1.3 Answers

1.3.1 Solution to 1.1 Example (Teaching Video)

(i) $$\int_1^e \frac{1}{x}\,dx = \Big[\, ln\, x \,\Big]_1^e$$

$$= ln\, e - ln\, 1$$

$$= 1 - 0$$

$$= 1$$

[2 marks]

(ii) $$\int_0^1 \frac{1}{x}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x}\,dx$$

$$= \lim_{t \to 0^+} \Big[\, ln\, x \,\Big]_t^1$$

$$= \lim_{t \to 0^+} \Big[\, ln\, 1 - ln\, t \,\Big]$$

$$= (-1) \lim_{t \to 0^+} \Big[\, ln\, t \,\Big]$$

Although $ln\, 0$ is undefined, as $t \to 0$ from above, $ln\, t \to -\infty$

$\therefore$ The integral does not converge; it's diverging.

[2 marks]

(iii) $$\int_e^\infty \frac{1}{x}\,dx = \lim_{t \to \infty} \int_e^t \frac{1}{x}\,dx$$

$$= \lim_{t \to \infty} \Big[\, ln\, x \,\Big]_e^t$$

$$= \lim_{t \to \infty} \Big[\, ln\, t - ln\, e \,\Big]$$

$$= \lim_{t \to \infty} \Big[\, ln\, t \,\Big] - 1$$

As $t \to \infty$, $ln\, t \to \infty$

$\therefore$ The integral does not converge; it's diverging.

[2 marks]

1.3.2 Solutions to 1.2 Exercise

Answer 1

(i)

$$\int_0^1 \frac{1}{x^2}\,dx = \int_0^1 x^{-2}\,dx$$

$$= \lim_{t \to 0^+} \int_t^1 x^{-2}\,dx$$

$$= \lim_{t \to 0^+} \left[-x^{-1} \right]_t^1$$

$$= (-1) \lim_{t \to 0^+} \left[\frac{1}{x} \right]_t^1$$

$$= (-1) \lim_{t \to 0^+} \left[\frac{1}{1} - \frac{1}{t} \right]$$

$$\text{As } t \to 0^+, \ \frac{1}{t} \to \infty$$

$\therefore$ The integral does not converge; it's diverging. ☐

[4 marks]

(ii)

$$\int_1^\infty \frac{1}{x^2}\,dx = \int_1^\infty x^{-2}\,dx$$

$$= \lim_{t \to \infty} \int_1^t x^{-2}\,dx$$

$$= \lim_{t \to \infty} \left[-x^{-1} \right]_1^t$$

$$= (-1) \lim_{t \to \infty} \left[\frac{1}{x} \right]_1^t$$

$$= (-1) \lim_{t \to \infty} \left[\frac{1}{t} - \frac{1}{1} \right]$$

$$\text{As } t \to \infty, \ \frac{1}{t} \to 0$$

$$= (-1) \times (-1)$$

$$= 1$$

$\therefore$ The integral does converge; it has a value of 1 ☐

[4 marks]

Answer 2

(i)

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-\frac{1}{2}}\,dx$$

$$= \lim_{t \to 0^+} \int_t^1 x^{-\frac{1}{2}}\,dx$$

$$= \lim_{t \to 0^+} \left[2x^{\frac{1}{2}} \right]_t^1$$

$$= 2 \lim_{t \to 0^+} \left[\sqrt{x} \right]_t^1$$

$$= 2 \lim_{t \to 0^+} \left[\sqrt{1} - \sqrt{t} \right]$$

As $t \to 0^+,\ \sqrt{t} \to 0$

$$= 2 \times 1$$

$$= 2$$

$\therefore$ The integral does converge; it has a value of 2 ☐

[4 marks]

(ii)

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \int_1^\infty x^{-\frac{1}{2}}\,dx$$

$$= \lim_{t \to \infty} \int_1^t x^{-\frac{1}{2}}\,dx$$

$$= \lim_{t \to \infty} \left[2x^{\frac{1}{2}} \right]_1^t$$

$$= 2 \lim_{t \to \infty} \left[\sqrt{x} \right]_1^t$$

$$= 2 \lim_{t \to \infty} \left[\sqrt{t} - \sqrt{1} \right]$$

As $t \to \infty,\ \sqrt{t} \to \infty$

$\therefore$ The integral does not converge; it is divergent ☐

[4 marks]

