Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

IMPROPER INTEGRALS

A three dimensional Möbius sculpture in solid aluminium
created by Sarang Sheth

IMPROPER INTEGRALS

Lesson 1

Further A-Level Pure Mathematics, Core 2
Improper Integrals

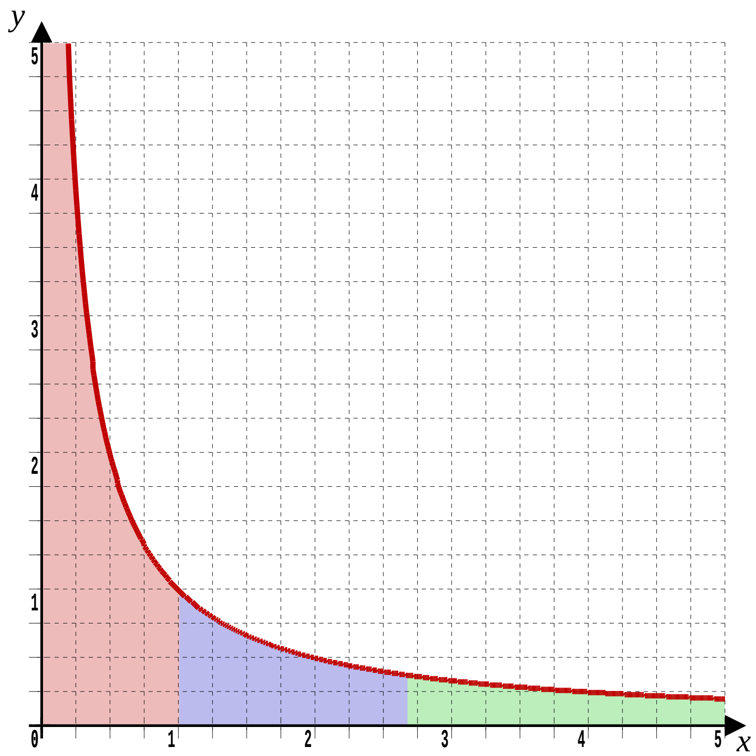
1.1 Divergent

Previously, our studies of integration have focussed on smooth, continuous and well behaved functions. It's time to tackle some of the issues that arise when trying to ease such restrictions and look at what are termed “Improper Integrals”.

As an example, consider the function graphed below, $y = \dfrac{1}{x}, \ x \neq 0$

We're going to consider the following three integrals,

(i) $\displaystyle\int_1^e \frac{1}{x}\,dx$ **(ii)** $\displaystyle\int_0^1 \frac{1}{x}\,dx$ **(iii)** $\displaystyle\int_e^\infty \frac{1}{x}\,dx$



Before watching the teaching video, consider the following;

Part (i) of this example is a straightforward calculation of the area coloured blue.

Part (ii) has an issue with the vertical asymptote along the y-axis.
Is the area described by the integral (a piece is shown coloured pink) of finite value ?

Part (iii) has an issue with the horizontal asymptote along the x-axis.
Is the area described by the integral (a piece is shown coloured green) of finite value ?

Teaching Video: http://www.NumberWonder.co.uk/v9100/1.mp4

[6 marks]

A metal enamelled infinity broach, of a snake eating itself

1.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 60

Question 1

(i) Show that $\int_0^1 \frac{1}{x^2}\,dx$ is divergent.

Make sure you show the limiting process clearly in your working.

[4 marks]

(ii) Show that $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent and find its value.

Make sure you show the limiting process clearly in your working.

[4 marks]

Question 2

(i) Show that $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ is convergent and find its value.

Make sure you show the limiting process clearly in your working.

[4 marks]

(ii) Show that $\int_1^\infty \frac{1}{\sqrt{x}}\, dx$ is divergent.

Make sure you show the limiting process clearly in your working.

[4 marks]

Question 3

(i) Given that $y = \sqrt{1 + x^2}$ find $\dfrac{dy}{dx}$

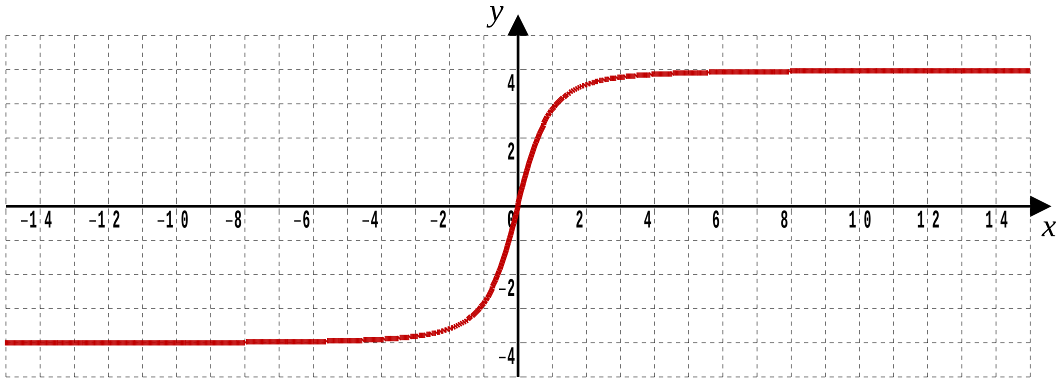
[2 marks]

(ii) Hence, or otherwise, show that $\displaystyle\int_0^\infty \frac{4x}{\sqrt{1 + x^2}}\, dx$ is divergent.

Make sure you show the limiting process clearly in your working.

[4 marks]

(iii) The graph of $f(x) = \dfrac{4x}{\sqrt{1 + x^2}}$ is shown below.



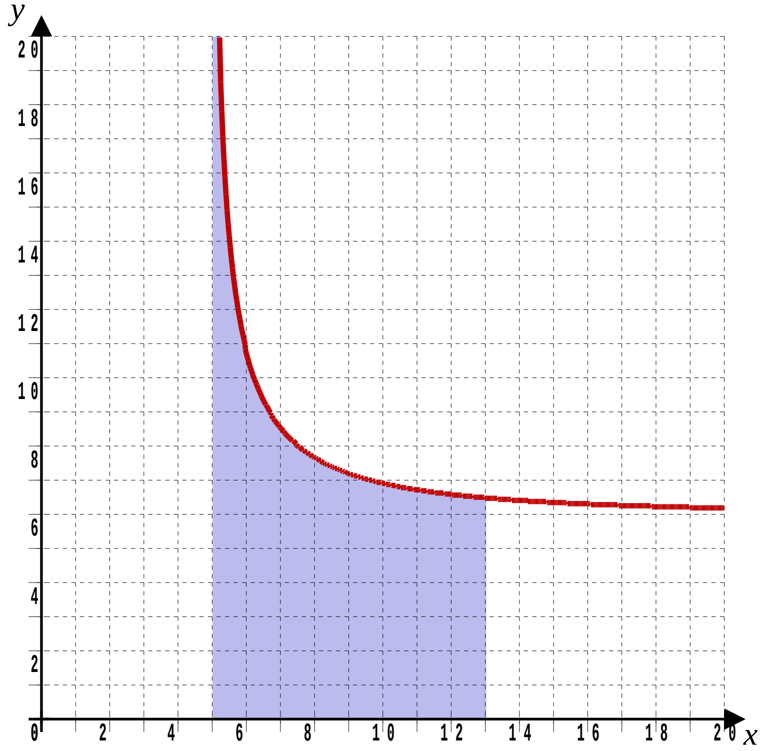
What features of the graph suggest that the part (ii) integral would be divergent ?

