2.3 Answers

Further A-Level Pure Mathematics, Core 2
Improper Integrals

Note :
To reduce "unhelpful clutter", in many solutions the constant of the integration is only inserted in the final line of working.

2.3.1 Solution to 2.1 Example (Teaching Video)

$$S = \int \frac{\ln x}{x}\, dx$$

$$= \int (\ln x)\left(\frac{1}{x}\right) dx$$

$$\quad \textbf{\textit{L}} \quad \textbf{\textit{I}} \qquad \textbf{\textit{D}} \quad \textbf{\textit{I}}$$

$$= (\ln x)(\ln x) - \int \left(\frac{1}{x}\right)(\ln x)\, dx$$

$$S = (\ln x)^2 - S$$

$$2S = (\ln x)^2$$

$$S = \frac{1}{2}(\ln x)^2 + c$$

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \lim_{t \to \infty} \int_1^t \frac{\ln x}{x}\, dx$$

$$= \frac{1}{2} \lim_{t \to \infty} \Big[(\ln x)^2 \Big]_1^t$$

$$= \frac{1}{2} \lim_{t \to \infty} \Big[(\ln t)^2 - (\ln 1)^2 \Big]$$

As $t \to \infty$, $\ln t \to \infty$

$\therefore$ The integral does not converge; it is divergent ☐

[6 marks]

This example is generalised in the final question of this lesson's exercise.

2.3.2 Solutions to 2.2 Exercise

Answer 1

$$\int f(x)\ dx = \int x\, e^{-x}\, dx$$

$$\quad\quad\quad\quad \boldsymbol{L} \quad \boldsymbol{I} \quad \boldsymbol{D} \quad \boldsymbol{I}$$

$$= x\left(-e^{-x}\right) - \int 1\left(-e^{-x}\right) dx$$

$$= -x\, e^{-x} + \int e^{-x}\, dx$$

$$= -x\, e^{-x} - e^{-x} + c$$

$$\int_0^{\infty} f(x)\ dx = \underset{t \to \infty}{limit} \int_0^t x\, e^{-x}\, dx$$

$$= \underset{t \to \infty}{limit} \left[-x\, e^{-x} - e^{-x}\right]_0^t$$

$$= \underset{t \to \infty}{limit} \left[-t\, e^{-t} - e^{-t} + 0 + e^{0}\right]$$

As $t \to \infty,\ e^{-t} \to 0$

$= 1$

$\therefore$ The integral is convergent; it has a value of 1

[6 marks]

Answer 2

(i) It's already known that $g(x) = e^{-x}$ is a strictly decreasing function that, as $x \to \infty$, $e^{-x} \to 0$

As $-1 \leqslant \sin x \leqslant 1$ with only spaced out isolated periodic points where $\sin x = \pm 1$ the multiplication of e^{-x} by $\sin x$ will not alter the overall tending towards zero.

That is, $x \to \infty$, $e^{-x} \sin x \to 0$ $\square$

[1 mark]

(ii)

$$S = \int e^{-x} \sin x \, dx$$

L I D I

$$= e^{-x}\left(-\cos x\right) - \int \left(-e^{-x}\right)\left(-\cos x\right) dx$$

$$= -e^{-x} \cos x - \int e^{-x} \cos x \, dx$$

L I D I

$$= -e^{-x} \cos x - \left[e^{-x} \sin x - \int \left(-e^{-x}\right)\left(\sin x\right) dx \right]$$

$$S = -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x \, dx$$

$$S = -e^{-x} \cos x - e^{-x} \sin x - S$$

$$2S = -e^{-x} \cos x - e^{-x} \sin x$$

$$S = -\frac{1}{2} e^{-x}\left(\cos x + \sin x\right) + c \qquad \square$$

[6 marks]

(iii)

$$\int_0^{\infty} e^{-x} \sin x \, dx = \lim_{t \to \infty} \int_0^{t} e^{-x} \sin x \, dx$$

$$= -\frac{1}{2} \lim_{t \to \infty} \Big[e^{-x} \cos x + e^{-x} \sin x \Big]_0^t$$

$$= -\frac{1}{2} \lim_{t \to \infty} \left[e^{-t} \cos t + e^{-t} \sin t - e^{0} \cos 0 - e^{0} \sin 0 \right]$$

$$= -\frac{1}{2} \left[-1 \right]$$

$$= \frac{1}{2}$$

[3 marks]

Answer 3

(i) As $x \to 0^+,\ x\,ln\,x \to 0$

[1 mark]

