4.3 Answers

Further A-Level Pure Mathematics, Core 2

Improper Integrals

4.3.1 Solution to 4.1 Example (Teaching Video)

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{(x-3)(x-2)}$$

$$\frac{1}{[\ (x-3)(x-2)\ ]} = \frac{A}{(x-3)} + \frac{B}{(x-2)}$$

Multiply through by the red brackets

$$\frac{1\ [\ \ldots\ldots\ ]}{[\ (x-3)(x-2)\ ]} = \frac{A\ [\ \ldots\ldots\ ]}{(x-3)} + \frac{B\ [\ \ldots\ldots\ ]}{(x-2)}$$

$$1 = A(x-2) + B(x-3)$$

$$\text{Let } x = 3 \Rightarrow A = 1$$

$$\text{Let } x = 2 \Rightarrow B = -1$$

$$\int_4^{\infty} \frac{1}{x^2 - 5x + 6}\,dx = \int_4^{\infty} \frac{1}{x-3} - \frac{1}{x-2}\,dx$$

$$= \lim_{t \to \infty} \int_4^{t} \frac{1}{x-3} - \frac{1}{x-2}\,dx$$

$$= \lim_{t \to \infty} \Big[\ \ln(x-3) - \ln(x-2)\ \Big]_4^{t}$$

$$= \lim_{t \to \infty} \left[\ \ln\left(\frac{x-3}{x-2}\right)\ \right]_4^{t}$$

$$= \lim_{t \to \infty} \left[\ \ln\left(\frac{t-3}{t-2}\right) - \ln\left(\frac{1}{2}\right)\ \right]$$

$$= \ln 1 - \ln 2^{-1}$$

$$= \ln 2$$

[8 marks]

4.3.2 Solutions to 4.2 Exercise

Answer 1

$$\frac{1}{x^2 + x - 2} = \frac{1}{(x+2)(x-1)}$$

$$\frac{1}{[\,(x-1)(x+2)\,]} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

Multiply through by the red brackets

$$\frac{1\,[\,\ldots\ldots\,]}{[\,(x-1)(x+2)\,]} = \frac{A\,[\,\ldots\ldots\,]}{(x-1)} + \frac{B\,[\,\ldots\ldots\,]}{(x+2)}$$

$$1 = A(x+2) + B(x-1)$$

$$\text{Let } x = 1 \Rightarrow A = \frac{1}{3}$$

$$\text{Let } x = -2 \Rightarrow B = -\frac{1}{3}$$

$$\int_2^{\infty} \frac{1}{x^2 + x - 2}\,dx = \frac{1}{3}\int_2^{\infty} \frac{1}{x-1} - \frac{1}{x+2}\,dx$$

$$= \frac{1}{3}\lim_{t\to\infty} \int_2^{t} \frac{1}{x-1} - \frac{1}{x+2}\,dx$$

$$= \frac{1}{3}\lim_{t\to\infty} \Big[\, ln(x-1) - ln(x+2)\,\Big]_2^t$$

$$= \frac{1}{3}\lim_{t\to\infty} \left[\, ln\left(\frac{x-1}{x+2}\right)\right]_2^t$$

$$= \frac{1}{3}\lim_{t\to\infty} \left[\, ln\left(\frac{t-1}{t+2}\right) - ln\left(\frac{1}{4}\right)\right]$$

$$= \frac{1}{3}\,ln\,1 - \frac{1}{3}\,ln\,2^{-2}$$

$$= \frac{2}{3}\,ln\,2 \qquad \Box$$

[8 marks]