[2 marks]

Question 4

The graph is of the function,

$$f(x) = \frac{6x}{\sqrt{x^2 - 25}} \qquad x \neq \pm 5$$



Show that $\int_5^{13} \frac{6x}{\sqrt{x^2 - 25}}\, dx$ converges and find its value.

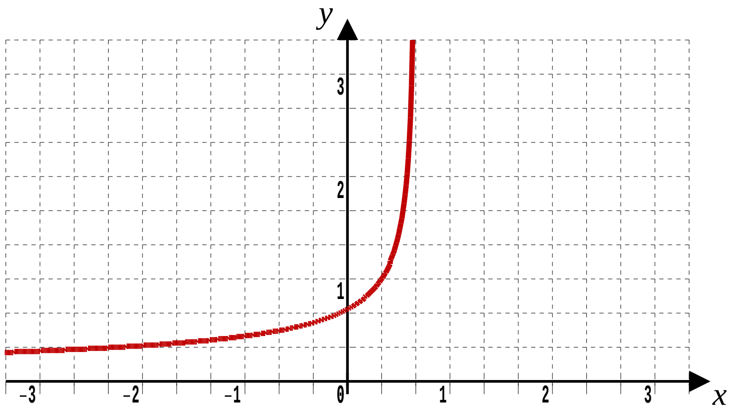
[6 marks]

Question 5

The graph is of the function $f(x) = \dfrac{1}{\sqrt{2 - 3x}}$ $\quad x \neq \dfrac{2}{3}$

Integration is to be used to find the area between this curve, the x-axis, the y-axis and the vertical asymptote $x = \dfrac{2}{3}$

(i) On the graph shade in the area to be found.



[1 mark]

(ii) Explain why the integral involved is an “improper integral”.

[1 mark]

(iii) Show that $\displaystyle\int_0^{\frac{2}{3}} \frac{1}{\sqrt{2 - 3x}}\, dx$ converges and find its value.

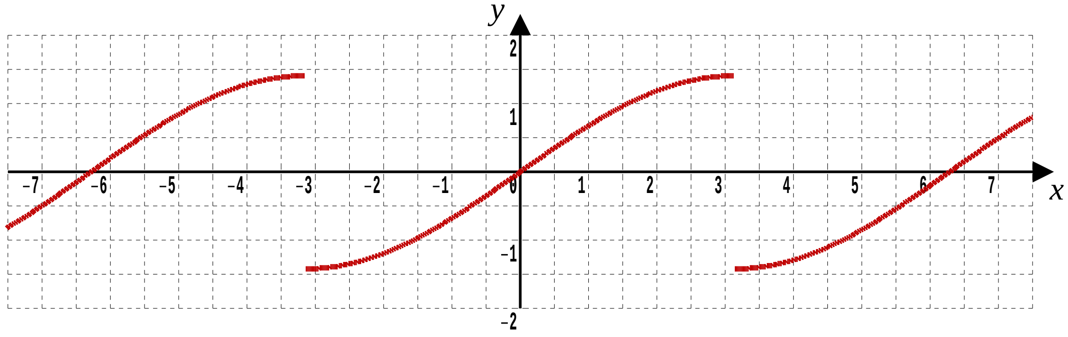
[6 marks]

Question 6

The graph is of the function $f(x) = \dfrac{\sin x}{\sqrt{1 + \cos x}}$ $\quad x \neq (2n + 1)\pi$

Integration is to be used to find the area between this curve, the x-axis, the y-axis and the line $x = \pi$

(i) On the graph shade in the area to be found.



[1 mark]

(ii) Explain why the integral involved is an "improper integral".

[1 mark]

(iii) Show that $\displaystyle\int_0^{\pi} \frac{\sin x}{\sqrt{1 + \cos x}}\, dx$ converges and find its value.

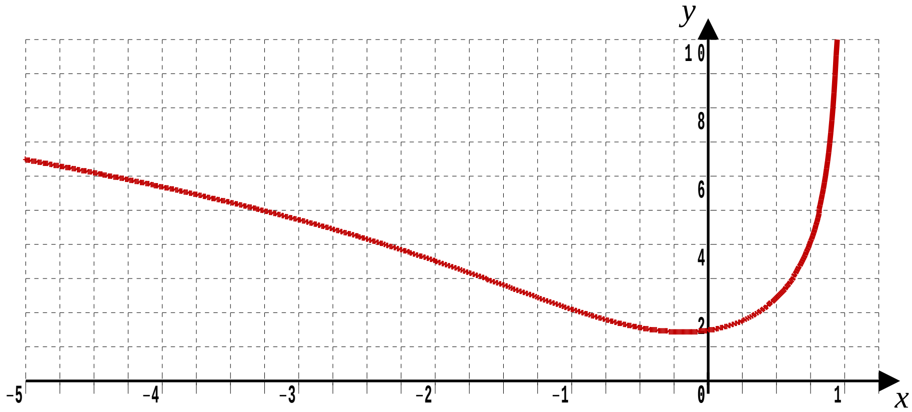
[6 marks]

Question 7

The graph is of the function $f(x) = \dfrac{3(x^2 + 1)}{\sqrt{4 - 3x - x^3}} \qquad x \neq 1$

Integration is to be used to find the area between this curve, the x-axis, the y-axis and the vertical asymptote at $x = 1$

(i) On the graph shade in the area to be found.



[1 mark]

(ii) Explain why the integral involved is an "improper integral".

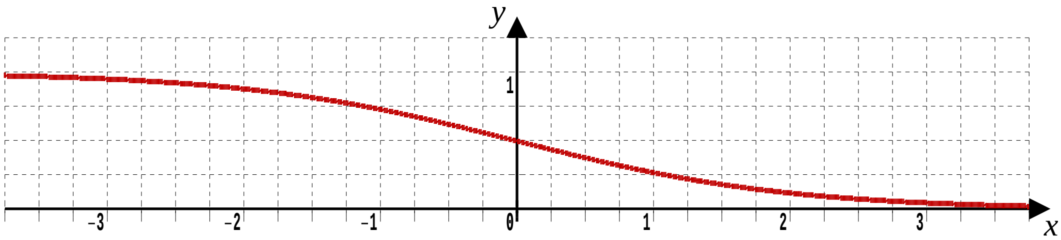
[1 mark]

(iii) Show that $\displaystyle\int_0^1 \frac{3(x^2 + 1)}{\sqrt{4 - 3x - x^3}}\,dx$ converges and find its value.

