

3.3 Answers

Further A-Level Pure Mathematics, Core 2

Improper Integrals

3.3.1 Solution to 3.1 Example (Teaching Video)

Let $x = 2 \sin \theta$ in which case $\frac{dx}{d\theta} = 2 \cos \theta$ and limits are $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$\begin{aligned}\int_0^2 \frac{1}{\sqrt{4-x^2}} dx &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{4-(2 \sin^2 \theta)}} \times 2 \cos \theta d\theta \\&= \int_0^{\frac{\pi}{2}} \frac{2 \cos \theta}{\sqrt{4(1-\sin^2 \theta)}} d\theta \\&= \int_0^{\frac{\pi}{2}} \frac{2 \cos \theta}{\sqrt{4} \sqrt{\cos^2 \theta}} d\theta \\&= \int_0^{\frac{\pi}{2}} 1 d\theta \\&= \left[\theta \right]_0^{\frac{\pi}{2}} \\&= \frac{\pi}{2}\end{aligned}$$

Notice that, in this case, the improper integral under the substitution became proper.

[6 marks]

3.3.2 Solution to 3.2 Exercise

Answer 1

- (i) If the area from $x = 0$ to $x = \infty$ is bounded then, because the function is symmetrical, the area from $x = -\infty$ to $x = \infty$ will be double it.

[1 mark]

- (ii)

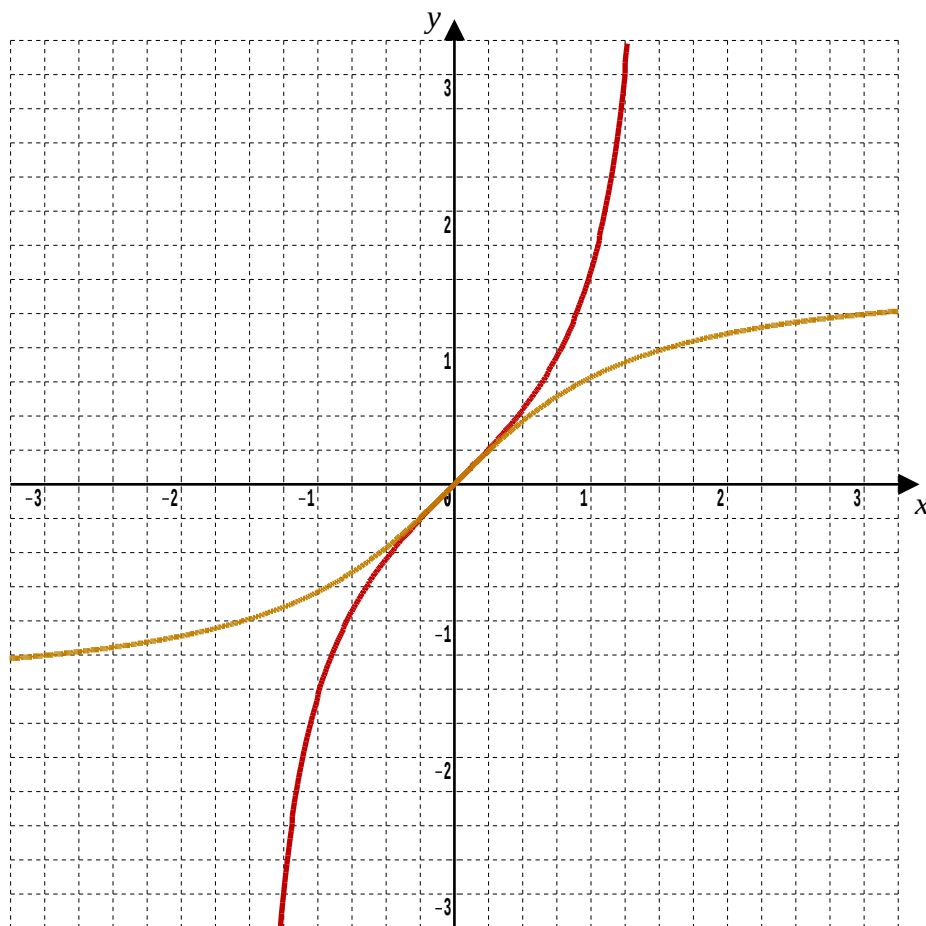
$$\begin{aligned}x &= \tan \theta \\&= \frac{\sin \theta}{\cos \theta} \\ \frac{dx}{d\theta} &= \frac{\cos \theta \cos \theta - (-\sin \theta) \sin \theta}{\cos^2 \theta} \text{ by the quotient rule} \\&= \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} \\&= \frac{1}{\cos^2 \theta} \\&= \sec^2 \theta \quad \square\end{aligned}$$

[2 marks]

(iii)