Answer 2

$$\frac{19}{6x^2 + 35x + 36} = \frac{19}{(3x+4)(2x+9)}$$

$$\frac{19}{[\,(3x+4)(2x+9)\,]} = \frac{A}{(3x+4)} + \frac{B}{(2x+9)}$$

Multiply through by the red brackets

$$\frac{19\,[\,\ldots\ldots\,]}{[\,(3x+4)(2x+9)\,]} = \frac{A\,[\,\ldots\ldots\,]}{(3x+4)} + \frac{B\,[\,\ldots\ldots\,]}{(2x+9)}$$

$$19 = A(2x+9) + B(3x+4)$$

$$\text{Let } x = -\frac{4}{3} \Rightarrow A = 3$$

$$\text{Let } x = -\frac{9}{2} \Rightarrow B = -2$$

$$\int_0^\infty \frac{19}{6x^2 + 35x + 36}\,dx = \int_0^\infty \frac{3}{3x+4} - \frac{2}{2x+9}\,dx$$

$$= \lim_{t \to \infty} \int_0^t \frac{3}{3x+4} - \frac{2}{2x+9}\,dx$$

$$= \lim_{t \to \infty} \Big[\, ln(3x+4) - ln(2x+9) \,\Big]_0^t$$

$$= \lim_{t \to \infty} \left[\, ln\left(\frac{3x+4}{2x+9}\right) \right]_0^t$$

$$= \lim_{t \to \infty} \left[\, ln\left(\frac{3t+4}{2t+9}\right) - ln\left(\frac{4}{9}\right) \right]$$

$$= ln\left(\frac{3}{2}\right) - ln\left(\frac{4}{9}\right)$$

$$= ln\left(\frac{3}{2} \div \frac{4}{9}\right)$$

$$= ln\left(\frac{3}{2} \times \frac{9}{4}\right)$$

$$= ln\left(\frac{3}{2}\right)^3$$

$$= 3\,ln\left(\frac{3}{2}\right) \qquad \square$$

[8 marks]

Answer 3

(i) Let $x = 2\,tan\,\theta$ in which case $\dfrac{dx}{d\theta} = 2\,sec^2\,\theta$

Making the substitution,

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4\,tan^2\,x + 4}\,2\,sec^2\,\theta\,d\theta$$
$$= \frac{2}{4}\int \frac{1}{tan^2\,\theta + 1}\,sec^2\,\theta\,d\theta$$
$$= \frac{1}{2}\int \frac{1}{sec^2\,\theta}\,sec^2\,\theta\,d\theta$$
$$= \frac{1}{2}\int 1\,d\theta$$
$$= \frac{1}{2}\,\theta + c$$
$$= \frac{1}{2}\,arctan\left(\frac{x}{2}\right) + c \qquad \square$$

[3 marks]

(ii)

$$\frac{32}{[\,(x^2+4)(x+2)\,]} = \frac{Ax+B}{(x^2+4)} + \frac{C}{(x+2)}$$

Multiply through by the red brackets

$$\frac{32\,[\,\dots\dots\,]}{[\,(x^2+4)(x+2)\,]} = \frac{(Ax+B)\,[\,\dots\dots\,]}{(x^2+4)} + \frac{C\,[\,\dots\dots\,]}{(x+2)}$$

$$32 = (Ax+B)(x+2) + C\left(x^2+4\right)$$

Let $x = -2 \Rightarrow C = 4$

Let $x = 0 \Rightarrow 32 = 2B + 4C \Rightarrow B = 8$

Let $x = 1 \Rightarrow 32 = (A+8)(3) + 4(5) \Rightarrow A = -4$

$$\int_2^\infty \frac{32}{(x^2+4)(x+2)}\,dx = \int_2^\infty \frac{8-4x}{x^2+4} + \frac{4}{x+2}\,dx$$
$$= \lim_{t\to\infty} \int_2^t 8\,\frac{1}{x^2+4} - 2\,\frac{2x}{x^2+4} + 4\,\frac{1}{x+2}\,dx$$
$$= \lim_{t\to\infty} \left[4\,arctan\left(\frac{x}{2}\right) - 2ln\left(x^2+4\right) + 4ln\,(x+2)\right]_2^t$$
$$= \lim_{t\to\infty} \left[4\,arctan\left(\frac{t}{2}\right) + 2ln\left(\frac{(t+2)^2}{t^2+4}\right) - \pi - 2ln\,(2)\right]$$
$$= 4 \times \frac{\pi}{2} + 2\,ln\,(1) - \pi - 2ln\,(2)$$
$$= \pi - 2ln\,(2)$$