[6 marks]

Question 8

The graph is of the function $f(x) = 1 - \dfrac{1}{e^{-x} + 1}$



If the following integral is convergent find its value, otherwise prove it's divergent,

$$\int_0^{\infty} \left(1 - \frac{1}{e^{-x} + 1}\right) dx$$

[6 marks]

1.3 Answers

1.3.1 Solution to 1.1 Example (Teaching Video)

(i) $$\int_1^e \frac{1}{x}\,dx = \left[\, ln\, x \,\right]_1^e$$

$$= ln\, e - ln\, 1$$

$$= 1 - 0$$

$$= 1$$

[2 marks]

(ii) $$\int_0^1 \frac{1}{x}\,dx = \underset{t \to 0^+}{limit} \int_t^1 \frac{1}{x}\,dx$$

$$= \underset{t \to 0^+}{limit} \left[\, ln\, x \,\right]_t^1$$

$$= \underset{t \to 0^+}{limit} \left[\, ln\, 1 - ln\, t \,\right]$$

$$= (-1) \underset{t \to 0^+}{limit} \left[\, ln\, t \,\right]$$

Although $ln\, 0$ is undefined, as $t \to 0$ from above, $ln\, t \to -\infty$

$\therefore$ The integral does not converge; it's diverging.

[2 marks]

(iii) $$\int_e^\infty \frac{1}{x}\,dx = \underset{t \to \infty}{limit} \int_e^t \frac{1}{x}\,dx$$

$$= \underset{t \to \infty}{limit} \left[\, ln\, x \,\right]_e^t$$

$$= \underset{t \to \infty}{limit} \left[\, ln\, t - ln\, e \,\right]$$

$$= \underset{t \to \infty}{limit} \left[\, ln\, t \,\right] - 1$$

As $t \to \infty$, $ln\, t \to \infty$

$\therefore$ The integral does not converge; it's diverging.

[2 marks]

1.3.2 Solutions to 1.2 Exercise

Answer 1

(i)

$$\int_0^1 \frac{1}{x^2}\,dx = \int_0^1 x^{-2}\,dx$$

$$= \underset{t \to 0^+}{limit} \int_t^1 x^{-2}\,dx$$

$$= \underset{t \to 0^+}{limit} \left[-x^{-1} \right]_t^1$$

$$= (-1) \underset{t \to 0^+}{limit} \left[\frac{1}{x} \right]_t^1$$

$$= (-1) \underset{t \to 0^+}{limit} \left[\frac{1}{1} - \frac{1}{t} \right]$$

$$\text{As } t \to 0^+, \ \frac{1}{t} \to \infty$$

$\therefore$ The integral does not converge; it's diverging. ☐

[4 marks]

(ii)

$$\int_1^\infty \frac{1}{x^2}\,dx = \int_1^\infty x^{-2}\,dx$$

$$= \underset{t \to \infty}{limit} \int_1^t x^{-2}\,dx$$

$$= \underset{t \to \infty}{limit} \left[-x^{-1} \right]_1^t$$

$$= (-1) \underset{t \to \infty}{limit} \left[\frac{1}{x} \right]_1^t$$

$$= (-1) \underset{t \to \infty}{limit} \left[\frac{1}{t} - \frac{1}{1} \right]$$

$$\text{As } t \to \infty, \ \frac{1}{t} \to 0$$

$$= (-1) \times (-1)$$

$$= 1$$

$\therefore$ The integral does converge; it has a value of 1 ☐

[4 marks]

Answer 2

(i)

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-\frac{1}{2}}\,dx$$

$$= \lim_{t \to 0^+} \int_t^1 x^{-\frac{1}{2}}\,dx$$

$$= \lim_{t \to 0^+} \left[\, 2\,x^{\frac{1}{2}} \,\right]_t^1$$

$$= 2 \lim_{t \to 0^+} \left[\sqrt{x}\, \right]_t^1$$

$$= 2 \lim_{t \to 0^+} \left[\, \sqrt{1} - \sqrt{t}\, \right]$$

As $t \to 0^+,\ \sqrt{t} \to 0$

$$= 2 \times 1$$

$$= 2$$

$\therefore$ The integral does converge; it has a value of 2 ☐

[4 marks]

(ii)

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \int_1^\infty x^{-\frac{1}{2}}\,dx$$

$$= \lim_{t \to \infty} \int_1^t x^{-\frac{1}{2}}\,dx$$

$$= \lim_{t \to \infty} \left[\, 2\,x^{\frac{1}{2}} \,\right]_1^t$$

$$= 2 \lim_{t \to \infty} \left[\sqrt{x}\, \right]_1^t$$

$$= 2 \lim_{t \to \infty} \left[\, \sqrt{t} - \sqrt{1}\, \right]$$

As $t \to \infty,\ \sqrt{t} \to \infty$

$\therefore$ The integral does not converge; it is divergent ☐

[4 marks]

Answer 3

(i)

$$y = \sqrt{1 + x^2}$$

$$y = \left(1 + x^2\right)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(1 + x^2\right)^{-\frac{1}{2}} \times 2x$$

$$= \frac{x}{\sqrt{1 + x^2}}$$

[2 marks]

(ii)

$$\int_0^{\infty} \frac{4x}{\sqrt{1 + x^2}}\,dx = 4\int_0^{\infty} \frac{x}{\sqrt{1 + x^2}}\,dx$$

$$= 4 \lim_{t \to \infty} \int_0^t \frac{x}{\sqrt{1 + x^2}}\,dx$$

$$= 4 \lim_{t \to \infty} \left[\sqrt{1 + x^2}\,\right]_0^t$$

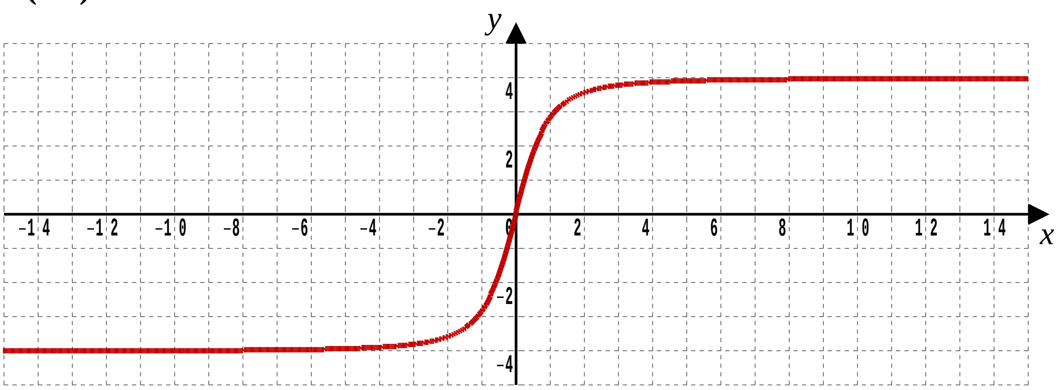
$$= 4 \lim_{t \to \infty} \left[\sqrt{1 + t^2} - \sqrt{1}\,\right]$$

As $t \to \infty,\ \sqrt{1 + t^2} \to \infty$

$\therefore$ The integral does not converge; it is divergent. ☐

[4 marks]

(iii)



The horizontal asymptote of $y = 4$ as $x \to \infty$ implies that, as the area to be found extends rightward, it resembles a rectangle of fixed width 4 and length x. The area of such a rectangle will $\to \infty$ as $x \to \infty$

[2 marks]