(ii)
$$\int ln\,x\,dx = \int \underset{\boldsymbol{L}}{ln\,x} \times \underset{\boldsymbol{I}}{} \underset{\boldsymbol{D}}{}\ \underset{\boldsymbol{I}}{1}\,dx$$

$$= (\,ln\,x\,)\,x - \int \frac{1}{x}\,x\,dx$$

$$= x\,ln\,x - \int 1\,dx$$

$$= x\,ln\,x - x + c$$

$$\int_0^1 ln\,x\,dx = \underset{t \to 0^+}{limit} \int_t^1 ln\,x\,dx$$

$$= \underset{t \to 0^+}{limit} \left[\, x\,ln\,x - x \,\right]_t^1$$

$$= \underset{t \to 0^+}{limit} \left[\, ln\,1 - 1 - t\,ln\,t + t \,\right]$$

$$= -1$$

$\therefore$ Area $= 1$

[6 marks]

Extra

L'Hôspital's Rule

For two functions $f\,(x)$ and $g\,(x)$, if either:

- $\underset{x \to a}{limit}\ f\,(x) = \underset{x \to a}{limit}\ g\,(x) = 0$
- $\underset{x \to a}{limit}\ f\,(x) = \pm\infty$ and $\underset{x \to a}{limit}\ g\,(x) = \pm\infty$

then, provided that $\underset{x \to a}{limit}\dfrac{f\,(x)}{g\,(x)}$ exists, $\underset{x \to a}{limit}\dfrac{f\,(x)}{g\,(x)} = \underset{x \to a}{limit}\dfrac{f'\,(x)}{g'\,(x)}$

A proof that, as $x \to 0^+,\ x\,ln\,x \to 0$ using L'Hôspital's Rule

$$\underset{x \to 0^+}{limit}\ x\,ln\,x = \underset{x \to 0^+}{limit}\ \frac{ln\,x}{\left(\frac{1}{x}\right)} = \underset{x \to 0^+}{limit}\ \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \times \frac{x^2}{x^2} = \underset{x \to 0^+}{limit}\ -\ x = 0$$

□

Answer 4

(i) As $x \to 0^+,\ x(\ln x)^2 \to 0$

[1 mark]

(ii)
$$\int (\ln x)^2\, dx = \int (\ln x)^2 \times 1\, dx$$

L I D I

$$= (\ln x)^2\, x - \int 2(\ln x)\frac{1}{x}\, x\, dx$$

$$= (\ln x)^2\, x - 2\int \ln x\, dx$$

$$= (\ln x)^2\, x - 2\left[x \ln x - x\right] + c$$

$$= x\,(\ln x)^2 - 2x \ln x + 2x + c$$

$$\int_0^1 (\ln x)^2\, dx = \lim_{t \to 0^+} \int_t^1 (\ln x)^2\, dx$$

$$= \lim_{t \to 0^+} \Big[x\,(\ln x)^2 - 2x \ln x + 2x\Big]_t^1$$

$$= \lim_{t \to 0^+} \Big[(\ln 1)^2 - 2\ln 1 + 2 - t\,(\ln t)^2 + 2t \ln t - 2t\Big]$$

$$= 2$$

$\therefore$ Area = 2

[6 marks]

Extra

A proof that, as $x \to 0^+,\ x\,(\ln x)^2 \to 0$ using L'Hôspital's Rule

$$\lim_{x \to 0^+} x\,(\ln x)^2 = \lim_{x \to 0^+} \frac{(\ln x)^2}{\left(\frac{1}{x}\right)}$$

$$= \lim_{x \to 0^+} \frac{2\,(\ln x)\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \times \frac{x}{x}$$

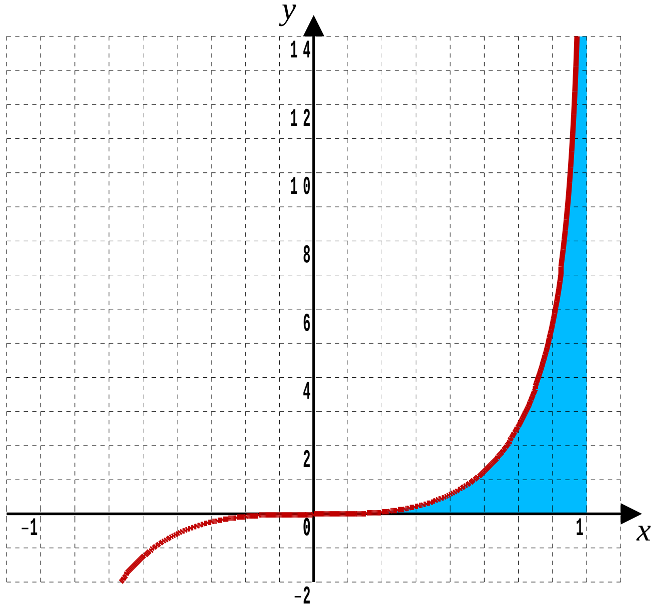
$$= -2 \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)}$$

$$= -2 \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \times \frac{x^2}{x^2}$$

$$= 2 \lim_{x \to 0^+} x = 0 \qquad \square$$

Answer 5

(i)



