{C}$

9.2 Two Useful Results

When working with complex numbers that are in Geometric Progression the following pair of results are worth keeping in mind,

---

Two Trigonometric Functions, as Exponentials

- $\cos(n\theta) = \dfrac{1}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}\right)$
- $\sin(n\theta) = \dfrac{1}{2\mathrm{i}}\left(e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta}\right)$

Proof

$$\begin{aligned}
\text{RHS} &= \frac{1}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}\right) \\
&= \frac{\cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(-n\theta) + \mathrm{i}\sin(-n\theta)}{2} && \text{using Euler's Relation} \\
&= \frac{\cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(n\theta) - \mathrm{i}\sin(n\theta)}{2} && \text{even and odd functions} \\
&= \frac{2\cos(n\theta)}{2} \\
&= \cos(n\theta) \\
&= \text{LHS} \qquad \square
\end{aligned}$$

A proof of the other result is very similar and is left as an exercise for the reader.

9.3 Example

Further A-Level Examination Question from January 2007, Q2(b)(ii) (MEI)

Express $e^{-\frac{\theta}{2}\mathrm{i}} + e^{\frac{\theta}{2}\mathrm{i}}$ in simplified trigonometric form and hence, or otherwise

show that $1 + e^{\theta\mathrm{i}} = 2\,e^{\frac{\theta}{2}\mathrm{i}}\,cos\left(\frac{\theta}{2}\right)$

Teaching Video : http://www.NumberWonder.co.uk/v9099/9.mp4

[4 marks]

9.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Express $e^{-\frac{\theta}{2}\mathrm{i}} - e^{\frac{\theta}{2}\mathrm{i}}$ in simplified trigonometric form and hence, or otherwise

show that $e^{\theta\mathrm{i}} - 1 = 2\mathrm{i}\,e^{\frac{\theta}{2}\mathrm{i}}\,sin\left(\frac{\theta}{2}\right)$

[4 marks]

Question 2

The convergent infinite series C and S are defined as,

$$C = 1 + \frac{1}{3}\cos\theta + \frac{1}{9}\cos 2\theta + \frac{1}{27}\cos 3\theta + \ldots$$

$$S = \frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \frac{1}{27}\sin 3\theta + \ldots.$$

(i) Show that $C + \mathrm{i}S = \dfrac{3}{3 - e^{\theta \mathrm{i}}}$

[4 marks]

(ii) Hence show that $C = \dfrac{9 - 3\cos\theta}{10 - 6\cos\theta}$ and find a similar expression for S

[4 marks]

Question 3

Further A-Level Examination Question from June 2019, Paper 2, Q4 (Edexcel)

The infinite series C and S are defined as,

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots.$$

Given that the series C and S are both convergent,

(a) Show that $C + \mathrm{i}S = \dfrac{2e^{\mathrm{i}\theta}}{2 - e^{4\mathrm{i}\theta}}$

[4 marks]

(ii) Hence show that $S = \dfrac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$

[4 marks]

Question 4

(i) Use Euler's Relation to express $e^{\mathrm{i}n\theta}$ in trigonometric form

[1 mark]

(ii) Use Euler's Relation to express $e^{-\mathrm{i}n\theta}$ in trigonometric form

[1 mark]

(iii) By treating your part (i) and part (ii) answers as a pair of simultaneous equation, and adding them together, prove that,

$$cos(n\theta) = \frac{1}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta} \right)$$

[1 mark]

(iv) In a similar manner, prove a similar result for $sin(n\theta)$

[1 mark]

Question 5

Show that $\sum_{k=0}^{7} (1 + \mathrm{i})^{k} = -15\mathrm{i}$

[4 marks]

Question 6

(i) Show that $\left(2 + e^{i\theta}\right)\left(2 + e^{-i\theta}\right) = 5 + 4\cos\theta$

[2 marks]

The convergent infinite series C and S are defined by

$$C = 1 - \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta - \frac{1}{8}\cos 3\theta + \ldots$$

$$S = \frac{1}{2}\sin\theta - \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots$$

(ii) By considering $C - iS$ show that $C = \dfrac{4 + 2\cos\theta}{5 + 4\cos\theta}$ and write down the corresponding expression for S

[4 marks]

Question 7

Further A-Level Examination Question from January 2013, FP2, Q2 (MEI)

(i) Show that, $1 + e^{2\theta \mathrm{i}} = 2\cos\theta\ (\cos\theta + \mathrm{i}\sin\theta)$

[2 marks]

(ii) The series C and S are defined as follows,

$$C = 1 + \binom{n}{1}\cos 2\theta + \binom{n}{2}\cos 4\theta + \ldots + \cos 2n\theta$$

$$S = \binom{n}{1}\sin 2\theta + \binom{n}{2}\sin 4\theta + \ldots + \sin 2n\theta$$

By considering $C + \mathrm{i}S$ show that $C = 2^n \cos^n\theta \cos n\theta$ and find a corresponding expression for S

[7 marks]

Question 8

Determine, in as simple a form as possible, the value of,

(i) $\displaystyle\sum_{k=0}^{3} \mathrm{i}^k$

[1 mark]

(ii) $\displaystyle\sum_{k=1}^{15} \mathrm{i}^k$

[2 marks]

(iii) $\displaystyle\sum_{k=0}^{12} z^k$ given that $z = e^{\frac{\pi}{2}\mathrm{i}}$

[2 marks]

(iv) $\displaystyle\sum_{k=0}^{121} z^k$ given that $z = e^{-\frac{\pi}{2}\mathrm{i}}$

[2 marks]

9.5 Answers

9.5.1 Solution to 9.3 Example (Teaching Video)

$$\begin{aligned} e^{-\frac{\theta}{2}\mathrm{i}} + e^{\frac{\theta}{2}\mathrm{i}} &= \cos\left(-\frac{\theta}{2}\right) + \mathrm{i}\sin\left(-\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right) + \mathrm{i}\sin\left(\frac{\theta}{2}\right) \\ &= \cos\left(\frac{\theta}{2}\right) - \mathrm{i}\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right) + \mathrm{i}\sin\left(\frac{\theta}{2}\right) \\ &= 2\cos\left(\frac{\theta}{2}\right) \end{aligned}$$