Answer 4

$$\int_5^{13} \frac{6x}{\sqrt{x^2 - 25}}\, dx = 3\int_5^{13} \frac{2x}{\sqrt{x^2 - 25}}\, dx$$

$$= 3 \lim_{t \to 5^+} \int_t^{13} (2x)\left(x^2 - 25\right)^{-\frac{1}{2}} dx$$

$$= 3 \lim_{t \to 5^+} \left[2\left(x^2 - 25\right)^{\frac{1}{2}} \right]_t^{13}$$

$$= 6 \lim_{t \to 5^+} \left[\sqrt{13^2 - 25} - \sqrt{t^2 - 25} \right]$$

As $t \to 5^+$, $\sqrt{t^2 - 25} \to 0$

$$= 6 \times 12$$

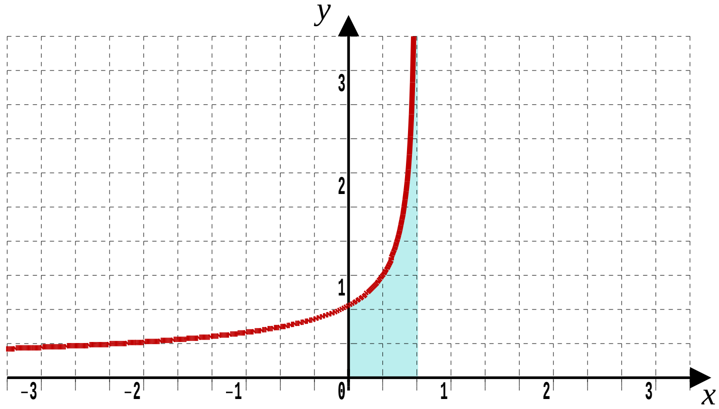
$$= 72$$

$\therefore$ The integral is convergent; it has a value of 72

[6 marks]

Answer 5

(i)



[1 mark]

(ii) The right hand boundary is of infinite height.

[1 mark]

(iii)

$$\int_0^{\frac{2}{3}} \frac{1}{\sqrt{2-3x}}\,dx = -\frac{1}{3}\int_0^{\frac{2}{3}} \frac{-3}{\sqrt{2-3x}}\,dx$$

$$= -\frac{1}{3}\lim_{t\to\left(\frac{2}{3}\right)^-}\int_0^t (-3)(2-3x)^{-\frac{1}{2}}\,dx$$

$$= -\frac{1}{3}\lim_{t\to\left(\frac{2}{3}\right)^-}\left[2(2-3x)^{\frac{1}{2}}\right]_0^t$$

$$= -\frac{2}{3}\lim_{t\to\left(\frac{2}{3}\right)^-}\left[\sqrt{2-3x}\right]_0^t$$

$$= -\frac{2}{3}\lim_{t\to\left(\frac{2}{3}\right)^-}\left[\sqrt{2-3t}-\sqrt{2}\right]$$

As $t \to \left(\frac{2}{3}\right)^-$, $\sqrt{2-3} \to 0$

$$= -\frac{2}{3}\times -\sqrt{2}$$

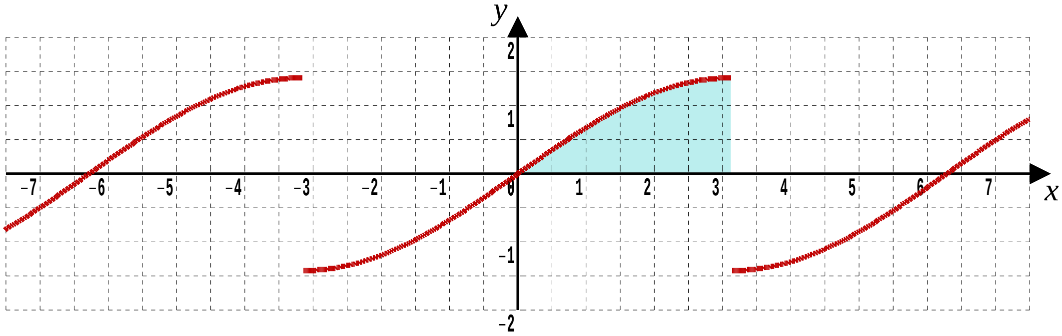
$$= \frac{2\sqrt{2}}{3}$$

$\therefore$ The integral is convergent; it has a value of $\dfrac{2\sqrt{2}}{3}$

[6 marks]

Answer 6

(i)



[1 mark]

(ii) The function has a discontinuity at its boundary $x = \pi$ where it jumps between the value of $+\sqrt{2}$ to the value $-\sqrt{2}$

[1 mark]

(iii)

$$\int_0^{\pi} \frac{\sin x}{\sqrt{1 + \cos x}}\, dx = (-1) \lim_{t \to \pi^-} \int_0^t (-\sin x)(1 + \cos x)^{-\frac{1}{2}}\, dx$$

$$= (-1) \lim_{t \to \pi^-} \left[2(1 + \cos x)^{\frac{1}{2}} \right]_0^t$$

$$= (-2) \lim_{t \to \pi^-} \left[\sqrt{1 + \cos x} \right]_0^t$$

$$= (-2) \lim_{t \to \pi^-} \left[\sqrt{1 + \cos t} - \sqrt{2} \right]$$

As $t \to \pi^-$, $\sqrt{1 + \cos t} \to 0$

$$= (-2) \times -\sqrt{2}$$

$$= 2\sqrt{2}$$

$\therefore$ The integral is convergent; it has a value of $2\sqrt{2}$

[6 marks]

Answer 7

(i)

[1 mark]

(ii) The rightmost boundary is along an asymptote and so the area specified has infinite height

[1 mark]

(iii)

$$\int_0^1 \frac{3\left(x^2+1\right)}{\sqrt{4-3x-x^3}}\,dx = (-1)\int_0^1 \frac{\left(-3x^2-3\right)}{\sqrt{4-3x-x^3}}\,dx$$

$$= (-1)\lim_{t\to 1^-}\int_0^t \left(-3-3x^2\right)\left(4-3x-x^3\right)^{-\frac{1}{2}}\,dx$$

$$= (-1)\lim_{t\to 1^-}\left[2\left(4-3x-x^3\right)^{\frac{1}{2}}\right]_0^t$$

$$= (-2)\lim_{t\to 1^-}\left[\sqrt{4-3t-t^3}-\sqrt{4}\right]$$

As $t \to 1^-$, $\sqrt{4-3t-t^3} \to 0$

$$= (-2)\times\left(-\sqrt{4}\right)$$

$$= 4$$

$\therefore$ The integral is convergent; it has a value of 4

[6 marks]

