6.2 Answers

Answer 1

$$\int_1^3 \frac{x - 1}{\sqrt{3 + 2x - x^2}} dx = \left(-\frac{1}{2}\right) \underset{t \to 3^-}{limit} \int_1^t \frac{2 - 2x}{\sqrt{3 + 2x - x^2}} dx$$

$$= \left(-\frac{1}{2}\right) \underset{t \to 3^-}{limit} \int_1^t (2 - 2x)(3 + 2x - x^2)^{-\frac{1}{2}} dx$$

Notice that this is of the form $f'(x)[f(x)]^n$ which is a “chain rule backwards”.

$$= \left(-\frac{1}{2}\right) \underset{t \to 3^-}{limit} \left[2(3 + 2x - x^2)^{\frac{1}{2}} \right]_1^t$$

$$= (-1) \underset{t \to 3^-}{limit} \left[\sqrt{3 + 2t - t^2} - \sqrt{3 + 2 - 1} \right]$$

$$= (-1)\left[0 - \sqrt{4} \right]$$

$$= 2$$

[6 marks]

Answer 2

Let $x = \frac{\sqrt{3}}{2} sin\,\theta$ in which case $\frac{dx}{d\theta} = \frac{\sqrt{3}}{2} cos\,\theta$

$$\int \frac{1}{\sqrt{3 - 4x^2}} dx = \int \frac{1}{\sqrt{3 - 3 sin^2\theta}} \times \frac{\sqrt{3}}{2} cos\,\theta\ d\theta$$

$$= \int \frac{1}{\sqrt{3(1 - sin^2\theta)}} \times \frac{\sqrt{3}}{2} cos\,\theta\ d\theta$$

$$= \int \frac{1}{\sqrt{3(cos^2\theta)}} \times \frac{\sqrt{3}}{2} cos\,\theta\ d\theta$$

$$= \frac{1}{2} \int 1\, d\theta$$

$$= \frac{1}{2}\theta + c$$

$$= \frac{1}{2} arcsin\left(\frac{2x}{\sqrt{3}}\right) + c$$

$$\int_0^{\frac{3}{4}} \frac{1}{\sqrt{3 - 4x^2}} dx = \frac{1}{2}\left[arcsin\left(\frac{2x}{\sqrt{3}}\right) \right]_0^{\frac{3}{4}}$$

$$= \frac{1}{2}\left[arcsin\left(\frac{\sqrt{3}}{2}\right) - arcsin(0) \right]$$

$$= \frac{\pi}{6}$$

[6 marks]

Answer 3

(i) Making use of Partial Fractions,

$$\frac{1}{[\,x(x+5)\,]} = \frac{A}{x} + \frac{B}{x+5}$$

Multiply through by the red bracket

$$\frac{1\,[\,\dots\dots\,]}{[\,x(x+5)\,]} = \frac{A\,[\,\dots\dots\,]}{x} + \frac{B\,[\,\dots\dots\,]}{x+5}$$

$$1 = A(x+5) + Bx$$

$$\text{Let } x = 0 \Rightarrow A = \frac{1}{5}$$

$$\text{Let } x = -5 \Rightarrow B = -\frac{1}{5}$$

$$\int \frac{1}{x(x+5)}\,dx = \frac{1}{5}\int \frac{1}{x} - \frac{1}{x+5}\,dx$$

$$= \frac{1}{5}[\,\ln x - \ln(x+5)\,]$$

$$= \frac{1}{5}\ln\left(\frac{x}{x+5}\right) + c$$

[4 marks]

(ii)

$$\int_5^{\infty} \frac{1}{x^2+5x}\,dx = \frac{1}{5}\lim_{t\to\infty}\left[\ln\left(\frac{x}{x+5}\right)\right]_5^t$$

$$= \frac{1}{5}\lim_{t\to\infty}\left[\ln\left(\frac{t}{t+5}\right) - \ln\left(\frac{5}{5+5}\right)\right]$$

$$= \frac{1}{5}\left[\ln 1 - \ln\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{5}\ln 2$$

That is, the integral converges and has a value of $\frac{1}{5}\ln 2$

[3 marks]

Answer 4

$$Volume \;=\; \pi \int_a^b x^2 \, dy$$

$$= \;\pi \int_0^{\infty} \frac{y^2}{(\,1 \,+\, 5y^3\,)^2} \; dy$$

$$= \;\frac{\pi}{15} \underset{t \to \infty}{limit} \int_0^t 15\,y^2 \left(\,1 \,+\, 5\,y^3\right)^{-2} \, dy$$

Notice that this is of the form $f'(x)\,[\,f(x)\,]^n$ which is a "chain rule backwards"

$$= \;\frac{\pi}{15} \underset{t \to \infty}{limit} \left[-\left(\,1 \,+\, 5\,y^3\right)^{-1}\,\right]_0^t$$

$$= \;\frac{\pi}{15} \underset{t \to \infty}{limit} \left[-\frac{1}{1 \,+\, 5\,t^3} \,+\, \frac{1}{1}\right]$$

$$= \;\frac{\pi}{15}$$

[6 marks]

Answer 5

(i)

$$a \tan y \;=\; x$$

Differentiate implicitly with respect to *x*, as instructed

$$a \sec^2 y \,\frac{dy}{dx} \;=\; 1$$

$$\frac{dy}{dx} \;=\; \frac{1}{a \sec^2 y} \qquad \text{use } 1 \,+\, \tan^2 y \;=\; \sec^2 y$$

$$= \;\frac{1}{a \,+\, a\tan^2 y} \times \frac{a}{a}$$

$$= \;\frac{a}{a^2 \,+\, (\,a \tan y\,)^2} \qquad \text{use } a \tan y \;=\; x$$

$$= \;\frac{a}{a^2 \,+\, x^2}$$

$\therefore$ If $y \;=\; \arctan\left(\dfrac{x}{a}\right)$ then $\dfrac{dy}{dx} \;=\; \dfrac{a}{a^2 \,+\, x^2}$ $\square$