To show that, $1 + e^{\theta\mathrm{i}} = 2e^{\frac{\theta}{2}\mathrm{i}}\cos\left(\frac{\theta}{2}\right)$

$$\begin{aligned} \text{RHS} &= 2e^{\frac{\theta}{2}\mathrm{i}}\cos\left(\frac{\theta}{2}\right) \\ &= e^{\frac{\theta}{2}\mathrm{i}}\,2\cos\left(\frac{\theta}{2}\right) \\ &= e^{\frac{\theta}{2}\mathrm{i}}\left(e^{-\frac{\theta}{2}\mathrm{i}} + e^{\frac{\theta}{2}\mathrm{i}}\right) \\ &= e^{0\mathrm{i}} + e^{\theta\mathrm{i}} \\ &= 1 + e^{\theta\mathrm{i}} \\ &= \text{LHS} \qquad \square \end{aligned}$$

[4 marks]

9.5.2 Solutions to 9.4 Exercise

Answer 1

$$
\begin{aligned}
e^{-\frac{\theta}{2}\mathrm{i}} - e^{\frac{\theta}{2}\mathrm{i}} &= cos\left(-\frac{\theta}{2}\right) + \mathrm{i}\, sin\left(-\frac{\theta}{2}\right) - cos\left(\frac{\theta}{2}\right) - \mathrm{i}\, sin\left(\frac{\theta}{2}\right) \\
&= cos\left(\frac{\theta}{2}\right) - \mathrm{i}\, sin\left(\frac{\theta}{2}\right) - cos\left(\frac{\theta}{2}\right) - \mathrm{i}\, sin\left(\frac{\theta}{2}\right) \\
&= -2\mathrm{i}\, sin\left(\frac{\theta}{2}\right)
\end{aligned}
$$

To show that, $e^{\theta\mathrm{i}} - 1 = 2\mathrm{i}\, e^{\frac{\theta}{2}\mathrm{i}}\, sin\left(\frac{\theta}{2}\right)$

$$
\begin{aligned}
\text{RHS} &= 2\mathrm{i}\, e^{\frac{\theta}{2}\mathrm{i}}\, sin\left(\frac{\theta}{2}\right) \\
&= e^{\frac{\theta}{2}\mathrm{i}}\, 2\mathrm{i}\, sin\left(\frac{\theta}{2}\right) \\
&= -e^{\frac{\theta}{2}\mathrm{i}} \left(-2\mathrm{i}\, sin\left(\frac{\theta}{2}\right)\right) \\
&= -e^{\frac{\theta}{2}\mathrm{i}} \left(e^{-\frac{\theta}{2}\mathrm{i}} - e^{\frac{\theta}{2}\mathrm{i}}\right) \\
&= -e^{0\mathrm{i}} + e^{\theta\mathrm{i}} \\
&= e^{\theta\mathrm{i}} - 1 \\
&= \text{LHS} \qquad \square
\end{aligned}
$$

[4 marks]

Answer 2

(i)

$$C + \mathrm{i}S = 1$$

$$+ \frac{1}{3}\left(\cos\theta + \mathrm{i}\sin\theta \right)$$

$$+ \frac{1}{9}\left(\cos 2\theta + \mathrm{i}\sin 2\theta \right)$$

$$+ \frac{1}{27}\left(\cos 3\theta + \mathrm{i}\sin 3\theta \right)$$

$$+ \ldots$$

$$= 1 + \frac{1}{3}e^{\mathrm{i}\theta} + \frac{1}{9}e^{\mathrm{i}2\theta} + \frac{1}{27}e^{\mathrm{i}3\theta} + \ldots$$

$$= \frac{1}{1 - \left(\frac{1}{3}e^{\mathrm{i}\theta} \right)} \times \frac{3}{3} \qquad \text{sum to infinity of a GP}$$

$$= \frac{3}{3 - e^{\mathrm{i}\theta}} \qquad \square$$

[4 marks]

(ii)

$$= \frac{3}{3 - \cos\theta - \mathrm{i}\sin\theta} \times \frac{3 - \cos\theta + \mathrm{i}\sin\theta}{3 - \cos\theta + \mathrm{i}\sin\theta}$$

$$= \frac{9 - 3\cos\theta + 3\mathrm{i}\sin\theta}{9 - 3\cos\theta - 3\mathrm{i}\sin\theta - 3\cos\theta + \cos^2\theta + \mathrm{i}\sin\theta\cos\theta + 3\mathrm{i}\sin\theta - \mathrm{i}\sin\theta\cos\theta - \mathrm{i}^2\sin^2\theta}$$

$$= \frac{9 - 3\cos\theta + 3\mathrm{i}\sin\theta}{9 - 3\cos\theta - 3\cos\theta + 1}$$

$$= \frac{9 - 3\cos\theta + 3\mathrm{i}\sin\theta}{10 - 6\cos\theta}$$

$$C + \mathrm{i}S = \frac{9 - 3\cos\theta + 3\mathrm{i}\sin\theta}{10 - 6\cos\theta}$$

$$= \frac{9 - 3\cos\theta}{10 - 6\cos\theta} + \mathrm{i}\,\frac{3\sin\theta}{10 - 6\cos\theta}$$

$$\therefore\ C = \frac{9 - 3\cos\theta}{10 - 6\cos\theta} \qquad \square$$

$$\text{and } S = \frac{3\sin\theta}{10 - 6\cos\theta}$$

[4 marks]