Answer 8

$$\int_0^\infty \left(1 - \frac{1}{e^{-x} + 1} \right) dx = \int_0^\infty \left(\frac{e^{-x} + 1}{e^{-x} + 1} - \frac{1}{e^{-x} + 1} \right) dx$$

$$= \int_0^\infty \frac{e^{-x}}{e^{-x} + 1} dx$$

$$= (-1) \int_0^\infty (-e^{-x})(e^{-x} + 1)^{-1}$$

$$= (-1) \lim_{t \to \infty} \int_0^t (-e^{-x})(e^{-x} + 1)^{-1} \, dx$$

$$= (-1) \lim_{t \to \infty} \left[\ln(e^{-x} + 1) \right]_0^t$$

$$= (-1) \lim_{t \to \infty} \left[\ln(e^{-t} + 1) - \ln(e^0 + 1) \right]$$

As $t \to \infty, \; e^{-t} \to 0$

$$= (-1) \left[\ln(1) - \ln(2) \right]$$

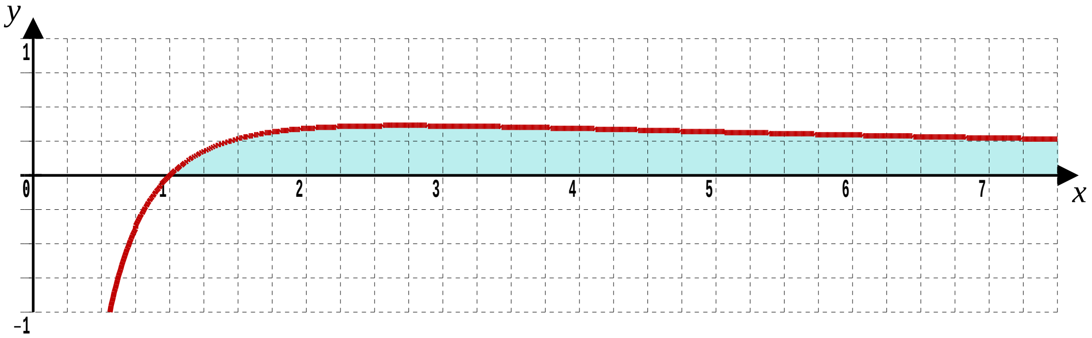
$$= \ln(2)$$

$\therefore$ The integral is convergent; it has a value of $\ln 2$

[6 marks]

2.1 By Parts

The graph is of the function $f(x) = \dfrac{\ln x}{x}$ and the interest is in finding the area shown shaded but extending off to infinity.
From the graph it's not clear if this area will have a finite value.



Teaching Video : http://www.NumberWonder.co.uk/v9100/2.mp4

[6 marks]

Mathematical Folklore
Dr Peter Lax has made outstanding, innovative and profound contributions to the theories of partial differential equation and numerical analysis. When awarded the National Medal of Science in 1986 a newspaper journalist asked what he did to deserve the medal. Lax responded, “I used Integration by Parts”, which was duly reported in the International press at the time.

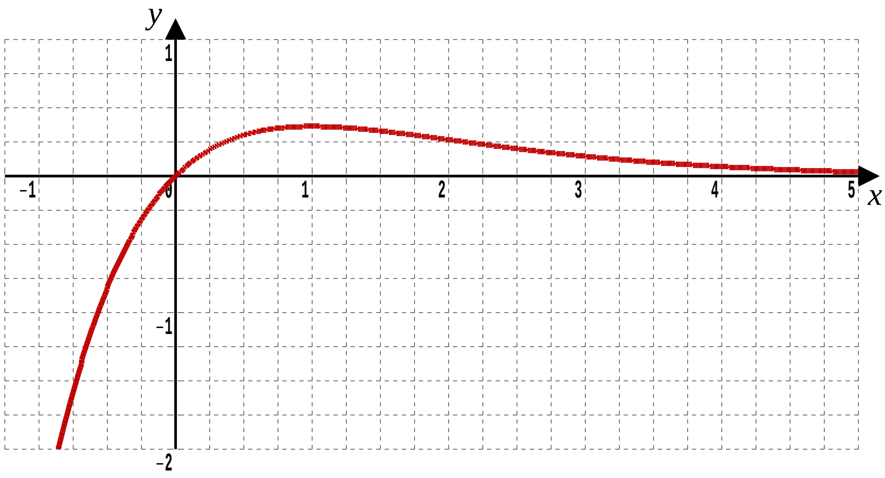
2.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 50

Question 1

The graph is of the function $f(x) = x e^{-x}$



Use integration by parts to determine the value of $\int_0^{\infty} f(x)\ dx$

You may assume that as $t \rightarrow \infty$, $e^{-t} \rightarrow 0$ and $t e^{-t} \rightarrow 0$

[6 marks]

Question 2

The graph is of the function $f(x) = e^{-x} \sin x$

(i) Explain, briefly, why as $x \to \infty$, $f(x) \to 0$

[1 mark]

(ii) Let $S = \int e^{-x} \sin x \, dx$

Use integration by parts twice to show that

$S = -\frac{1}{2} e^{-x} (\cos x + \sin x) + c$ where c is a constant

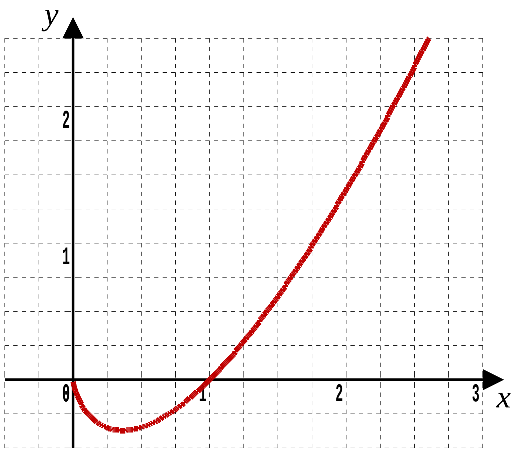
[6 marks]

(iii) Determine the value of $\int_0^{\infty} e^{-x} \sin x \, dx$

[3 marks]

Question 3

The graph is of the function $f(x) = x \ln x$

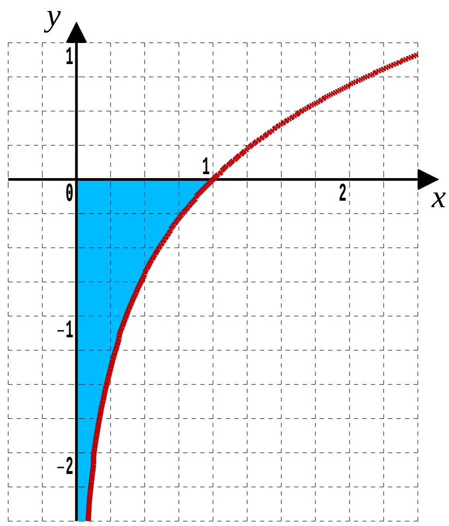


(i) As $x \to 0^+$ what does the graph suggest that $x \ln x$ will tend to ?

[1 mark]

Proving the part (i) suggestion requires L'Hôspital's Rule, covered in the “Further Pure 1” optional unit of the Further A-Level Mathematics course, on page 152.

The graph is of the function $g(x) = ln\,x$

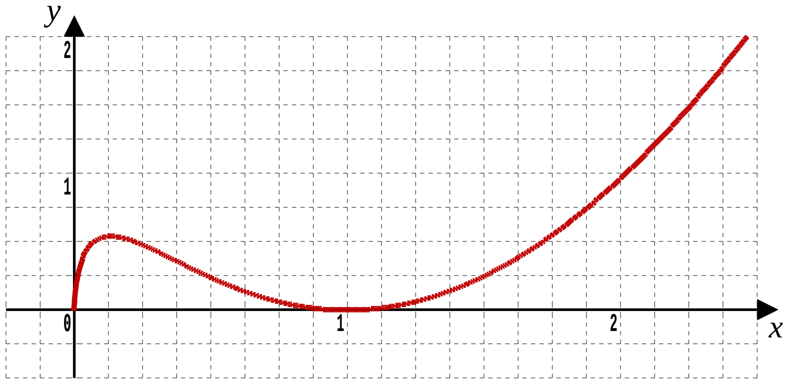


(ii) Find the shaded area enclosed by the graph of the function $g(x) = ln\,x$ and the coordinate axes.