[1 mark]

(ii) **(a)**

$$y = (1 - x^2)^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2}(1 - x^2)^{\frac{1}{2}}(-2x)$$

$$= -3x(1 - x^2)^{\frac{1}{2}}$$

$$= -3x\sqrt{1 - x^2}$$

(b)

$$y = (1 - x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x)$$

$$= -x(1 - x^2)^{-\frac{1}{2}}$$

$$= -\frac{x}{\sqrt{1 - x^2}}$$

[2 marks]

(iii) Without the limits for now,

$$\int \frac{x^3}{\sqrt{1 - x^2}}\,dx = \int (x^2)\left(\frac{x}{\sqrt{1 - x^2}}\right)dx$$

L ***I*** ***D*** ***I***

$$= (x^2)(-1)(1 - x^2)^{\frac{1}{2}} - \int (2x)(-1)(1 - x^2)^{\frac{1}{2}}\,dx$$

$$= -x^2(1 - x^2)^{\frac{1}{2}} - \frac{2}{3}\int(-3x)(1 - x^2)^{\frac{1}{2}}\,dx$$

$$= -x^2(1 - x^2)^{\frac{1}{2}} - \frac{2}{3}(1 - x^2)^{\frac{3}{2}} + c$$

$$\int_0^1 \frac{x^3}{\sqrt{1 - x^2}}\, dx = \underset{t \to 0^+}{limit} \int_t^1 \frac{x^3}{\sqrt{1 - x^2}}\, dx$$

$$= \underset{t \to 0^+}{limit} \left[-x^2 \left(1 - x^2 \right)^{\frac{1}{2}} - \frac{2}{3} \left(1 - x^2 \right)^{\frac{3}{2}} \right]_t^1$$

$$= \underset{t \to 0^+}{limit} \left[0 + t^2 \left(1 - t^2 \right)^{\frac{1}{2}} + \frac{2}{3} \left(1 - t^2 \right) \right]_t^1$$

$$= \frac{2}{3}$$

[6 marks]

Answer 6

(i)

$$S = \int \frac{(\, ln\, x\,)^k}{x}\, dx$$

$$= \int (\, ln\, x\,)^k \left(\frac{1}{x} \right) dx$$

L ***I*** ***D*** ***I***

$$= (\, ln\, x\,)^k \, \left(ln\, x \right) - \int k\, (\, ln\, x\,)^{k-1} \left(\frac{1}{x} \right) \left(ln\, x \right) dx$$

$$= (\, ln\, x\,)^{k+1} - k \int (\, ln\, x\,)^k \left(\frac{1}{x} \right) dx$$

$$S = (\, ln\, x\,)^{k+1} - k\, S$$

$$(\, 1 + k\,)\, S = (\, ln\, x\,)^{k+1}$$

$$S = \frac{(\, ln\, x\,)^{k+1}}{k + 1} + c \qquad \square$$

[5 marks]

(ii)

$$\int_0^1 \frac{(\ln x)^k}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{(\ln x)^k}{x}\,dx$$

$$= \frac{1}{k+1} \lim_{t\to 0^+} \Big[(\ln x)^k \Big]_t^1$$

$$= \frac{1}{k+1} \lim_{t\to 0^+} \Big[0 - (\ln t)^k \Big]$$

As $t \to 0^+$, $\ln t \to -\infty$

Furthermore, $k \geqslant 1$ so $(\ln t)^k \to -\infty$

$\therefore$ The integral does not converge; it is divergent ☐

[3 marks]

$$\int_1^\infty \frac{(\ln x)^k}{x}\,dx = \lim_{t\to \infty} \int_1^t \frac{(\ln x)^k}{x}\,dx$$

$$= \frac{1}{k+1} \lim_{t\to \infty} \Big[(\ln x)^k \Big]_1^t$$

$$= \frac{1}{k+1} \lim_{t\to \infty} \Big[(\ln t)^k - 0 \Big]$$

As $t \to \infty$, $\ln t \to \infty$

Furthermore, $k \geqslant 1$ so $(\ln t)^k \to \infty$

$\therefore$ The integral does not converge; it is divergent ☐

[3 marks]