3.3 Answers

3.3.1 Solution to 3.1 Example (Teaching Video)

Let $x = 2\sin\theta$ in which case $\dfrac{dx}{d\theta} = 2\cos\theta$ and limits are $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

$$\int_0^2 \frac{1}{\sqrt{4 - x^2}}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{4 - (2\sin^2\theta)}} \times 2\cos\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\cos\theta}{\sqrt{4(1 - \sin^2\theta)}}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\cos\theta}{\sqrt{4}\sqrt{\cos^2\theta}}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 1\,d\theta$$

$$= [\,\theta\,]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

Notice that, in this case, the improper integral under the substitution became proper.

[6 marks]

3.3.2 Solution to 3.2 Exercise

Answer 1

(i) If the area from $x = 0$ to $x = \infty$ is bounded then, because the function is symmetrical, the area from $x = -\infty$ to $x = \infty$ will be double it.

[1 mark]

(ii)

$$x = \tan\theta$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$\frac{dx}{d\theta} = \frac{\cos\theta\cos\theta - (-\sin\theta)\sin\theta}{\cos^2\theta} \quad \text{by the quotient rule}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta}$$

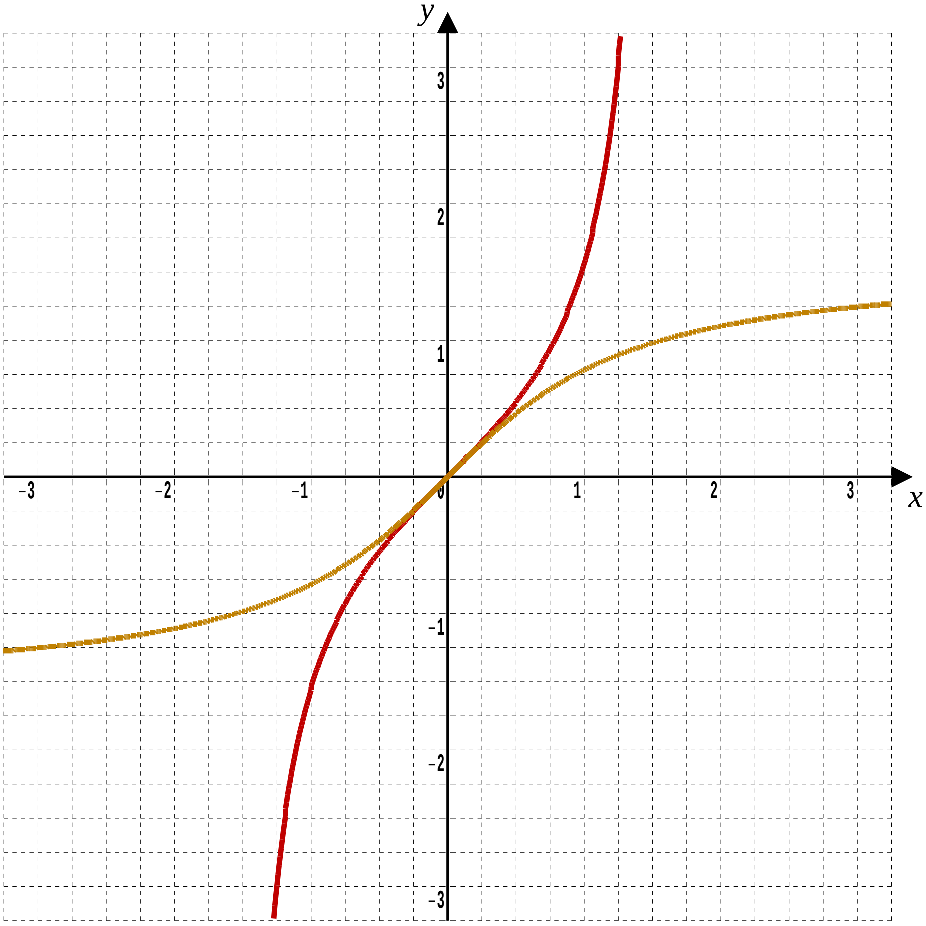
$$= \sec^2\theta \qquad \square$$

[2 marks]

(iii)

Let $x = \tan\theta$ in which case $\dfrac{dx}{d\theta} = \sec^2\theta$

For the limits it may help to have the graph of $\theta = \arctan x$ handy;