[6 marks]

Question 4

The graph is of the function $f(x) = x\,(\ln x)^2$

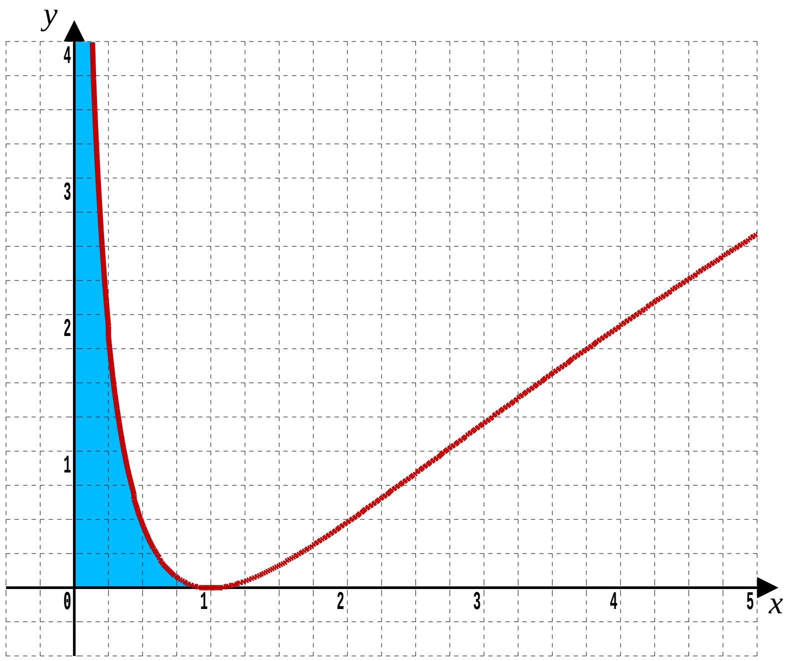


(i) As $x \to 0^+$ what does the graph suggest that $x(\ln x)^2$ will tend to ?

[1 mark]

Again, proving the part (i) suggestion requires L'Hôspital's Rule, covered in the "Further Pure 1" optional unit of the Further A-Level Mathematics course.

The graph is of the function $g(x) = (\ln x)^2$



(ii) Find the shaded area enclosed by the graph of the function $g(x) = (\ln x)^2$ and the coordinate axes.

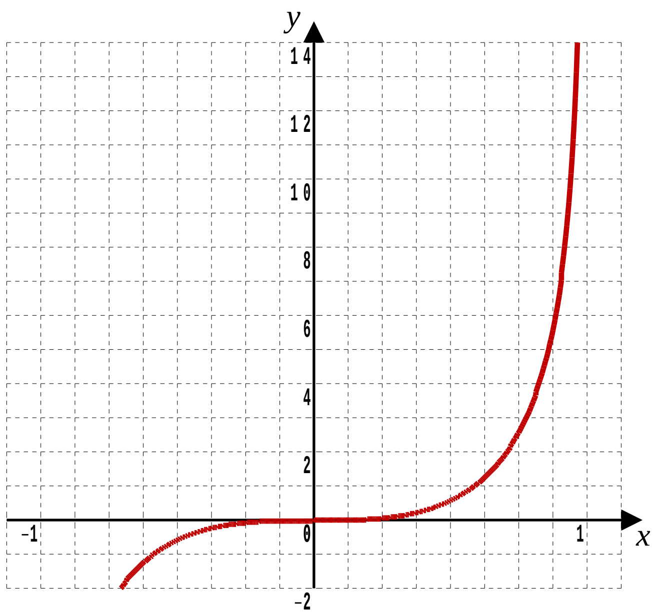
[6 marks]

Question 5

This question is about using integration by parts to show that the following improper integral is convergent, and finding its value,

$$\int_0^1 \frac{x^3}{\sqrt{1 - x^2}}\, dx$$

(i) On the graph, shade in the area to be found,



[1 mark]

(ii) Differentiate each of the following,

(a) $y = \left(1 - x^2\right)^{\frac{3}{2}}$ **(b)** $y = \left(1 - x^2\right)^{\frac{1}{2}}$

[2 marks]

(iii) By rewriting the integral in the following way,

$$\int_0^1 1\frac{x^3}{\sqrt{1 - x^2}}\,dx = \int_0^1 \left(x^2\right)\left(\frac{x}{\sqrt{1 - x^2}}\right)dx$$

and then applying integration by parts, determine its value.

[6 marks]

Question 6

(i) Given that $S = \int \frac{(\ln x)^k}{x} \, dx$ prove that $S = \frac{(\ln x)^{k+1}}{k + 1} + c$ where k is a positive integer and c is the constant of integration

[5 marks]

(ii) Hence show that both $\int_0^1 \frac{(\ln x)^k}{x} \, dx$ and $\int_1^\infty \frac{(\ln x)^k}{x} \, dx$ are divergent where k is a positive integer.

[6 marks]

2.3 Answers

Note :
To reduce "unhelpful clutter", in many solutions the constant of the integration is only inserted in the final line of working.

2.3.1 Solution to 2.1 Example (Teaching Video)

$$
\begin{aligned}
S &= \int \frac{\ln x}{x}\,dx \\
&= \int (\ln x)\left(\frac{1}{x}\right)\,dx \\
& \qquad\; \boldsymbol{L} \qquad \boldsymbol{I} \qquad\qquad \boldsymbol{D} \qquad \boldsymbol{I} \\
&= (\ln x)(\ln x) - \int \left(\frac{1}{x}\right)(\ln x)\,dx \\
S &= (\ln x)^2 - S \\
2S &= (\ln x)^2 \\
S &= \frac{1}{2}(\ln x)^2 + c
\end{aligned}
$$

$$
\begin{aligned}
\int_1^\infty \frac{\ln x}{x}\,dx &= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx \\
&= \frac{1}{2}\lim_{t\to\infty}\Big[\,(\ln x)^2\,\Big]_1^t \\
&= \frac{1}{2}\lim_{t\to\infty}\Big[\,(\ln t)^2 - (\ln 1)^2\,\Big]
\end{aligned}
$$

As $t \to \infty$, $\ln t \to \infty$

$\therefore$ The integral does not converge; it is divergent $\square$

[6 marks]

This example is generalised in the final question of this lesson's exercise.