Answer 3
(i)

$$C + \mathrm{i}S = \cos\theta + \mathrm{i}\sin\theta$$
$$+ \frac{1}{2}(\cos 5\theta + \mathrm{i}\sin 5\theta)$$
$$+ \frac{1}{4}(\cos 9\theta + \mathrm{i}\sin 9\theta)$$
$$+ \frac{1}{8}(\cos 13\theta + \mathrm{i}\sin 13\theta)$$
$$+ \ldots$$
$$= e^{\mathrm{i}\theta} + \frac{1}{2}e^{\mathrm{i}5\theta} + \frac{1}{4}e^{\mathrm{i}9\theta} + \frac{1}{8}e^{\mathrm{i}13\theta} + \ldots$$
$$= \frac{e^{\mathrm{i}\theta}}{1 - \left(\frac{1}{2}e^{\mathrm{i}4\theta}\right)} \times \frac{2}{2} \qquad \text{sum to infinity of a GP}$$
$$= \frac{2e^{\mathrm{i}\theta}}{2 - e^{\mathrm{i}4\theta}} \qquad \square$$

[4 marks]

(ii)

$$\frac{2e^{\mathrm{i}\theta}}{2 - e^{\mathrm{i}4\theta}} = \frac{2\cos\theta + 2\mathrm{i}\sin\theta}{2 - \cos 4\theta - \mathrm{i}\sin 4\theta} \times \frac{2 - \cos 4\theta + \mathrm{i}\sin 4\theta}{2 - \cos 4\theta + \mathrm{i}\sin 4\theta}$$

The "clutter" can be reduced by noting that all of the imaginary terms in the denominator cancel each other out when complex conjugates multiply.
Also, as only the imaginary part of the answer is sought, there is no need to work out any of the real terms in the numerator.

$$\mathrm{i}S = \frac{2\mathrm{i}\cos\theta\sin 4\theta + 4\mathrm{i}\sin\theta - 2\mathrm{i}\sin\theta\cos 4\theta}{4 - 2\cos 4\theta - 2\cos 4\theta + \cos^2 4\theta + \sin^2 4\theta}$$

Carefully shuffling the pieces around yields,

$$S = \frac{4\sin\theta + 2(\sin 4\theta\cos\theta - \cos 4\theta\sin\theta)}{4 - 2\cos 4\theta - 2\cos 4\theta + 1}$$
$$= \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta} \qquad \square$$

[4 marks]

Answer 4

(i) $e^{\mathrm{i}n\theta} = \cos n\theta + \mathrm{i}\sin n\theta$

[1 mark]

(ii) $e^{-\mathrm{i}n\theta} = \cos n\theta - \mathrm{i}\sin n\theta$

[1 mark]

(iii)

$$\begin{aligned} e^{\mathrm{i}n\theta} &= \cos n\theta + \mathrm{i}\sin n\theta \\ \text{ADD} \quad e^{-\mathrm{i}n\theta} &= \cos n\theta - \mathrm{i}\sin n\theta \end{aligned}$$

$$e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta} = 2\cos n\theta$$

$$\therefore \quad \cos n\theta = \frac{1}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}\right) \qquad \square$$

[1 mark]

(iv)

$$\begin{aligned} e^{\mathrm{i}n\theta} &= \cos n\theta + \mathrm{i}\sin n\theta \\ \text{SUBTRACT} \quad e^{-\mathrm{i}n\theta} &= \cos n\theta - \mathrm{i}\sin n\theta \end{aligned}$$

$$e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta} = 2\mathrm{i}\sin n\theta$$

$$\therefore \quad \sin n\theta = \frac{1}{2}\left(e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta}\right) \qquad \square$$

[1 mark]

Answer 5

$$\begin{aligned} \sum_{k=0}^{7}(1+\mathrm{i})^k &= \sum_{k=0}^{7}\left(\sqrt{2}\,e^{\frac{\pi}{4}\mathrm{i}}\right)^k \\ &= 1 + \sqrt{2}\,e^{\frac{\pi}{4}\mathrm{i}} + \left(\sqrt{2}\,e^{\frac{\pi}{4}\mathrm{i}}\right)^2 + \ldots + \left(\sqrt{2}\,e^{\frac{\pi}{4}\mathrm{i}}\right)^7 \\ &= \frac{1\left(1 - \left(\sqrt{2}\,e^{\frac{\pi}{4}\mathrm{i}}\right)^8\right)}{\left(1 - \left(\sqrt{2}\,e^{\frac{\pi}{4}\mathrm{i}}\right)\right)} \\ &= \frac{1 - 16\,e^{\frac{8\pi}{4}\mathrm{i}}}{1 - \sqrt{2}\cos\left(\frac{\pi}{4}\right) - \sqrt{2}\,\mathrm{i}\sin\left(\frac{\pi}{4}\right)} \\ &= \frac{1 - 16\cos(2\pi) - 16\,\mathrm{i}\sin(2\pi)}{1 - 1 - \mathrm{i}} \times \frac{\mathrm{i}}{\mathrm{i}} \\ &= \mathrm{i} - 16\mathrm{i} \\ &= -15\mathrm{i} \qquad \square \end{aligned}$$

[4 marks]