Red : $y = \tan x$ Gold : $y = \arctan x$

The limits $x = 0$, $x = \infty$ become $\theta = 0$, $\theta = \dfrac{\pi}{2}$

$$\int_0^\infty \frac{1}{1 + x^2}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^2\theta} \times \sec^2\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sec^2\theta} \times \sec^2\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} 1\ d\theta$$

$$= \Big[\ \theta\ \Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} \qquad\qquad \therefore \int_{-\infty}^{\infty} f(x)\ dx = \pi \qquad \square$$

[6 marks]

Answer 2

(i) Let $x = 6 + u^2$ in which case $\dfrac{dx}{du} = 2u$

The limits $x = 6, \quad x = 7$ become $\theta = 0, \quad \theta = 1$

$$\int_6^7 \frac{(x - 4)(5x + 2)}{\sqrt{x - 6}}\, dx = \int_0^1 \frac{\left(u^2 + 2\right)\left(5u^2 + 32\right)}{\sqrt{u^2}}\, 2u\, du$$

$$= 2\int_0^1 5u^4 + 42u^2 + 64\, du$$

$$= 2\left[\, u^5 + 14u^3 + 64u \,\right]_0^1$$

$$= 158$$

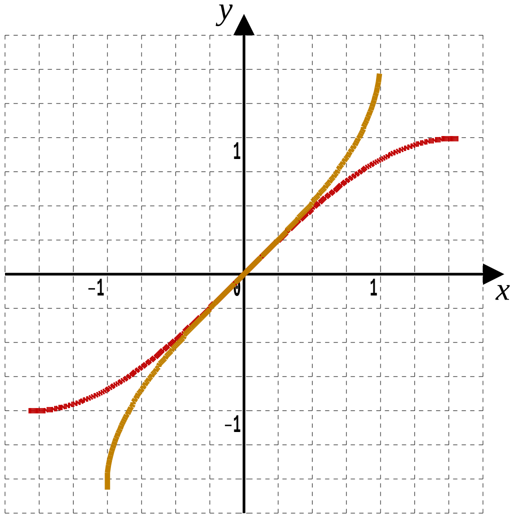
[5 marks]

(ii) The function looks to be both increasing and convex and so, as $x \to \infty$ the area $\to \infty$ indicating the function from 7 to ∞ is divergent.

[2 marks]

Answer 3

Let $x = \sin\theta + 2$ in which case $\dfrac{dx}{d\theta} = \cos\theta$



Red : $y = \sin x$ Gold : $y = \arcsin x$

The limits $x = 1$, $x = 3$ become $\theta = -\dfrac{\pi}{2}$, $\theta = \dfrac{\pi}{2}$

$$\int_1^3 \frac{x}{\sqrt{(x-1)(3-x)}}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\theta + 2}{\sqrt{(\sin\theta + 1)(1 - \sin\theta)}} \cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\theta + 2}{\sqrt{1 - \sin^2\theta}} \cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\theta + 2}{\sqrt{\cos^2\theta}} \cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta + 2\, d\theta$$

$$= \Big[-\cos\theta + 2\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left[-\cos\left(\frac{\pi}{2}\right) + \pi + \cos\left(-\frac{\pi}{2}\right) + \pi\right]$$

$= 2\pi$ (As $\cos x$ is an even function)

[7 marks]

Answer 4

Let $x = \frac{1}{2} sin\,\theta$ in which case $\frac{dx}{d\theta} = \frac{1}{2} cos\,\theta$

The limits $x = \pm\frac{1}{2}$ become $-\frac{\pi}{2}, \frac{\pi}{2}$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^2}{\sqrt{1-4x^2}}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\frac{1}{2}\sin\theta\right)^2}{\sqrt{1-4\left(\frac{1}{2}\sin\theta\right)^2}}\,\frac{1}{2}\cos\theta\,d\theta$$

$$= \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2\theta\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta$$

$$= \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\,d\theta$$

$$= \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1-\cos 2\theta\;\;d\theta$$

$$= 2\times\frac{1}{16}\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left[\frac{\pi}{2}-0-0+0\right]$$

$$= \frac{\pi}{16}$$

[7 marks]