Let $x = \tan \theta$ in which case $\frac{dx}{d\theta} = \sec^2 \theta$

For the limits it may help to have the graph of $\theta = \arctan x$ handy;



Red : $y = \tan x$ Gold : $y = \arctan x$

The limits $x = 0, \quad x = \infty$ become $\theta = 0, \quad \theta = \frac{\pi}{2}$

$$\begin{aligned} \int_0^\infty \frac{1}{1+x^2} dx &= \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^2 \theta} \times \sec^2 \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\sec^2 \theta} \times \sec^2 \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} 1 d\theta \\ &= [\theta]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} \quad \therefore \int_{-\infty}^\infty f(x) dx = \pi \quad \square \end{aligned}$$

[6 marks]

Answer 2

(i) Let $x = 6 + u^2$ in which case $\frac{dx}{du} = 2u$

The limits $x = 6, \quad x = 7$ become $\theta = 0, \quad \theta = 1$

$$\begin{aligned}\int_6^7 \frac{(x-4)(5x+2)}{\sqrt{x-6}} dx &= \int_0^1 \frac{(u^2+2)(5u^2+32)}{\sqrt{u^2}} 2u du \\ &= 2 \int_0^1 5u^4 + 42u^2 + 64u du \\ &= 2 \left[u^5 + 14u^3 + 64u \right]_0^1 \\ &= 158\end{aligned}$$

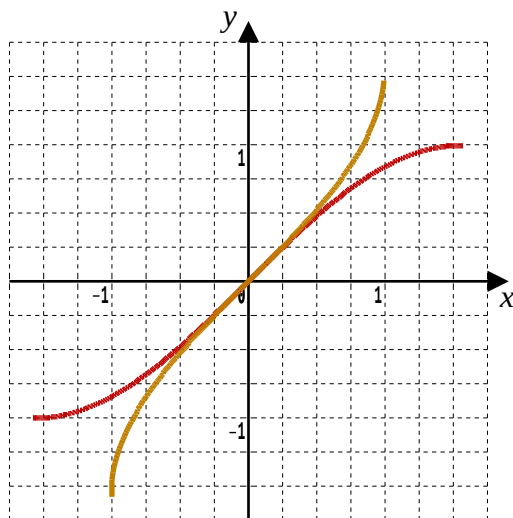
[5 marks]

(ii) The function looks to be both increasing and convex and so, as $x \rightarrow \infty$ the area $\rightarrow \infty$ indicating the function from 7 to ∞ is divergent.

[2 marks]

Answer 3

Let $x = \sin \theta + 2$ in which case $\frac{dx}{d\theta} = \cos \theta$



Red : $y = \sin x$ Gold : $y = \arcsin x$

The limits $x = 1$, $x = 3$ become $\theta = -\frac{\pi}{2}$, $\theta = \frac{\pi}{2}$

$$\begin{aligned}
 \int_1^3 \frac{x}{\sqrt{(x-1)(3-x)}} dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin \theta + 2}{\sqrt{(\sin \theta + 1)(1 - \sin \theta)}} \cos \theta d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin \theta + 2}{\sqrt{1 - \sin^2 \theta}} \cos \theta d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin \theta + 2}{\sqrt{\cos^2 \theta}} \cos \theta d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \theta + 2 d\theta \\
 &= \left[-\cos \theta + 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
 &= \left[-\cos\left(\frac{\pi}{2}\right) + \pi + \cos\left(-\frac{\pi}{2}\right) + \pi \right] \\
 &= 2\pi \quad (\text{As } \cos x \text{ is an even function})
 \end{aligned}$$

[7 marks]

Answer 4

Let $x = \frac{1}{2} \sin \theta$ in which case $\frac{dx}{d\theta} = \frac{1}{2} \cos \theta$

The limits $x = \pm \frac{1}{2}$ become $-\frac{\pi}{2}, \frac{\pi}{2}$

$$\begin{aligned}
 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^2}{\sqrt{1-4x^2}} dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\frac{1}{2} \sin \theta\right)^2}{\sqrt{1-4\left(\frac{1}{2} \sin \theta\right)^2}} \frac{1}{2} \cos \theta d\theta \\
 &= \frac{1}{8} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 \theta \cos \theta}{\sqrt{1-\sin^2 \theta}} d\theta \\
 &= \frac{1}{8} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \theta d\theta \\
 &= \frac{1}{8} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta \right) d\theta \\
 &= \frac{1}{16} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 - \cos 2\theta d\theta \\
 &= 2 \times \frac{1}{16} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}} \\
 &= \frac{1}{8} \left[\frac{\pi}{2} - 0 - 0 + 0 \right] \\
 &= \frac{\pi}{16}
 \end{aligned}$$

[7 marks]