Answer 6

(i)

$$\begin{aligned}
\text{LHS} &= (2 + e^{i\theta})(2 + e^{-i\theta}) \\
&= 4 + 2e^{i\theta} + 2e^{-i\theta} + e^{0i} \\
&= 4 + 2\cos\theta + 2i\sin\theta + 2\cos(-\theta) + 2i\sin(-\theta) + 1 \\
&= 5 + 2\cos\theta + 2i\sin\theta + 2\cos\theta - 2i\sin\theta \qquad \text{even and odd functions} \\
&= 5 + 4\cos\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned}
C - iS &= 1 \\
&\quad - \frac{1}{2}(\cos\theta + i\sin\theta) \\
&\quad + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) \\
&\quad - \frac{1}{8}(\cos 3\theta + i\sin 3\theta) \\
&\quad + \ldots \\
&= 1 - \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{i2\theta} - \frac{1}{8}e^{i3\theta} + \ldots \\
&= \frac{1}{1 + \frac{1}{2}e^{i\theta}} \times \frac{2}{2} \\
&= \frac{2}{2 + e^{i\theta}} \\
&= \frac{2}{2 + \cos\theta + i\sin\theta} \times \frac{2 + \cos\theta - i\sin\theta}{2 + \cos\theta - i\sin\theta} \\
&= \frac{4 + 2\cos\theta - 2i\sin\theta}{4 + 2\cos\theta + 2\cos\theta + \cos^2\theta + \sin^2\theta} \\
&= \frac{4 + 2\cos\theta - 2i\sin\theta}{5 + 4\cos\theta}
\end{aligned}$$

$$\therefore C = \frac{4 + 2\cos\theta}{5 + 4\cos\theta} \quad \text{and} \quad S = \frac{\sin\theta}{5 + 4\cos\theta}$$

Notice that the minus sign was already in the expression $C - iS$

[4 marks]

Answer 7

(i)

$$\begin{aligned}
\text{LHS} &= 1 + e^{2\theta \mathrm{i}} \\
&= 1 + \cos 2\theta + \mathrm{i} \sin 2\theta \\
&= \cos^2 \theta + \sin^2 \theta + \cos^2 \theta - \sin^2 \theta + \mathrm{i}\, 2 \sin \theta \cos \theta \\
&= 2 \cos^2 \theta + 2\mathrm{i} \sin \theta \cos \theta \\
&= 2 \cos \theta \ (\cos \theta + \mathrm{i} \sin \theta) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned}
C + \mathrm{i}S &= 1 \\
&\quad + \binom{n}{1} (\cos 2\theta + \sin 2\theta) \\
&\quad + \binom{n}{2} (\cos 4\theta + \sin 4\theta) \\
&\quad + \ldots \\
&\quad + \binom{n}{n} (\cos 2n\theta + \mathrm{i} \sin 2n\theta) \\
&= \binom{n}{0} e^{0\mathrm{i}} + \binom{n}{1} e^{2\theta \mathrm{i}} + \binom{n}{2} e^{4\theta \mathrm{i}} + \ldots + \binom{n}{n} e^{2n\theta \mathrm{i}} \\
&= \binom{n}{0} \left(e^{2\theta \mathrm{i}} \right)^0 + \binom{n}{1} \left(e^{2\theta \mathrm{i}} \right)^1 + \binom{n}{2} \left(e^{2\theta \mathrm{i}} \right)^2 + \ldots + \binom{n}{n} \left(e^{2\theta \mathrm{i}} \right)^n \\
&= \left(1 + e^{2\theta \mathrm{i}} \right)^n && \text{Binomial Theorem “backwards”} \\
&= (2 \cos \theta \ (\cos \theta + \mathrm{i} \sin \theta))^n && \text{using the part (i) result} \\
&= 2^n \cos^n \theta \ (\cos \theta + \mathrm{i} \sin \theta)^n \\
&= 2^n \cos^n \theta \ (\cos n\theta + \mathrm{i} \sin n\theta) && \text{by de Moivre's Theorem}
\end{aligned}$$

$$\therefore \ C = 2^n \cos^n \theta \cos n\theta \qquad \square$$

$$\text{and } \ S = 2^n \sin^n \theta \sin n\theta$$

[7 marks]

Answer 8

(i)

$$\sum_{k=0}^{3} \mathrm{i}^k = \mathrm{i}^0 + \mathrm{i}^1 + \mathrm{i}^2 + \mathrm{i}^3$$

$$= 1 + \mathrm{i} - 1 - \mathrm{i}$$

$$= 0$$

[1 mark]

(ii)

$$\sum_{k=0}^{15} \mathrm{i}^k = \sum_{k=0}^{3} \mathrm{i}^k - \mathrm{i}^0 + \sum_{k=4}^{7} \mathrm{i}^k + \sum_{k=8}^{11} \mathrm{i}^k + \sum_{k=12}^{15} \mathrm{i}^k$$

$$= 0 - 1 + 0 + 0 + 0$$

$$= -1$$

[2 marks]

(iii)

$$e^{\frac{\pi}{2}\mathrm{i}} = \mathrm{i} \qquad \therefore \sum_{k=0}^{12} z^k = \sum_{k=0}^{12} \mathrm{i}^k$$

$$= \sum_{k=0}^{3} \mathrm{i}^k + \sum_{k=4}^{7} \mathrm{i}^k + \sum_{k=8}^{11} \mathrm{i}^k + \mathrm{i}^{12}$$

$$= \sum_{m=0}^{2} \left(\sum_{k=4m}^{4m+3} \mathrm{i}^k \right) + \mathrm{i}^{12}$$

$$= \sum_{m=0}^{2} (0) + 1$$

$$= 1$$

[2 marks]

(iv)

$$e^{-\frac{\pi}{2}\mathrm{i}} = -\mathrm{i} \qquad \therefore \sum_{k=0}^{121} z^k = \sum_{k=0}^{121} (-\mathrm{i})^k$$

$$= \sum_{m=0}^{29} \left(\sum_{k=4m}^{4m+3} \mathrm{i}^k \right) + (-\mathrm{i})^{120} + (-\mathrm{i})^{121}$$

$$= \sum_{m=0}^{29} (0) + 1 - \mathrm{i}$$

$$= 1 - \mathrm{i}$$

[2 marks]