Answer 3

(i)

$$y = \sqrt{1 + x^2}$$

$$y = \left(1 + x^2\right)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(1 + x^2\right)^{-\frac{1}{2}} \times 2x$$

$$= \frac{x}{\sqrt{1 + x^2}}$$

[2 marks]

(ii)

$$\int_0^{\infty} \frac{4x}{\sqrt{1 + x^2}}\, dx = 4 \int_0^{\infty} \frac{x}{\sqrt{1 + x^2}}\, dx$$

$$= 4 \lim_{t \to \infty} \int_0^{t} \frac{x}{\sqrt{1 + x^2}}\, dx$$

$$= 4 \lim_{t \to \infty} \left[\sqrt{1 + x^2} \right]_0^t$$

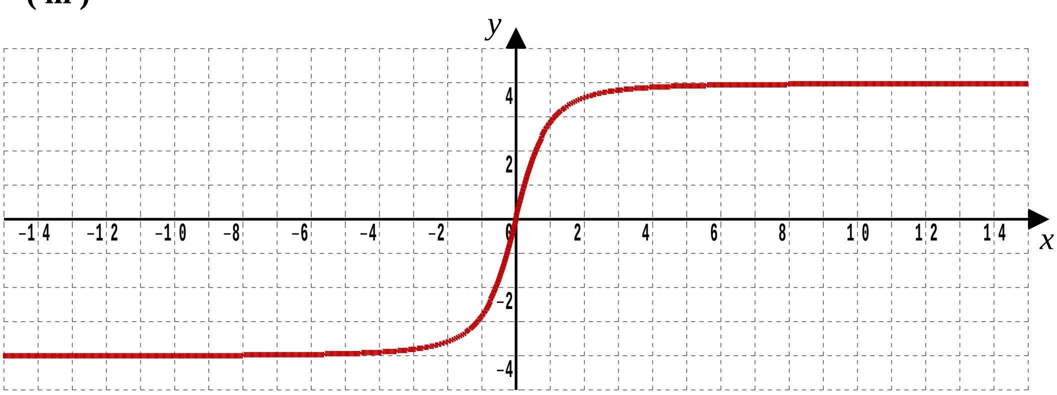
$$= 4 \lim_{t \to \infty} \left[\sqrt{1 + t^2} - \sqrt{1} \right]$$

As $t \to \infty$, $\sqrt{1 + t^2} \to \infty$

$\therefore$ The integral does not converge; it is divergent. $\square$

[4 marks]

(iii)



The horizontal asymptote of $y = 4$ as $x \to \infty$ implies that, as the area to be found extends rightward, it resembles a rectangle of fixed width 4 and length x. The area of such a rectangle will $\to \infty$ as $x \to \infty$

[2 marks]

Answer 4

$$\int_5^{13} \frac{6x}{\sqrt{x^2 - 25}}\, dx = 3\int_5^{13} \frac{2x}{\sqrt{x^2 - 25}}\, dx$$

$$= 3 \lim_{t \to 5^+} \int_t^{13} (2x)\left(x^2 - 25\right)^{-\frac{1}{2}}\, dx$$

$$= 3 \lim_{t \to 5^+} \left[2\left(x^2 - 25\right)^{\frac{1}{2}} \right]_t^{13}$$

