5.4 Answers

5.4.1 Solution to Example 5.2 (Teaching Video)

$$\begin{aligned} Volume &= \pi \int_a^b x^2\, dy \\ &= \pi \int_2^{\infty} \frac{1}{y\,(\ln y)^2}\, dy \\ &= \pi \lim_{t \to \infty} \int_2^t \left(\frac{1}{y}\right)(\ln y)^{-2}\, dy \end{aligned}$$

Notice that this is of the form $f'(y)\,[\,f(y)\,]^n$ which is a “chain rule backwards”

$$\begin{aligned} &= \pi \lim_{t \to \infty} \Big[-(\ln y)^{-1} \Big]_2^t \\ &= \pi \lim_{t \to \infty} \left[-\frac{1}{\ln t} + \frac{1}{\ln 2} \right] \\ &= \frac{\pi}{\ln 2} \end{aligned}$$

[5 marks]

5.4.2 Solutions to Exercise 5.3

Answer 1

$$\begin{aligned} Volume &= \pi \int_a^b x^2\, dy \\ &= \pi \int_1^{\infty} \frac{(y+1)^2}{y^6}\, dy \\ &= \pi \lim_{t \to \infty} \int_1^t \frac{y^2 + 2y + 1}{y^6}\, dy \\ &= \pi \lim_{t \to \infty} \int_1^t y^{-4} + 2y^{-5} + y^{-6}\, dy \\ &= \pi \lim_{t \to \infty} \left[-\frac{1}{3y^3} - \frac{2}{4y^4} - \frac{1}{5y^5} \right]_1^t \\ &= \pi \lim_{t \to \infty} \left[-\frac{1}{3t^3} - \frac{1}{2t^4} - \frac{1}{5t^5} + \frac{1}{3} + \frac{1}{2} + \frac{1}{5} \right] \\ &= \frac{31\pi}{30} \end{aligned}$$

[5 marks]

Answer 2

$$Volume = \pi \int_a^b y^2\, dx$$
$$= \pi \int_0^\infty \frac{x}{(1 + 4x^2)^2}\, dx$$
$$= \frac{\pi}{8} \underset{t \to \infty}{limit} \int_0^t 8x \left(1 + 4x^2\right)^{-2}\, dx$$

Notice that this is of the form $f'(x)[\, f(x)\,]^n$ which is a “chain rule backwards”

$$= \frac{\pi}{8} \underset{t \to \infty}{limit} \left[- \frac{1}{1 + 4x^2} \right]_0^t$$
$$= \frac{\pi}{8} \left[- \frac{1}{1 + 4t^2} + 1 \right]$$
$$= \frac{\pi}{8}$$

[5 marks]

Answer 3

$$Volume = \pi \int_a^b y^2\, dx$$
$$= \pi \int x^2 e^{-2x}\, dx$$

$$\begin{array}{cccc} \boldsymbol{L} & \boldsymbol{I} & \boldsymbol{D} & \boldsymbol{I} \end{array}$$
$$= \pi \left[x^2 \left(-\frac{1}{2}\right) e^{-2x} - \int 2x \left(-\frac{1}{2}\right) e^{-2x}\, dx \right]$$
$$= \pi \left[-\frac{1}{2} x^2 e^{-2x} + \int x\, e^{-2x}\, dx \right]$$

$$\begin{array}{cccc} \boldsymbol{L} & \boldsymbol{I} & \boldsymbol{D} & \boldsymbol{I} \end{array}$$
$$= \pi \left[-\frac{1}{2} x^2 e^{-2x} + x\left(-\frac{1}{2}\right) e^{-2x} - \int 1 \left(-\frac{1}{2}\right) e^{-2x}\, dx \right]$$
$$= \frac{\pi}{2} \left[-x^2 e^{-2x} - x e^{-2x} + \int e^{-2x}\, dx \right]$$
$$= \frac{\pi}{2} \left[-x^2 e^{-2x} - x e^{-2x} - \frac{1}{2} e^{-2x} + c \right]$$

$$\therefore\ \pi \int_0^\infty x^2 e^{-2x}\, dx = \frac{\pi}{2} \underset{t \to \infty}{limit} \left[-t^2 e^{-2t} - t e^{-2t} - \frac{1}{2} e^{-2t} + \frac{1}{2} \right]_0^t$$
$$= \frac{\pi}{4}$$

[7 marks]

Use L'Hôspital's Rule to prove that $\underset{t \to \infty}{limit}\ t^n e^{-2t} = 0$ for $n \in \mathbb{Z}^+$

Answer 4

(i) Let $x = 3\,tan\,\theta$ in which case $\dfrac{dx}{d\theta} = 3\,sec^2\,\theta$

Making the substitution,

$$\begin{aligned}
\int \frac{1}{x^2 + 9}\,dx &= \int \frac{1}{9\,tan^2\,x + 9}\,3\,sec^2\,\theta\,d\theta \\
&= \frac{3}{9}\int \frac{1}{tan^2\,\theta + 1}\,sec^2\,\theta\,d\theta \\
&= \frac{1}{3}\int \frac{1}{sec^2\,\theta}\,sec^2\,\theta\,d\theta \\
&= \frac{1}{3}\int 1\,d\theta \\
&= \frac{1}{3}\,\theta + c \\
&= \frac{1}{3}\,arctan\left(\frac{x}{3}\right) + c
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^{\infty} \frac{1}{x^2 + 9}\,dx &= \frac{1}{3}\,\underset{t \to \infty}{limit}\left[\,arctan\left(\frac{x}{3}\right)\right]_0^t \\
&= \frac{1}{3}\,\underset{t \to \infty}{limit}\left[\,arctan\left(\frac{t}{3}\right) - arctan\left(\frac{0}{3}\right)\right] \\
&= \frac{1}{3}\left[\frac{\pi}{2} - 0\right] \\
&= \frac{\pi}{6}
\end{aligned}$$