Lesson 10

Further A-Level Pure Mathematics, Core 2
Complex Numbers II

10.1 Revision

Any solution based entirely on graphical or numerical methods is not acceptable

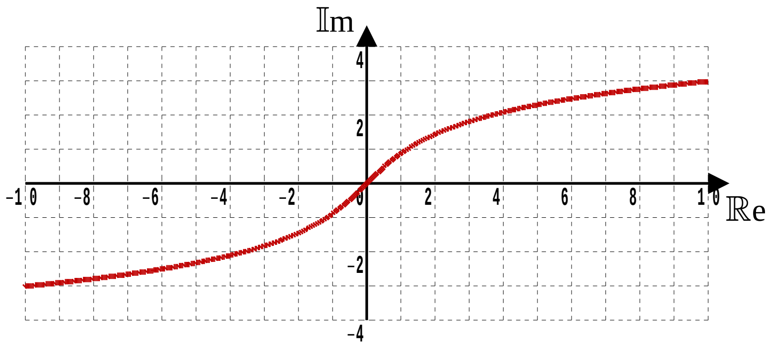
Marks Available : 40

Question 1

(i) Simplify $\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)$

[1 mark]

(ii) The graph is of the function $f(x) = ln\left(x+\sqrt{x^2+1}\right)$



Given that the graph would seem to have half turn rotational symmetry about the origin does this suggest that the function is even or odd ?

[1 mark]

(iii) Prove that the function is either an even function or an odd function, as dictated by your answer to part (ii)

[4 marks]

Question 2

Show that $\dfrac{\cos 5\theta + \mathrm{i}\sin 5\theta}{\cos 2\theta + \mathrm{i}\sin 2\theta}$ can be expressed in the form $\cos n\theta + \mathrm{i}\sin n\theta$, where n is an integer to be found

[4 marks]

Question 3

Prove that $\dfrac{e^{\mathrm{i}\theta} - 1}{e^{\mathrm{i}\theta} + 1} = \mathrm{i}\tan\left(\dfrac{x}{2}\right)$

[5 marks]

Question 4

Further A-Level Examination Question from June 2018, FP2, Q2(a) (MEI)

(i) Use de Moivre's theorem to prove that

$$cot\ 4\theta = \frac{1 - 6\,tan^2\,\theta + tan^4\,\theta}{4\,tan\,\theta\left(1 - tan^2\,\theta\right)}$$

[5 marks]

(ii) Hence express the roots of the equation

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

in exact trigonometric form

[4 marks]

Question 5

Further A-Level Examination Question from June 2009, FP2, Q3(b) (MEI)

The infinite series C and S are defined as follows,

$$C = \cos\theta + \frac{1}{3}\cos 3\theta + \frac{1}{9}\cos 5\theta + \ldots$$

$$S = \sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{9}\sin 5\theta + \ldots$$

By considering $C + \mathrm{i}S$, show that $C = \dfrac{3\cos\theta}{5 - 3\cos 2\theta}$

and find a similar expression for S

[11 marks]

Question 6

Given that $z = e^{\frac{\pi}{n}i}$ where n is a positive integer, prove that

$$1 + z + z^2 + ... + z^n = i\,cot\left(\frac{\pi}{2n}\right)$$

[5 marks]

10.2 Answers

Answer 1

(i)

$$\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right) = x^2+1-x^2 \quad \text{difference of two squares}$$
$$= 1$$

[1 mark]

(ii) Odd

[1 mark]

(iii)

$$f(x) = ln\left(x+\sqrt{x^2+1}\right)$$

$$\therefore \quad f(-x) = ln\left((-x)+\sqrt{(-x)^2+1}\right)$$
$$= ln\left((-x)+\sqrt{x^2+1}\right)$$
$$= ln\left(\sqrt{x^2+1}-x\right)$$
$$= ln\left(\frac{\sqrt{x^2+1}-x}{1}\times\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)$$
$$= ln\left(\frac{1}{\sqrt{x^2+1}+x}\right) \quad \text{using the part (i) result}$$
$$= ln\left(\sqrt{x^2+1}+x\right)^{-1}$$
$$= -ln\left(x+\sqrt{x^2+1}\right)$$
$$= -f(x)$$

which is the definition of an odd function

$\therefore \quad f(x)$ is an odd function ☐

[4 marks]

Answer 2

METHOD 1

$$\frac{\cos 5\theta + \mathrm{i}\sin 5\theta}{\cos 2\theta + \mathrm{i}\sin 2\theta} = \frac{e^{\mathrm{i}5\theta}}{e^{\mathrm{i}2\theta}}$$

$$= e^{\mathrm{i}3\theta}$$

$$= \cos 3\theta + \mathrm{i}\sin 3\theta$$

METHOD 2

$$\frac{\cos 5\theta + \mathrm{i}\sin 5\theta}{\cos 2\theta + \mathrm{i}\sin 2\theta}$$

$$= \frac{\cos 5\theta + \mathrm{i}\sin 5\theta}{\cos 2\theta + \mathrm{i}\sin 2\theta} \times \frac{\cos 2\theta - \mathrm{i}\sin 2\theta}{\cos 2\theta - \mathrm{i}\sin 2\theta}$$

$$= \frac{\cos 5\theta \cos 2\theta + \sin 5\theta \sin 2\theta + \mathrm{i}(\sin 5\theta \cos 2\theta - \cos 5\theta \sin 2\theta)}{\cos^2 2\theta - \mathrm{i}\cos 2\theta \sin 2\theta + \mathrm{i}\sin 2\theta \cos 2\theta + \sin^2 2\theta}$$

$$= \frac{\cos(5\theta - 2\theta) + \mathrm{i}\sin(5\theta - 2\theta)}{1}$$

$$= \cos 3\theta + \mathrm{i}\sin 3\theta$$

[4 marks]

Answer 3

$$\text{LHS} = \frac{e^{\mathrm{i}\theta} - 1}{e^{\mathrm{i}\theta} + 1}$$

$$= \frac{e^{\mathrm{i}\theta} - 1}{e^{\mathrm{i}\theta} + 1} \times \frac{e^{-\mathrm{i}\frac{\theta}{2}}}{e^{-\mathrm{i}\frac{\theta}{2}}}$$

$$= \frac{e^{\mathrm{i}\frac{\theta}{2}} - e^{-\mathrm{i}\frac{\theta}{2}}}{e^{\mathrm{i}\frac{\theta}{2}} + e^{-\mathrm{i}\frac{\theta}{2}}}$$

$$= \frac{2\mathrm{i}\sin\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)}$$

$$= \mathrm{i}\tan\left(\frac{\theta}{2}\right) = \text{RHS} \qquad \square$$

[5 marks]