[9 marks]

Answer 4

$$\int_{\frac{1}{2}}^{\infty} \frac{1 - x^2}{4x^4 + x^2}\, dx = \int_{\frac{1}{2}}^{\infty} \frac{1 - x^2}{x^2 (4x^2 + 1)}\, dx$$

Partial fractions,

$$\frac{1 - x^2}{[\, x^2 (4x^2 + 1)\,]} = \frac{Ax + B}{x^2} + \frac{Cx + D}{4x^2 + 1}$$

$$\frac{(1 - x^2)[\, \ldots \ldots\,]}{[\, x^2 (4x^2 + 1)\,]} = \frac{(Ax + B)[\, \ldots \ldots]}{x^2} + \frac{(Cx + D)[\, \ldots \ldots\,]}{4x^2 + 1}$$

$$1 - x^2 = (Ax + B)(4x^2 + 1) + (Cx + D)x^2$$

Coefficients of x^0 : $1 = B$

Coefficients of x : $0 = A$

Coefficients of x^2 : $(-1) = 4B + D \Rightarrow D = -5$

Coefficients of x^3 : $0 = A + C \Rightarrow C = 0$

$$\int_{\frac{1}{2}}^{\infty} \frac{1 - x^2}{4x^4 + x^2}\, dx = \int_{\frac{1}{2}}^{\infty} \frac{1}{x^2}\, dx - \int_{\frac{1}{2}}^{\infty} \frac{5}{4x^2 + 1}\, dx$$

$$\int_{\frac{1}{2}}^{\infty} \frac{1}{x^2}\, dx = \lim_{t \to \infty} \int_{\frac{1}{2}}^{t} x^{-2}\, dt$$

$$= \lim_{t \to \infty} \left[-\frac{1}{x} \right]_{\frac{1}{2}}^{t}$$

$$= \lim_{t \to \infty} \left[-\frac{1}{t} + 2 \right]$$

$$= 2$$

Let $x = \frac{1}{2} \tan\theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{2} \sec^2\theta$

$$\int_{\frac{1}{2}}^{\infty} \frac{5}{4x^2 + 1}\, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{5}{\tan^2\theta + 1} \times \frac{1}{2} \sec^2\theta\, d\theta$$

$$= \frac{5}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1\, d\theta$$

$$= \frac{5}{2} [\, \theta\,]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{5\pi}{8}$$

$$\therefore \int_{\frac{1}{2}}^{\infty} \frac{1 - x^2}{4x^4 + x^2}\, dx = 2 - \frac{5\pi}{8} \qquad \square$$

[10 marks]

Answer 5

$$\frac{3-x}{\left[(x-2)^2(x^2+4)\right]} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+4}$$

$$\frac{(3-x)[\ldots\ldots]}{\left[(x-2)^2(x^2+4)\right]} = \frac{A[\ldots\ldots]}{(x-2)} + \frac{B[\ldots\ldots]}{(x-2)^2} + \frac{(Cx+D)[\ldots\ldots]}{x^2+4}$$

$$3-x = A(x-2)\left(x^2+4\right) + B\left(x^2+4\right) + (Cx+D)(x-2)^2$$

$$\text{Let } x = 2 \Rightarrow 1 = 8B \Rightarrow B = \frac{1}{8}$$

$$\begin{aligned}\text{Let } x = 2i \Rightarrow 3-2i &= (2iC+D)(2i-2)^2 \\ &= (2iC+D)(-4-8i+4) \\ &= (2iC+D)(-8i) \\ &= 16C - 8iD\end{aligned}$$