[4 marks]

(ii)

$$\int \frac{1}{x^2 + 9}\,dx = \int \left(\frac{1}{x^2 + 9}\right)(\,1\,)\,dx$$

L ***I*** ***D*** ***I***

$$\begin{aligned}
&= \left(\frac{1}{x^2 + 9}\right)x - \int (-1)\left(x^2 + 9\right)^{-2} 2x \times x\,dx \\
&= \left(\frac{x}{x^2 + 9}\right) + 2\int \frac{x^2 + 9 - 9}{(\,x^2 + 9\,)^2}\,dx \\
&= \left(\frac{x}{x^2 + 9}\right) + 2\int \frac{x^2 + 9}{(\,x^2 + 9\,)^2}\,dx - 2\int \frac{9}{(\,x^2 + 9\,)^2}\,dx
\end{aligned}$$

$$\int \frac{1}{x^2 + 9}\,dx = \frac{x}{x^2 + 9} + 2\int \frac{1}{x^2 + 9}\,dx - 18\int \frac{1}{(\,x^2 + 9\,)^2}\,dx \qquad \square$$

[4 marks]

(iii)

$$Volume = \pi \int_a^b y^2 \, dx$$

$$= \pi \int_0^{\infty} \frac{1}{(x^2 + 9)^2} \, dx$$

$$= \frac{\pi}{18} \lim_{t \to \infty} \left[\frac{x}{x^2 + 9} \right]_0^t + \frac{\pi}{18} \lim_{t \to \infty} \int_0^t \frac{1}{x^2 + 9} \, dx \qquad \text{from part (ii)}$$

$$= \frac{\pi}{18} \lim_{t \to \infty} \left[\frac{t}{t^2 + 9} - 0 \right] + \frac{\pi}{18} \times \frac{\pi}{6} \qquad \text{from part (i)}$$

$$= \frac{\pi^2}{108}$$

[4 marks]

Answer 5

Using Partial Fractions,

$$\frac{4y + 1}{[\, y^2 (y + 1)^2 \,]} = \frac{Ay + B}{y^2} + \frac{Cy + D}{(y + 1)^2}$$

Multiplying through by the red bracket

$$\frac{(4y + 1)[\, \ldots \ldots \,]}{[\, y^2 (y + 1)^2 \,]} = \frac{(Ay + B)[\, \ldots \ldots \,]}{y^2} + \frac{(Cy + D)\, [\, \ldots \ldots \,]}{(y + 1)^2}$$

$$4y + 1 = (Ay + B)(y + 1)^2 + (Cy + D)y^2$$

$$\text{Let } y = 0 \Rightarrow 1 = B$$

$$\text{Let } y = -1 \Rightarrow -3 = (-C + D)(1) \Rightarrow D = C - 3$$

$$\text{Coefficients of } y^3 : 0 = A + C$$

$$\text{Let } y = 1 \Rightarrow 5 = (A + B)\, 4 + (C + D)$$

$$5 = -4C + 4 + (C + C - 3)$$

$$4 = -2C$$

$$C = -2$$

$$\therefore A = 2, \; B = 1, \; C = -2 \text{ and } D = -5$$

$$\therefore \frac{4y + 1}{y^2 (y + 1)^2} = \frac{2y + 1}{y^2} - \frac{2y + 5}{(y + 1)^2}$$

$$= \frac{2}{y} + \frac{1}{y^2} - \frac{2y + 2}{y^2 + 2y + 1} - 3(1)(y + 1)^{-2}$$

$$\pi \int_1^{\infty} \frac{4y + 1}{y^2 (y + 1)^2}\, dy = \pi \lim_{t \to \infty} \int_1^t \frac{2}{y} + \frac{1}{y^2} - \frac{2y + 2}{y^2 + 2y + 1} - 3(1)(y + 1)^{-2}\, dy$$

$$= \pi \lim_{t \to \infty} \left[2 \ln y - \frac{1}{y} - \ln\left(y^2 + 2y + 1 \right) + \frac{3}{y + 1} \right]_1^t$$

$$= \pi \lim_{t \to \infty} \left[\ln\left(\frac{y^2}{y^2 + 2y + 1} \right) - \frac{1}{y} + \frac{3}{y + 1} \right]_1^t$$

$$= \pi \lim_{t \to \infty} \left[\ln\left(\frac{t^2}{t^2 + 2t + 1} \right) - \frac{1}{t} + \frac{3}{t + 1} - \ln\left(\frac{1}{4} \right) + 1 - \frac{3}{2} \right]$$

$$= \pi \left[\ln 1 - 0 + 0 - \ln\left(\frac{1}{4} \right) + 1 - \frac{3}{2} \right]$$

$$= \pi \left[-\ln 4^{-1} - \frac{1}{2} \right]$$

$$= \pi \left(\ln 4 - \frac{1}{2} \right)$$

$$= \pi \left(2 \ln 2 - \frac{1}{2} \right)$$

[11 marks]