$$= 6 \lim_{t \to 5^+} \left[\sqrt{13^2 - 25} - \sqrt{t^2 - 25} \right]$$

As $t \to 5^+$, $\sqrt{t^2 - 25} \to 0$

$$= 6 \times 12$$

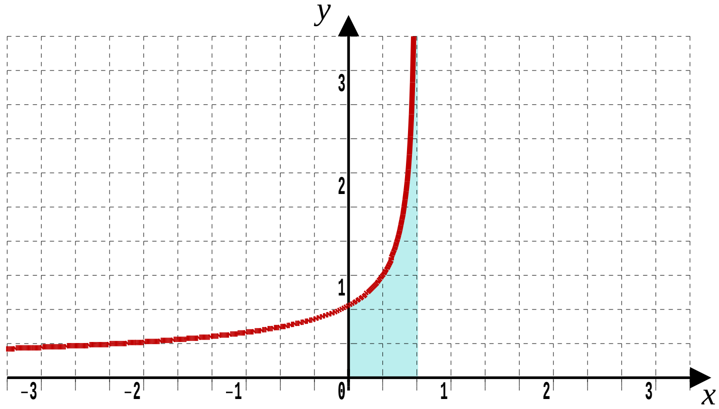
$$= 72$$

$\therefore$ The integral is convergent; it has a value of 72

[6 marks]

Answer 5

(i)



[1 mark]

(ii) The right hand boundary is of infinite height.

[1 mark]

(iii)

$$\int_0^{\frac{2}{3}} \frac{1}{\sqrt{2-3x}}\,dx = -\frac{1}{3}\int_0^{\frac{2}{3}} \frac{-3}{\sqrt{2-3x}}\,dx$$

$$= -\frac{1}{3}\underset{t\to\left(\frac{2}{3}\right)^-}{\text{limit}} \int_0^t (-3)(2-3x)^{-\frac{1}{2}}\,dx$$

$$= -\frac{1}{3}\underset{t\to\left(\frac{2}{3}\right)^-}{\text{limit}} \left[2(2-3x)^{\frac{1}{2}}\right]_0^t$$

$$= -\frac{2}{3}\underset{t\to\left(\frac{2}{3}\right)^-}{\text{limit}} \left[\sqrt{2-3x}\right]_0^t$$

$$= -\frac{2}{3}\underset{t\to\left(\frac{2}{3}\right)^-}{\text{limit}} \left[\sqrt{2-3t}-\sqrt{2}\right]$$

As $t \to \left(\frac{2}{3}\right)^-$, $\sqrt{2-3} \to 0$

$$= -\frac{2}{3}\times-\sqrt{2}$$

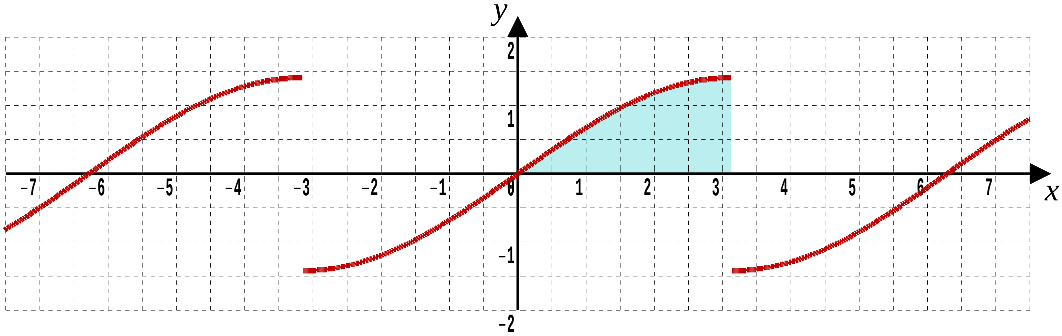
$$= \frac{2\sqrt{2}}{3}$$

$\therefore$ The integral is convergent; it has a value of $\dfrac{2\sqrt{2}}{3}$

[6 marks]

Answer 6

(i)



[1 mark]

(ii) The function has a discontinuity at its boundary $x = \pi$ where it jumps between the value of $+\sqrt{2}$ to the value $-\sqrt{2}$

[1 mark]