Answer 5

(i)

$$\begin{aligned}
 \text{LHS} &= (b-x)(x-a) \\
 &= (b-a\cos^2 \theta - b\sin^2 \theta)(a\cos^2 \theta + b\sin^2 \theta - a) \\
 &= (b(1-\sin^2 \theta) - a\cos^2 \theta)((-a)(1-\cos^2 \theta) + b\sin^2 \theta) \\
 &= (b\cos^2 \theta - a\cos^2 \theta)(b\sin^2 \theta - a\sin^2 \theta) \\
 &= ((b-a)\cos^2 \theta)((b-a)\sin^2 \theta) \\
 &= (b-a)^2 \cos^2 \theta \sin^2 \theta \\
 &= \text{RHS} \quad \square
 \end{aligned}$$

[3 marks]

(ii) For the lower limit, when $x = a$,

$$a = a \cos^2 \theta + b \sin^2 \theta$$

$$a(1 - \cos^2 \theta) = b \sin^2 \theta$$

$$b \sin^2 \theta - a \sin^2 \theta = 0$$

$$(b - a) \sin^2 \theta = 0$$

Either $b - a = 0$ which is not so as $a < b$

$$\text{Or } \sin^2 \theta = 0$$

$$\theta = 0$$

For the upper limit, when $x = b$

$$b = a \cos^2 \theta + b \sin^2 \theta$$

$$b(1 - \sin^2 \theta) = a \cos^2 \theta$$

$$b \cos^2 \theta - a \cos^2 \theta = 0$$

$$(b - a) \cos^2 \theta = 0$$

Either $b - a = 0$ which is not so as $a < b$

$$\text{Or } \cos^2 \theta = 0$$

$$\theta = \frac{\pi}{2}$$

Let $x = a \cos^2 \theta + b \sin^2 \theta$

$$\Rightarrow \frac{dx}{d\theta} = 2a \cos \theta (-\sin \theta) + 2b \sin \theta \cos \theta$$

$$= 2b \cos \theta \sin \theta - 2a \cos \theta \sin \theta$$

$$= 2(b - a) \cos \theta \sin \theta$$

The limits $x = a, b$ become $0, \frac{\pi}{2}$

$$\int_a^b \frac{1}{\sqrt{(b-x)(x-a)}} dx = \int_0^{\frac{\pi}{2}} \frac{2(b-a) \cos \theta \sin \theta}{\sqrt{(b-a)^2 \cos^2 \theta \sin^2 \theta}} d\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} 1 d\theta$$

$$= 2 \left[\theta \right]_0^{\frac{\pi}{2}}$$

$$= 2 \left[\frac{\pi}{2} - 0 \right]$$

$$= \pi \quad \square$$

[8 marks]

Answer 6

Let $x = \sec \theta$ in which case $\frac{dx}{d\theta} = \tan \theta \sec \theta$

(Easily proven using the chain rule)

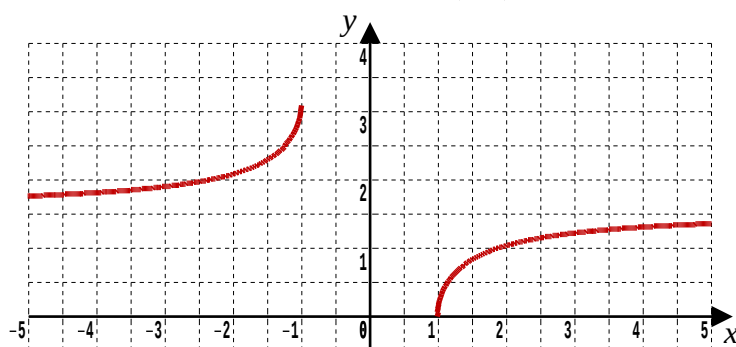
Graph Plotting

To draw the graphs of $\operatorname{arcsec} x$, $\operatorname{arccsc} x$ and $\operatorname{arccot} x$, or calculate their value at a particular point, use the relationships,

$$\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$$

$$\operatorname{arccsc} x = \arcsin\left(\frac{1}{x}\right)$$

$$\operatorname{arccot} x = \arctan\left(\frac{1}{x}\right)$$



Graph of $y = \operatorname{arcsec} x$

The limits $x = 2, \infty$ become $\frac{\pi}{3}, \frac{\pi}{2}$

$$\begin{aligned} \int_2^{\infty} \frac{1}{x\sqrt{x^2-1}} dx &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sec \theta \sqrt{\sec^2 \theta - 1}} \tan \theta \sec \theta d\theta \\ &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\tan \theta}{\sqrt{\tan^2 \theta}} d\theta \\ &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1 d\theta \\ &= \left[\theta \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= \left[\frac{\pi}{2} - \frac{\pi}{3} \right] \\ &= \frac{\pi}{6} \end{aligned}$$

[9 marks]

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