Answer 5

(i)

$$\begin{aligned}
\text{LHS} &= (b-x)(x-a)\\
&= \left(b-a\cos^2\theta-b\sin^2\theta\right)\left(a\cos^2\theta+b\sin^2\theta-a\right)\\
&= \left(b\left(1-\sin^2\theta\right)-a\cos^2\theta\right)\left((-a)\left(1-\cos^2\theta\right)+b\sin^2\theta\right)\\
&= \left(b\cos^2\theta-a\cos^2\theta\right)\left(b\sin^2\theta-a\sin^2\theta\right)\\
&= \left((b-a)\cos^2\theta\right)\left((b-a)\sin^2\theta\right)\\
&= (b-a)^2\cos^2\theta\sin^2\theta\\
&= \text{RHS} \qquad\qquad \square
\end{aligned}$$

[3 marks]

(ii) For the lower limit, when $x = a$,

$$a = a\cos^2\theta + b\sin^2\theta$$

$$a\left(1 - \cos^2\theta\right) = b\sin^2\theta$$

$$b\sin^2\theta - a\sin^2\theta = 0$$

$$\left(b - a\right)\sin^2\theta = 0$$

Either $b - a = 0$ which is not so as $a < b$

Or $\sin^2\theta = 0$

$$\theta = 0$$

For the upper limit, when $x = b$

$$b = a\cos^2\theta + b\sin^2\theta$$

$$b\left(1 - \sin^2\theta\right) = a\cos^2\theta$$

$$b\cos^2\theta - a\cos^2\theta = 0$$

$$\left(b - a\right)\cos^2\theta = 0$$

Either $b - a = 0$ which is not so as $a < b$

Or $\cos^2\theta = 0$

$$\theta = \frac{\pi}{2}$$

Let $x = a\cos^2\theta + b\sin^2\theta$

$$\Rightarrow \frac{dx}{d\theta} = 2a\cos\theta\,(-\sin\theta) + 2b\sin\theta\cos\theta$$

$$= 2b\cos\theta\sin\theta - 2a\cos\theta\sin\theta$$

$$= 2(b - a)\cos\theta\sin\theta$$

The limits $x = a,\ b$ become $0,\ \frac{\pi}{2}$

$$\int_a^b \frac{1}{\sqrt{(b - x)(x - a)}}\,dx = \int_0^{\frac{\pi}{2}} \frac{2(b - a)\cos\theta\sin\theta}{\sqrt{(b - a)^2\cos^2\theta\sin^2\theta}}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1\,d\theta$$

$$= 2\left[\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[\frac{\pi}{2} - 0\right]$$

$$= \pi \qquad \square$$

[8 marks]

Answer 6

Let $x = sec\ \theta$ in which case $\dfrac{dx}{d\theta} = tan\ \theta\ sec\ \theta$

(Easily proven using the chain rule)

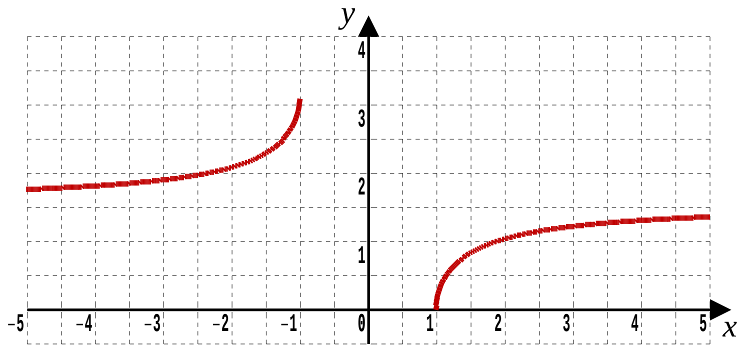
Graph Plotting

To draw the graphs of *arcsec x*, *arccsc x* and *arccot x*, or calculate their value at a particular point, use the relationships,

$$arcsec\ x = arccos\left(\frac{1}{x}\right)$$

$$arccsc\ x = arcsin\left(\frac{1}{x}\right)$$

$$arccot\ x = arctan\left(\frac{1}{x}\right)$$



Graph of $y = arcsec\ x$

The limits $x = 2,\ \infty$ become $\dfrac{\pi}{3},\ \dfrac{\pi}{2}$

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2 - 1}}dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{sec\ \theta\sqrt{sec^2\ \theta - 1}}\ tan\ \theta\ sec\ \theta\ d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{tan\ \theta}{\sqrt{tan^2\ \theta}}\ d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1\ d\theta$$

$$= \Big[\ \theta\ \Big]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \left[\ \frac{\pi}{2} - \frac{\pi}{3}\ \right]$$

$$= \frac{\pi}{6}$$

[9 marks]