$$\therefore C = \frac{3}{16} \text{ and } D = \frac{1}{4}$$

$$\text{Coefficients of } x^3 : 0 = A + C \Rightarrow A = -\frac{3}{16}$$

$$\begin{aligned}96\int_{2\sqrt{3}}^{\infty}\frac{3-x}{(x-2)^2(x^2+4)}\,dx &= \int_{2\sqrt{3}}^{\infty}\frac{-18}{x-2} + \frac{12}{(x-2)^2} + \frac{18x+24}{x^2+4}\,dx \\ &= \int_{2\sqrt{3}}^{\infty}\frac{-18}{x-2} + \frac{12}{(x-2)^2} + \frac{18x}{x^2+4} + \frac{24}{x^2+4}\,dx\end{aligned}$$

$$\begin{aligned}\int_{2\sqrt{3}}^{\infty}\frac{12}{(x-2)^2}\,dx &= 12\lim_{t\to\infty}\int_{2\sqrt{3}}^{t}(x-2)^{-2}\,dx \\ &= 12\lim_{t\to\infty}\left[-(x-2)^{-1}\right]_{2\sqrt{3}}^{t} \\ &= 12\lim_{t\to\infty}\left[-\frac{1}{t-2} + \frac{1}{2\sqrt{3}-2}\right] \\ &= \frac{6}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1} \\ &= \frac{6(\sqrt{3}+1)}{3-1} \\ &= 3\sqrt{3}+3\end{aligned}$$

$$\int_{2\sqrt{3}}^{\infty} \frac{18x}{x^2 + 4} - \frac{18}{x-2}\, dx = \underset{t \to \infty}{limit} \int_{2\sqrt{3}}^{t} 9 \times \frac{2x}{x^2 + 4} - 18 \times \frac{1}{x-2}\, dx$$

$$= \underset{t \to \infty}{limit} \left[9\, ln\left(x^2 + 4 \right) - 18\, ln\left(x - 2 \right) \right]_{2\sqrt{3}}^{t}$$

$$= 9 \underset{t \to \infty}{limit} \left[ln\left(\frac{x^2 + 4}{(x-2)^2} \right) \right]_{2\sqrt{3}}^{t}$$

$$= 9 \underset{t \to \infty}{limit} \left[ln\left(\frac{t^2 + 4}{(t-2)^2} \right) - ln\left(\frac{16}{4(\sqrt{3} - 1)^2} \right) \right]$$

$$= 9 \left[ln\, 1 - ln\left(\frac{4}{4 - 2\sqrt{3}} \right) \right]$$

$$= 9 \left[0 - ln\left(\frac{2}{2 - \sqrt{3}} \right) \right]$$

$$= 9\, ln\left(\frac{2 - \sqrt{3}}{2} \right)$$

$$= 9\, ln\left(1 - \frac{\sqrt{3}}{2} \right)$$

Let $x = 2\, tan\, \theta$ in which case $\frac{dx}{d\theta} = 2\, sec^2\, \theta$

The limits $2\sqrt{3},\ \infty$ become $\frac{\pi}{3},\ \frac{\pi}{2}$

$$\int_{2\sqrt{3}}^{\infty} \frac{24}{x^2 + 4}\, dx = 6 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{4}{4\, tan^2\, \theta + 4}\, 2\, sec^2\, \theta\, d\theta$$

$$= 12 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{sec^2\, \theta}\, sec^2\, \theta\, d\theta$$

$$= 12 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1\, d\theta$$

$$= 12 \left[\theta \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= 12 \left[\frac{\pi}{2} - \frac{\pi}{3} \right] = 2\pi$$

Thus $96 \int_{2\sqrt{3}}^{\infty} \frac{3 - x}{(x-2)^2 (x^2 + 4)}\, dx = 2\pi + 3 + 3\sqrt{3} + 9\, ln\left(1 - \frac{\sqrt{3}}{2} \right)$ □

[12 marks]