2.3.2 Solutions to 2.2 Exercise

Answer 1

$$\int f(x)\ dx = \int x\, e^{-x}\, dx$$

$$\begin{array}{cccc} \boldsymbol{L} & \boldsymbol{I} & \boldsymbol{D} & \boldsymbol{I} \end{array}$$

$$= x\left(-e^{-x}\right) - \int 1\left(-e^{-x}\right) dx$$

$$= -x\, e^{-x} + \int e^{-x}\, dx$$

$$= -x\, e^{-x} - e^{-x} + c$$

$$\int_0^{\infty} f(x)\ dx = \underset{t \to \infty}{limit} \int_0^{t} x\, e^{-x}\, dx$$

$$= \underset{t \to \infty}{limit} \left[-x\, e^{-x} - e^{-x}\right]_0^t$$

$$= \underset{t \to \infty}{limit} \left[-t\, e^{-t} - e^{-t} + 0 + e^{0}\right]$$

As $t \to \infty,\ e^{-t} \to 0$

$$= 1$$

$\therefore$ The integral is convergent; it has a value of 1

[6 marks]

Answer 2

(i) It's already known that $g(x) = e^{-x}$ is a strictly decreasing function that, as $x \to \infty,\ e^{-x} \to 0$

As $-1 \leqslant \sin x \leqslant 1$ with only spaced out isolated periodic points where $\sin x = \pm 1$ the multiplication of e^{-x} by $\sin x$ will not alter the overall tending towards zero.

That is, $x \to \infty,\ e^{-x} \sin x \to 0$ □

[1 mark]

(ii)

$$S = \int e^{-x} \sin x \, dx$$

$$\textbf{\textit{L}} \quad \textbf{\textit{I}} \qquad\qquad \textbf{\textit{D}} \qquad \textbf{\textit{I}}$$

$$= e^{-x}\left(-\cos x\right) - \int \left(-e^{-x}\right)\left(-\cos x\right) dx$$

$$= -e^{-x} \cos x - \int e^{-x} \cos x \, dx$$

$$\textbf{\textit{L}} \quad \textbf{\textit{I}} \qquad \textbf{\textit{D}} \qquad \textbf{\textit{I}}$$

$$= -e^{-x} \cos x - \left[e^{-x} \sin x - \int \left(-e^{-x}\right)\left(\sin x\right) dx \right]$$

$$S = -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x \, dx$$

$$S = -e^{-x} \cos x - e^{-x} \sin x - S$$

$$2S = -e^{-x} \cos x - e^{-x} \sin x$$

$$S = -\frac{1}{2} e^{-x}\left(\cos x + \sin x \right) + c \qquad \square$$

[6 marks]

(iii)

$$\int_0^{\infty} e^{-x} \sin x \, dx = \lim_{t \to \infty} \int_0^{t} e^{-x} \sin x \, dx$$

$$= -\frac{1}{2} \lim_{t \to \infty} \Big[e^{-x} \cos x + e^{-x} \sin x \Big]_0^t$$

$$= -\frac{1}{2} \lim_{t \to \infty} \left[e^{-t} \cos t + e^{-t} \sin t - e^{0} \cos 0 - e^{0} \sin 0 \right]$$

$$= -\frac{1}{2} \left[-1 \right]$$

$$= \frac{1}{2}$$

[3 marks]

Answer 3

(i) As $x \to 0^+, \ x \ln x \to 0$

[1 mark]

(ii) $\int \ln x \, dx = \int \ln x \times 1 \, dx$

L I D I

$$= (\ln x)\, x - \int \frac{1}{x}\, x \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + c$$

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$

$$= \lim_{t \to 0^+} \Big[x \ln x - x \Big]_t^1$$

$$= \lim_{t \to 0^+} \Big[\ln 1 - 1 - t \ln t + t \Big]$$

$$= -1$$

$\therefore$ Area $= 1$

[6 marks]

Extra

L'Hôspital's Rule

For two functions $f(x)$ and $g(x)$, if either:

- $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$
- $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

then, provided that $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

A proof that, as $x \to 0^+, \ x \ln x \to 0$ using L'Hôspital's Rule

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} = \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \times \frac{x^2}{x^2} = \lim_{x \to 0^+} - x = 0$$

□

Answer 4

(i) As $x \to 0^+, \ x(\ln x)^2 \to 0$

[1 mark]

(ii) $$\int (\ln x)^2 \, dx = \int (\ln x)^2 \times 1 \, dx$$

L I D I

$$= (\ln x)^2 x - \int 2(\ln x)\frac{1}{x} x \, dx$$

$$= (\ln x)^2 x - 2\int \ln x \, dx$$

$$= (\ln x)^2 x - 2\left[x \ln x - x \right] + c$$

$$= x(\ln x)^2 - 2x \ln x + 2x + c$$

$$\int_0^1 (\ln x)^2 \, dx = \underset{t \to 0^+}{limit} \int_t^1 (\ln x)^2 \, dx$$

$$= \underset{t \to 0^+}{limit} \left[x(\ln x)^2 - 2x \ln x + 2x \right]_t^1$$

$$= \underset{t \to 0^+}{limit} \left[(\ln 1)^2 - 2\ln 1 + 2 - t(\ln t)^2 + 2t \ln t - 2t \right]$$

$$= 2$$

$\therefore$ Area $= 2$

[6 marks]

Extra

A proof that, as $x \to 0^+, \ x(\ln x)^2 \to 0$ using L'Hôspital's Rule

$$\underset{x \to 0^+}{limit} \ x(\ln x)^2 = \underset{x \to 0^+}{limit} \frac{(\ln x)^2}{\left(\frac{1}{x}\right)}$$

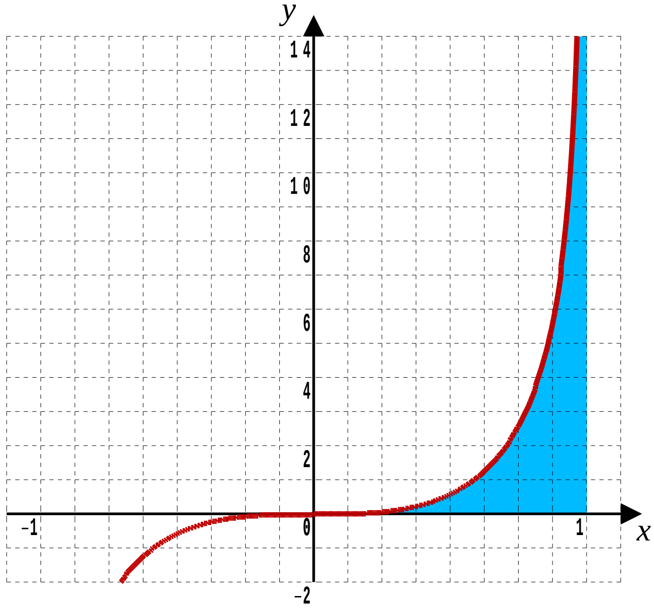
$$= \underset{x \to 0^+}{limit} \frac{2(\ln x)\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \times \frac{x}{x}$$

$$= -2 \underset{x \to 0^+}{limit} \frac{\ln x}{\left(\frac{1}{x}\right)}$$

$$= -2 \underset{x \to 0^+}{limit} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \times \frac{x^2}{x^2}$$

$$= 2 \underset{x \to 0^+}{limit} \ x = 0 \qquad \square$$

Answer 5
(i)



[1 mark]

(ii) **(a)** $y = (1 - x^2)^{\frac{3}{2}}$

$$\frac{dy}{dx} = \frac{3}{2}(1 - x^2)^{\frac{1}{2}}(-2x)$$
$$= -3x(1 - x^2)^{\frac{1}{2}}$$
$$= -3x\sqrt{1 - x^2}$$