(iii)

$$\int_0^{\pi} \frac{\sin x}{\sqrt{1 + \cos x}}\, dx = (-1) \lim_{t \to \pi^-} \int_0^t (-\sin x)(1 + \cos x)^{-\frac{1}{2}}\, dx$$

$$= (-1) \lim_{t \to \pi^-} \left[2(1 + \cos x)^{\frac{1}{2}} \right]_0^t$$

$$= (-2) \lim_{t \to \pi^-} \left[\sqrt{1 + \cos x} \right]_0^t$$

$$= (-2) \lim_{t \to \pi^-} \left[\sqrt{1 + \cos t} - \sqrt{2} \right]$$

As $t \to \pi^-$, $\sqrt{1 + \cos t} \to 0$

$$= (-2) \times -\sqrt{2}$$

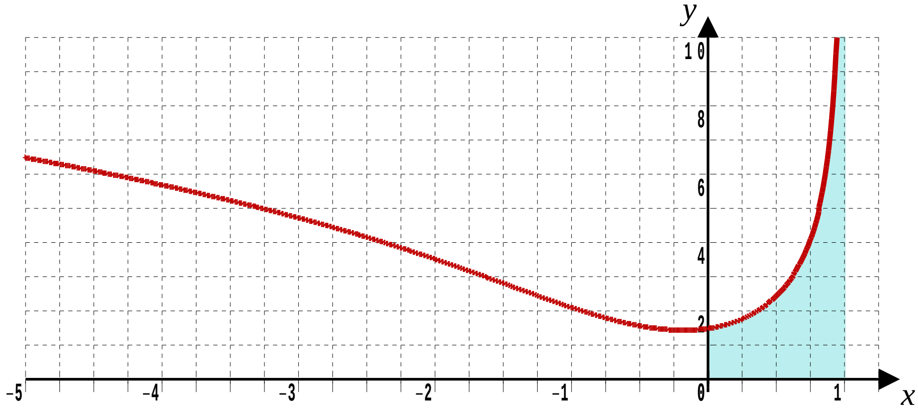
$$= 2\sqrt{2}$$

$\therefore$ The integral is convergent; it has a value of $2\sqrt{2}$

[6 marks]

Answer 7

(i)



[1 mark]

(ii) The rightmost boundary is along an asymptote and so the area specified has infinite height

[1 mark]

(iii)

$$\int_0^1 \frac{3\left(x^2+1\right)}{\sqrt{4-3x-x^3}}dx = (-1)\int_0^1 \frac{\left(-3x^2-3\right)}{\sqrt{4-3x-x^3}}dx$$

$$= (-1) \lim_{t \to 1^-} \int_0^t \left(-3-3x^2\right)\left(4-3x-x^3\right)^{-\frac{1}{2}} dx$$

$$= (-1) \lim_{t \to 1^-} \left[2\left(4-3x-x^3\right)^{\frac{1}{2}}\right]_0^t$$

$$= (-2) \lim_{t \to 1^-} \left[\sqrt{4-3t-t^3} - \sqrt{4}\right]$$

As $t \to 1^-$, $\sqrt{4-3t-t^3} \to 0$

$$= (-2) \times \left(-\sqrt{4}\right)$$

$$= 4$$

$\therefore$ The integral is convergent; it has a value of 4

[6 marks]

Answer 8

$$\int_0^{\infty} \left(1 - \frac{1}{e^{-x} + 1} \right) dx = \int_0^{\infty} \left(\frac{e^{-x} + 1}{e^{-x} + 1} - \frac{1}{e^{-x} + 1} \right) dx$$

$$= \int_0^{\infty} \frac{e^{-x}}{e^{-x} + 1}\, dx$$

$$= (-1) \int_0^{\infty} \left(-e^{-x} \right) \left(e^{-x} + 1 \right)^{-1}$$

$$= (-1) \underset{t \to \infty}{limit} \int_0^{t} \left(-e^{-x} \right) \left(e^{-x} + 1 \right)^{-1} dx$$

$$= (-1) \underset{t \to \infty}{limit} \left[ln\left(e^{-x} + 1 \right) \right]_0^t$$

$$= (-1) \underset{t \to \infty}{limit} \left[ln\left(e^{-t} + 1 \right) - ln\left(e^{0} + 1 \right) \right]$$

As $t \to \infty, \; e^{-t} \to 0$

$$= (-1) \left[ln(1) - ln(2) \right]$$

$$= ln(2)$$

$\therefore$ The integral is convergent; it has a value of $ln\, 2$

[6 marks]