Answer 4

(i) $\left(\cos\theta + \mathrm{i}\sin\theta\right)^4 = 1 \times (\cos\theta)^4 \times \left(\mathrm{i}\sin\theta\right)^0$ Binomial theorem

$$+ 4 \times (\cos\theta)^3 \times \left(\mathrm{i}\sin\theta\right)^1$$

$$+ 6 \times (\cos\theta)^2 \times \left(\mathrm{i}\sin\theta\right)^2$$

$$+ 4 \times (\cos\theta)^1 \times \left(\mathrm{i}\sin\theta\right)^3$$

$$+ 1 \times (\cos\theta)^0 \times \left(\mathrm{i}\sin\theta\right)^4$$

$$\cos(4\theta) + \mathrm{i}\sin(4\theta) = \cos^4\theta + \mathrm{i}\,4\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta$$
$$- \mathrm{i}\,4\cos\theta\sin^3\theta + \sin^4\theta$$

Matching real parts : $\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

Matching imaginary parts : $\sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$$\begin{aligned}
\text{LHS} &= \cot 4\theta \\
&= \frac{\cos 4\theta}{\sin 4\theta} \\
&= \frac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta} \times \frac{\frac{1}{\cos^4\theta}}{\frac{1}{\cos^4\theta}} \\
&= \frac{1 - 6\tan^2\theta + \tan^4\theta}{4\tan\theta - 4\tan^3\theta} \\
&= \frac{1 - 6\tan^2\theta + \tan^4\theta}{4\tan\theta\left(1 - \tan^2\theta\right)} = \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

(ii)

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

Let $x = tan\,\theta$ in which case,

$$tan^4\theta + 4\,tan^3\theta - 6\,tan^2\theta - 4\,tan\,\theta + 1 = 0$$

$$tan^4\theta - 6\,tan^2\theta + 1 = 4\,tan\,\theta - 4\,tan^3\theta$$

$$\frac{1 - 6\,tan^2\theta + tan^4\theta}{4\,tan\,\theta - 4\,tan^3\theta} = 1$$

$$cot\,4\theta = 1$$

$$tan\,4\theta = 1$$

$$4\theta = \frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{9\pi}{4},\ \frac{13\pi}{4},\ \ldots$$

$$\theta = \frac{\pi}{16},\ \frac{5\pi}{16},\ \frac{9\pi}{16},\ \frac{13\pi}{16},\ \ldots$$

$$x = tan\left(\frac{\pi}{16}\right),\ tan\left(\frac{5\pi}{16}\right),\ tan\left(\frac{9\pi}{16}\right),\ tan\left(\frac{13\pi}{16}\right),\ \ldots$$

Further solutions will simply start to duplicate these four

[4 marks]

Answer 5

$$C - \mathrm{i}S = (\cos\theta + \mathrm{i}\sin\theta)$$

$$+ \frac{1}{3}(\cos 3\theta + \mathrm{i}\sin 3\theta)$$

$$+ \frac{1}{9}(\cos 5\theta + \mathrm{i}\sin 5\theta)$$

$$+ \ \ldots$$

$$= e^{\mathrm{i}\theta} + \frac{1}{3}e^{\mathrm{i}3\theta} + \frac{1}{9}e^{\mathrm{i}5\theta} + \ldots$$

$$= \frac{e^{\mathrm{i}\theta}}{1 - \frac{1}{3}e^{\mathrm{i}2\theta}} \times \frac{3}{3}$$

$$= \frac{3e^{\mathrm{i}\theta}}{3 - e^{\mathrm{i}2\theta}}$$

$$= \frac{3\cos\theta + 3\mathrm{i}\sin\theta}{3 - \cos 2\theta - \mathrm{i}\sin 2\theta} \times \frac{3 - \cos 2\theta + \mathrm{i}\sin 2\theta}{3 - \cos 2\theta + \mathrm{i}\sin 2\theta}$$

$$= \frac{9\cos\theta - 3\cos\theta\cos 2\theta + 3\mathrm{i}\cos\theta\sin 2\theta + 9\mathrm{i}\sin\theta - 3\mathrm{i}\sin\theta\cos 2\theta - 3\sin\theta\sin 2\theta}{9 - 3\cos 2\theta - 3\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}$$

$$= \frac{9\cos\theta - 3(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta) + 9\mathrm{i}\sin\theta + 3\mathrm{i}(\sin 2\theta\cos\theta - \cos 2\theta\sin\theta)}{10 - 6\cos 2\theta}$$

$$= \frac{9\cos\theta - 3\cos\theta + 9\mathrm{i}\sin\theta + 3\mathrm{i}\sin\theta}{10 - 6\cos 2\theta}$$

$$= \frac{6\cos\theta + 12\mathrm{i}\sin\theta}{10 - 6\cos 2\theta}$$

$$= \frac{3\cos\theta + 6\mathrm{i}\sin\theta}{5 - 3\cos 2\theta}$$

$$\therefore C = \frac{3\cos\theta}{5 - 3\cos 2\theta} \qquad \square$$

$$\text{and } S = \frac{6\sin\theta}{5 - 3\cos 2\theta}$$

[11 marks]

