Answer 1

Let $x = \frac{1}{2} \tan\theta$ in which case $\frac{dx}{d\theta} = \frac{1}{2} \sec^2\theta$

$$\int \frac{1}{(1 + 4x^2)^{\frac{3}{2}}}\,dx = \int \frac{1}{(1 + \tan^2\theta)^{\frac{3}{2}}} \times \frac{1}{2} \sec^2\theta\;d\theta$$

$$= \int \frac{1}{(\sec^2\theta)^{\frac{3}{2}}} \times \frac{1}{2} \sec^2\theta\;d\theta$$

$$= \frac{1}{2} \int \frac{1}{\sec^3\theta} \times \sec^2\theta\,d\theta$$

$$= \frac{1}{2} \int \cos\theta\,d\theta$$

$$= \frac{1}{2} \sin\theta + c$$

$$= \frac{1}{2} \sin(\arctan(2x)) + c$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(1 + 4x^2)^{\frac{3}{2}}}\,dx = \frac{1}{2}\left[\sin(\arctan(2x))\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[\sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right]$$

$$= \frac{\sqrt{2}}{2}$$

[6 marks]

Answer 2

$$Volume = \pi \int_a^b x^2 \, dy$$

$$= \pi \int_0^{\infty} y\, e^{-2y} \, dy$$

$$= \pi \lim_{t \to \infty} \int_0^t y\, e^{-2y} \, dy$$

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$$= \pi \lim_{t \to \infty} \left[y \left(-\frac{1}{2}\right) e^{-2y} - \int 1 \left(-\frac{1}{2}\right) e^{-2y} \, dy \right]_0^t$$

$$= \frac{\pi}{2} \lim_{t \to \infty} \left[-y\, e^{-2y} + \int e^{-2y} \, dy \right]_0^t$$

$$= \frac{\pi}{2} \lim_{t \to \infty} \left[-y\, e^{-2y} + \left(-\frac{1}{2}\right) e^{-2y} \right]_0^t$$

$$= \frac{\pi}{2} \lim_{t \to \infty} \left[-t\, e^{-2t} - \frac{1}{2} e^{-2t} + 0 + \frac{1}{2} \right]$$

$$= \frac{\pi}{4}$$

[6 marks]

Answer 3

Partial fractions;

$$\frac{8x - 12}{[\,(2x^2+3)(x+1)\,]} = \frac{Ax+B}{2x^2+3} + \frac{C}{x+1}$$

Multiply through by the red bracket

$$\frac{(8x-12)[\,\dots\dots\,]}{[\,(2x^2+3)(x+1)\,]} = \frac{(Ax+B)\,[\,\dots\dots\,]}{2x^2+3} + \frac{C\,[\,\dots\dots\,]}{x+1}$$

$$8x - 12 = \left(Ax+B\right)\left(x+1\right) + C\left(2x^2+3\right)$$

$$\text{Let } x = -1 \Rightarrow C = -4$$

$$\text{Let } x = 0 \Rightarrow B = 0$$

$$\text{Let } x = 1 \Rightarrow A = 8$$

$$\int \frac{8x-12}{(2x^2+3)(x+1)}\,dx = \int \frac{8x}{2x^2+3} - \frac{4}{x+1}\,dx$$

$$= 2\int 4x\left(2x^2+3\right)^{-1} - \frac{4}{x+1}\,dx$$

$$= 2\,ln\left(2x^2+3\right) - 4\,ln(x+1) + c$$

$$= 2\left[\,ln\left(2x^2+3\right) - ln\,(x+1)^2\,\right] + c$$

$$= 2\,ln\left(\frac{2x^2+3}{(x+1)^2}\right) + c$$

$$\int_0^\infty \frac{8x-12}{(2x^2+3)(x+1)}\,dx = 2\lim_{t\to\infty}\left[\,ln\left(\frac{2x^2+3}{(x+1)^2}\right)\right]_0^t$$

$$= 2\lim_{t\to\infty}\left[\,ln\left(\frac{2t^2+3}{(t+1)^2}\right) - ln\left(\frac{3}{1}\right)\right]$$

$$= 2\left[\,ln\,2 - ln\,3\right]$$

$$= 2\,ln\left(\frac{2}{3}\right)$$

$$= ln\left(\frac{4}{9}\right)$$

$$\text{That is, } k = \frac{4}{9}$$

[7 marks]

Answer 4

(i)

$$y = arctan\sqrt{x}$$

$$tan\, y = \sqrt{x}$$

$$(tan\, y)^2 = x$$

Differentiate implicitly with respect to x

$$2\, tan\, y\, sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2\, tan\, y\, sec^2 y}$$

$$= \frac{1}{2\, tan\, y(1 + tan^2 y)}$$

$$= \frac{1}{2\sqrt{x}\,(1 + x)}$$

[3 marks]

(ii) From part (i),

$$\int_0^1 \frac{1}{\sqrt{x}\,(x + 1)}\, dx = 2\int_0^1 \frac{1}{2\sqrt{x}\,(x + 1)}\, dx$$

$$= 2\Big[arctan\sqrt{x}\, \Big]_0^1$$

$$= 2\,\Big[arctan\sqrt{1} - arctan\, 0 \Big]$$

$$= 2 \times \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$

[3 marks]

Answer 5

Let $x^2 = sec\,\theta$, and differentiate implicitly with respect to θ to get,

$$2\,x \frac{dx}{d\theta} = sec\,\theta\; tan\,\theta$$

$$\frac{dx}{d\theta} = \frac{sec\,\theta\; tan\,\theta}{2\sqrt{sec\,\theta}}$$

$$= \frac{1}{2}\sqrt{sec\,\theta}\; tan\,\theta$$

For the lower limit $\quad x^2 = sec\,\theta$

$$1 = \frac{1}{cos\,\theta}$$

$$cos\,\theta = 1$$

$$\theta = 0$$

For the upper limit $\quad x^2 = sec\,\theta$

$$(\sqrt{2})^2 = \frac{1}{cos\,\theta}$$

$$cos\,\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

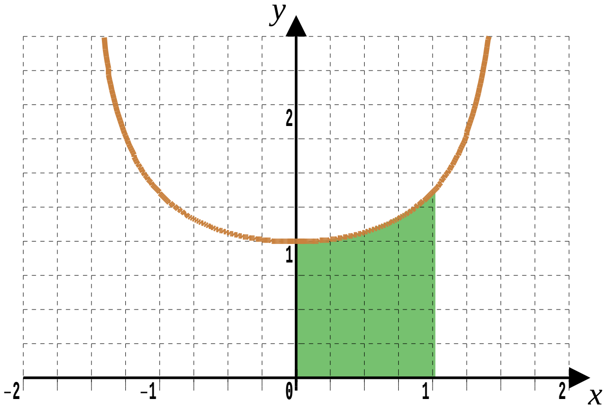
Making the substitution,

$$\int_1^{\sqrt{2}} \frac{1}{\sqrt{x^4 - 1}}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{sec^2\,\theta - 1}} \times \frac{1}{2}\sqrt{sec\,\theta}\,tan\,\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{tan^2\,\theta}} \times \frac{1}{2}\sqrt{sec\,\theta}\,tan\,\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{cos\,\theta}}\,d\theta$$

This integral is a proper one and could be interpreted as finding the finite area shaded green under the curve $y = \frac{1}{\sqrt{cos\,x}}$



Thus the original improper integral $\int_1^{\sqrt{2}} \frac{1}{\sqrt{x^4 - 1}}\,dx$ must be convergent $\square$

[8 marks]