[3 marks]

(ii) From part (i),

$$\int \frac{a}{a^2 + x^2}\, dx = arctan\left(\frac{x}{a} \right) + c$$

$$\int \frac{1}{x^2 - 4x + 8}\, dx = \int \frac{1}{(x-2)^2 + 4}\, dx$$

$$= \frac{1}{2} \int \frac{2}{2^2 + (x-2)^2}\, dx$$

$$= \frac{1}{2}\, arctan\left(\frac{x-2}{2} \right) + c$$

$$\int_0^4 \frac{1}{x^2 - 4x + 8}\, dx = \frac{1}{2} \left[arctan\left(\frac{x-2}{2} \right) \right]_0^4$$

$$= \frac{1}{2} \left[arctan\, 1 - arctan(-1) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right]$$

$$= \frac{\pi}{4}$$

[4 marks]

(iii)

$$\int arctan\, x\, dx = \int arctan\, x\ (1)\ dx$$

L ***I*** ***D*** ***I***

$$= arctan\, x\ (x) - \int \frac{1}{1 + x^2}\ (x)\ dx$$

$$= x\, arctan\, x - \frac{1}{2} \int 2x \left(1 + x^2 \right)^{-1} dx$$

Notice that this is of the form $f'(x)[\, f(x)\,]^n$ which is a “chain rule backwards”

$$= x\, arctan\, x - \frac{1}{2}\, ln\left(1 + x^2 \right) + c$$

[4 marks]

(iv)

$$\int_0^\infty arctan\, x\, dx = \lim_{t \to \infty} \left[x\, arctan\, x - \frac{1}{2}\, ln\left(1 + x^2 \right) \right]_0^t$$

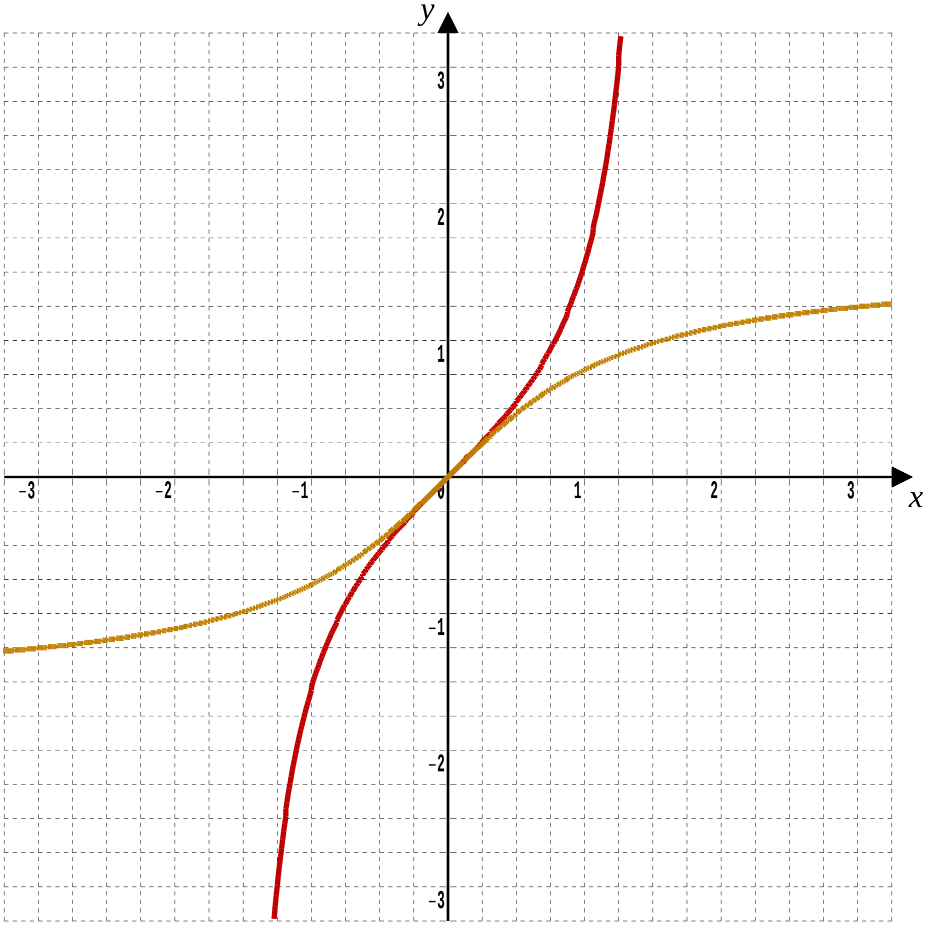
$$= \lim_{t \to \infty} \left[t\, arctan\, t - \frac{1}{2}\, ln\left(1 + t^2 \right) - 0 - 0 \right]$$

As $t \to \infty$, $t\, arctan\, t \to \infty$ and $ln\left(1 + t^2 \right) \to \infty$

Thus this improper integral is not convergent and so has no value

[2 marks]

(v)



Red : $y = \tan x$ Gold : $y = \arctan x$

The graph of $\arctan x \rightarrow \frac{\pi}{2}$ as $x \rightarrow \infty$ with the result that the area given by $\int_0^t \arctan x \, dx$ will be roughly the same as that of a rectangle measuring $\left(\frac{\pi}{2}\right)$ by t for large values of t

Thus $Area \rightarrow \infty$ as $t \rightarrow \infty$ further endorsing the fact that the integral is divergent.

[2 marks]