Answer 6

$$
\begin{aligned}
\text{LHS} &= 1 + z + z^2 + \ldots + z^n \\
&= 1 + e^{\frac{\pi}{n}\mathrm{i}} + \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^2 + \ldots + \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^n \\
&= \frac{1\left(1 - \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^{n+1}\right)}{1 - e^{\frac{\pi}{n}\mathrm{i}}} \\
&= \frac{1 - e^{\frac{\pi(n+1)}{n}\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}} \\
&= \frac{1 - e^{\frac{\pi n}{n}\mathrm{i}}\, e^{\frac{\pi}{n}\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}} \\
&= \frac{1 - e^{\pi\mathrm{i}}\, e^{\frac{\pi}{n}\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}} \\
&= \frac{1 + e^{\frac{\pi}{n}\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}} \times \frac{e^{-\frac{\pi}{2n}\mathrm{i}}}{e^{-\frac{\pi}{2n}\mathrm{i}}} \\
&= \frac{e^{-\frac{\pi}{2n}\mathrm{i}} + e^{\frac{\pi}{2n}\mathrm{i}}}{e^{-\frac{\pi}{2n}\mathrm{i}} - e^{\frac{\pi}{2n}\mathrm{i}}} \\
&= -\frac{e^{\frac{\pi}{2n}\mathrm{i}} + e^{-\frac{\pi}{2n}\mathrm{i}}}{e^{\frac{\pi}{2n}\mathrm{i}} - e^{-\frac{\pi}{2n}\mathrm{i}}} \\
&= -\frac{2\cos\left(\frac{\pi}{2n}\right)}{2\mathrm{i}\sin\left(\frac{\pi}{2n}\right)} \times \frac{\mathrm{i}}{\mathrm{i}} \\
&= \mathrm{i}\cot\left(\frac{\pi}{2n}\right) \\
&= \text{RHS}
\end{aligned}
$$

This makes use of the following two results from Lesson 9 : Compare with Q3

Two Trigonometric Functions, as Exponentials

- $\cos(n\theta) = \frac{1}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}\right)$
- $\sin(n\theta) = \frac{1}{2\mathrm{i}}\left(e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta}\right)$

[5 marks]

Lesson 11

Further A-Level Pure Mathematics, Core 2

Complex Numbers II

11.1 Test

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 35

Question 1

Given that $g(x) = \dfrac{f(x) + f(-x)}{2}$ and $h(x) = \dfrac{f(x) - f(-x)}{2}$ prove that,

(i) $g(x)$ is an even function

[2 marks]

(ii) $h(x)$ is an odd function

[2 marks]

(iii) $g(x) + h(x) = f(x)$

[1 mark]

(iv) Taking all of parts (i), (ii) and (iii) into account, what has been proved is true for every function $f(x)$?

[1 mark]

Question 2

Show that $\dfrac{\cos 2\theta + \text{i}\sin 2\theta}{\cos 9\theta - \text{i}\sin 9\theta}$ can be expressed in the form $\cos n\theta + \text{i}\sin n\theta$,

where n is an integer to be found

[4 marks]

Question 3

Prove that $\dfrac{1 + e^{\text{i}\theta}}{1 - e^{\text{i}\theta}} = \text{i}\cot\left(\dfrac{\theta}{2}\right)$

[5 marks]

Question 4

Further A-Level Examination Question, 2017, SAM, Core 2, Q4 (Edexcel)

A complex number z has modulus 1 and argument θ

(a) Show that,

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad n \in \mathbb{Z}^+$$

[2 marks]

(b) Hence, show that,

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

[5 marks]

Question 5

Further A-Level Examination Question from June 2014, FP2, Q2(a) (MEI)

The infinite series C and S are defined as follows,

$$C = a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \ldots$$

$$S = a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \ldots$$

where a is a real number and $|a| < 1$

By considering $C + \mathrm{i}S$, show that $S = \dfrac{a\sin\theta}{1 - 2a\cos\theta + a^2}$

Find a corresponding expression for C

[8 marks]

Question 6

Given that $z = e^{\frac{\pi}{n}\mathrm{i}}$ where n is a positive integer, prove that,

$$1 + z + z^2 + ... + z^{2n-1} = 0$$

[5 marks]

11.2 Answers

Further A-Level Pure Mathematics, Core 2
Complex Numbers II

Answer 1

(i) $g(x) = \dfrac{f(x) + f(-x)}{2} \Rightarrow g(-x) = \dfrac{f(-x) + f(x)}{2}$

$$= \frac{f(x) + f(-x)}{2}$$

$$= g(x) \qquad \therefore\ g(x) \text{ is an even function}$$

[2 marks]

(ii) $h(x) = \dfrac{f(x) - f(-x)}{2} \Rightarrow h(-x) = \dfrac{f(-x) - f(x)}{2}$

$$= -\frac{f(x) - f(-x)}{2}$$

$$= -h(x) \qquad \therefore\ h(x) \text{ is an odd function}$$

[2 marks]

(iii)

$$\begin{aligned}
\text{LHS} &= g(x) + h(x) \\
&= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\
&= \frac{f(x) + f(-x) + f(x) - f(-x)}{2} \\
&= \frac{2f(x)}{2} \\
&= f(x) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[1 mark]

(iv) Every function, $f(x)$, can be written as the sum of an even function and an odd function

[1 mark]

Answers 2

METHOD 1

$$\frac{\cos 2\theta + \mathrm{i}\sin 2\theta}{\cos 9\theta - \mathrm{i}\sin 9\theta} = \frac{\cos 2\theta + \mathrm{i}\sin 2\theta}{\cos(-9\theta) + \mathrm{i}\sin(-9\theta)}$$

(As *cos* is an even function, and *sin* is an odd function)

$$= \frac{e^{\mathrm{i}2\theta}}{e^{-\mathrm{i}9\theta}}$$

$$= e^{\mathrm{i}11\theta}$$

$$= \cos 11\theta + \mathrm{i}\sin 11\theta$$

METHOD 2

$$\frac{\cos 2\theta + \mathrm{i}\sin 2\theta}{\cos 9\theta - \mathrm{i}\sin 9\theta}$$

$$= \frac{\cos 2\theta + \mathrm{i}\sin 2\theta}{\cos 9\theta - \mathrm{i}\sin 9\theta} \times \frac{\cos 9\theta + \mathrm{i}\sin 9\theta}{\cos 9\theta + \mathrm{i}\sin 9\theta}$$

$$= \frac{\cos 9\theta\cos 2\theta - \sin 9\theta\sin 2\theta + \mathrm{i}(\sin 9\theta\cos 2\theta + \cos 9\theta\sin 2\theta)}{\cos^2 9\theta - \mathrm{i}\cos 9\theta\sin 9\theta + \mathrm{i}\sin 9\theta\cos 9\theta + \sin^2 9\theta}$$

$$= \frac{\cos(9\theta + 2\theta) + \mathrm{i}\sin(9\theta + 2\theta)}{1}$$

$$= \cos 11\theta + \mathrm{i}\sin 11\theta$$

[4 marks]

Answer 3

$$\text{LHS} = \frac{1 + e^{\mathrm{i}\theta}}{1 - e^{\mathrm{i}\theta}}$$

$$= \frac{1 + e^{\mathrm{i}\theta}}{1 - e^{\mathrm{i}\theta}} \times \frac{e^{-\mathrm{i}\frac{\theta}{2}}}{e^{-\mathrm{i}\frac{\theta}{2}}}$$

$$= \frac{e^{-\mathrm{i}\frac{\theta}{2}} + e^{\mathrm{i}\frac{\theta}{2}}}{e^{-\mathrm{i}\frac{\theta}{2}} - e^{\mathrm{i}\frac{\theta}{2}}}$$

$$= -\frac{e^{\mathrm{i}\frac{\theta}{2}} + e^{-\mathrm{i}\frac{\theta}{2}}}{e^{\mathrm{i}\frac{\theta}{2}} - e^{-\mathrm{i}\frac{\theta}{2}}}$$

$$= -\frac{2\cos\left(\frac{\theta}{2}\right)}{2\mathrm{i}\sin\left(\frac{\theta}{2}\right)} \times \frac{\mathrm{i}}{\mathrm{i}}$$

$$= \mathrm{i}\cot\left(\frac{\theta}{2}\right)$$

[5 marks]

Answer 4

(a)

$$\text{LHS} = z^n + \frac{1}{z^n}$$

$$= e^{\mathrm{i}n\theta} + \frac{1}{e^{\mathrm{i}n\theta}}$$

$$= e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}$$

$$= \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$$

As *cos* is an even function and *sin* odd

$$= \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$$

$$= 2\cos n\theta$$

$$= \text{RHS} \qquad \square$$

[2 marks]

(b) Apply the binomial theorem to $\left(z + \frac{1}{z}\right)^4$

$$\left(z + \frac{1}{z}\right)^4 = 1 \times z^4 \times \left(\frac{1}{z}\right)^0 + 4 \times z^3 \times \left(\frac{1}{z}\right)^1 + 6 \times z^2 \times \left(\frac{1}{z}\right)^2 + 4 \times z^1 \times \left(\frac{1}{z}\right)^3 + 1 \times z^0 \times \left(\frac{1}{z}\right)^4$$

$$\therefore \left(z + \frac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + 4\left(\frac{1}{z^2}\right) + \left(\frac{1}{z^4}\right)$$

$$(2\cos\theta)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$$

$$8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$$

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3) \qquad \square$$

[4 marks]

Answer 5

$$C + \mathrm{i}S = a(\cos\theta + \mathrm{i}\sin\theta)$$
$$+ a^2(\cos 2\theta + \mathrm{i}\sin 2\theta)$$
$$+ a^3(\cos 3\theta + \mathrm{i}\sin 3\theta)$$
$$+ \ldots$$
$$= a e^{\mathrm{i}\theta} + a^2 e^{\mathrm{i}2\theta} + a^3 e^{\mathrm{i}3\theta} + \ldots$$
$$= \frac{a e^{\mathrm{i}\theta}}{1 - a e^{\mathrm{i}\theta}}$$
$$= \frac{a\cos\theta + a\mathrm{i}\sin\theta}{1 - a\cos\theta - a\mathrm{i}\sin\theta} \times \frac{1 - a\cos\theta + a\mathrm{i}\sin\theta}{1 - a\cos\theta + a\mathrm{i}\sin\theta}$$
$$= \frac{a\cos\theta - a^2\cos^2\theta + a^2\mathrm{i}\cos\theta\sin\theta + a\mathrm{i}\sin\theta - a^2\mathrm{i}\sin\theta\cos\theta - a^2\sin^2\theta}{1 - a\cos\theta - a\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}$$
$$= \frac{a\cos\theta - a^2(\cos^2\theta + \sin^2\theta) + a\mathrm{i}\sin\theta}{1 - 2\cos\theta + a^2\left(\cos^2\theta + \sin^2\theta\right)}$$
$$= \frac{a\cos\theta - a^2 + a\mathrm{i}\sin\theta}{1 - 2\cos\theta + a^2}$$
$$\therefore\ C = \frac{a\cos\theta - a^2}{1 - 2\cos\theta + a^2}$$
$$\text{and}\quad S = \frac{a\sin\theta}{1 - 2\cos\theta + a^2} \qquad \square$$

[8 marks]

Answer 6

$$\text{LHS} = 1 + z + z^2 + \ldots + z^{2n-1}$$

$$= 1 + e^{\frac{\pi}{n}\mathrm{i}} + \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^2 + \ldots + \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^{2n-1}$$

$$= \frac{1\left(1 - \left(e^{\frac{\pi}{n}\mathrm{i}}\right)^{2n}\right)}{1 - e^{\frac{\pi}{n}\mathrm{i}}}$$

$$= \frac{1 - e^{\frac{2n\pi}{n}\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}}$$

$$= \frac{1 - e^{2\pi\mathrm{i}}}{1 - e^{\frac{\pi}{n}\mathrm{i}}}$$

$$= \frac{1 - 1}{1 - e^{\frac{\pi}{n}\mathrm{i}}}$$

$$= \text{RHS}$$