(b) $y = (1 - x^2)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x)$$
$$= -x(1 - x^2)^{-\frac{1}{2}}$$
$$= -\frac{x}{\sqrt{1 - x^2}}$$

[2 marks]

(iii) Without the limits for now,

$$\int \frac{x^3}{\sqrt{1 - x^2}}\,dx = \int (x^2)\left(\frac{x}{\sqrt{1 - x^2}}\right)dx$$

L ***I*** ***D*** ***I***

$$= (x^2)(-1)(1 - x^2)^{\frac{1}{2}} - \int (2x)(-1)(1 - x^2)^{\frac{1}{2}}\,dx$$
$$= -x^2(1 - x^2)^{\frac{1}{2}} - \frac{2}{3}\int(-3x)(1 - x^2)^{\frac{1}{2}}\,dx$$
$$= -x^2(1 - x^2)^{\frac{1}{2}} - \frac{2}{3}(1 - x^2)^{\frac{3}{2}} + c$$

$$\int_0^1 \frac{x^3}{\sqrt{1 - x^2}}\, dx = \underset{t \to 0^+}{limit} \int_t^1 \frac{x^3}{\sqrt{1 - x^2}}\, dx$$

$$= \underset{t \to 0^+}{limit} \left[-x^2 (1 - x^2)^{\frac{1}{2}} - \frac{2}{3} (1 - x^2)^{\frac{3}{2}} \right]_t^1$$

$$= \underset{t \to 0^+}{limit} \left[0 + t^2 (1 - t^2)^{\frac{1}{2}} + \frac{2}{3} (1 - t^2) \right]_t^1$$

$$= \frac{2}{3}$$

[6 marks]

Answer 6
(i)

$$S = \int \frac{(\ln x)^k}{x}\, dx$$

$$= \int (\ln x)^k \left(\frac{1}{x}\right) dx$$

L ***I*** ***D*** ***I***

$$= (\ln x)^k \ (\ln x) - \int k (\ln x)^{k-1} \left(\frac{1}{x}\right) (\ln x)\ dx$$

$$= (\ln x)^{k+1} - k \int (\ln x)^k \left(\frac{1}{x}\right) dx$$

$$S = (\ln x)^{k+1} - k S$$

$$(1 + k) S = (\ln x)^{k+1}$$

$$S = \frac{(\ln x)^{k+1}}{k + 1} + c \qquad \square$$

[5 marks]

(ii)

$$\int_0^1 \frac{(\ ln\, x\ )^k}{x}\, dx \ = \ \underset{t \to 0^+}{limit} \int_t^1 \frac{(\ ln\, x\ )^k}{x}\, dx$$

$$= \frac{1}{k\ +\ 1} \underset{t \to 0^+}{limit} \left[\ (\ ln\, x\ )^k \ \right]_t^1$$

$$= \frac{1}{k\ +\ 1} \underset{t \to 0^+}{limit} \left[\ 0\ -\ (\ ln\, t\ )^k \ \right]$$

As $t \ \to \ 0^+,\ \ ln\, t \ \to \ -\infty$

Furthermore, $k \ \geqslant \ 1$ so $(\ ln\, t\ )^k \ \to \ -\infty$

$\therefore$ The integral does not converge; it is divergent ☐

[3 marks]

$$\int_1^\infty \frac{(\ ln\, x\ )^k}{x}\, dx \ = \ \underset{t \to \infty}{limit} \int_1^t \frac{(\ ln\, x\ )^k}{x}\, dx$$

$$= \frac{1}{k\ +\ 1} \underset{t \to \infty}{limit} \left[\ (\ ln\, x\ )^k \ \right]_1^t$$

$$= \frac{1}{k\ +\ 1} \underset{t \to \infty}{limit} \left[\ (\ ln\, t\ )^k \ -\ 0 \right]$$

As $t \ \to \ \infty,\ \ ln\, t \ \to \ \infty$

Furthermore, $k \ \geqslant \ 1$ so $(\ ln\, t\ )^k \ \to \ \infty$

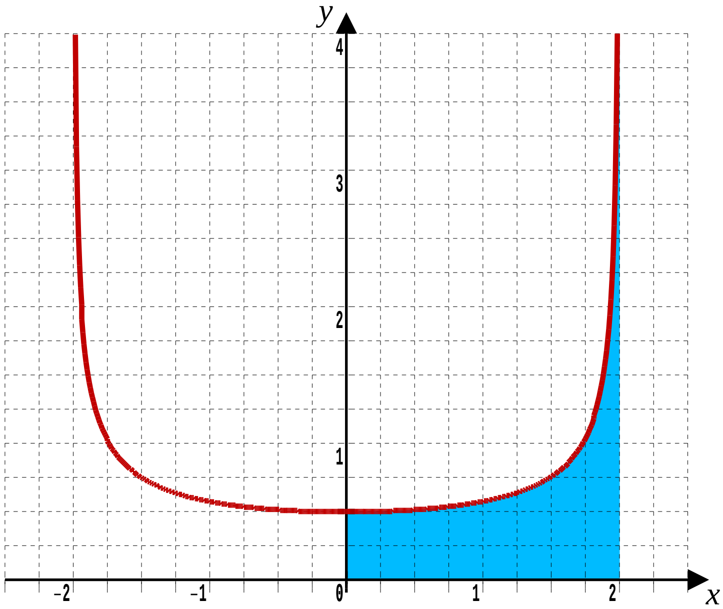
$\therefore$ The integral does not converge; it is divergent ☐

[3 marks]

Lesson 3

3.1 Substitution

The graph is of the function $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$ and the task is to find the area shown shaded from $x = 0$ to the vertical asymptote at $x = 2$



Teaching video : http://www.NumberWonder.co.uk/v9100/3.mp4

[7 marks]

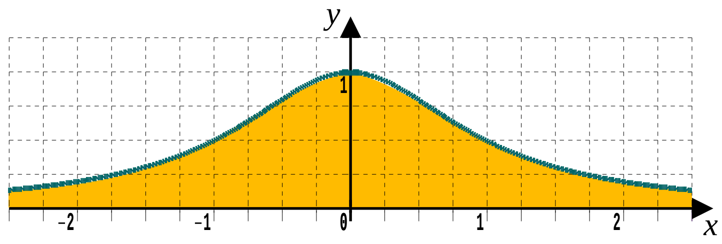
3.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

The graph is of the function $f(x) = \dfrac{1}{1 + x^2}$ and the task to find $\int_{-\infty}^{\infty} f(x)\, dx$



(i) The function is even. Explain how this fact can simplify the task.

[1 mark]

(ii) Prove that the derivative with respect to θ of $\tan\theta$ is $\sec^2\theta$

[2 marks]

(iii) Use the substitution $x = \tan\theta$ to show that $\int_{-\infty}^{\infty} f(x)\, dx = \pi$

A sketch of the graph of $\theta = \arctan x$ may help with changing the limits.

[6 marks]

Question 2

The graph is of $y = \dfrac{(x-4)(5x+2)}{\sqrt{x-6}}$

(i) Use the substitution $x = 6 + u^2$ to determine the value of the improper integral $\displaystyle\int_6^7 \frac{(x-4)(5x+2)}{\sqrt{x-6}}\,dx$

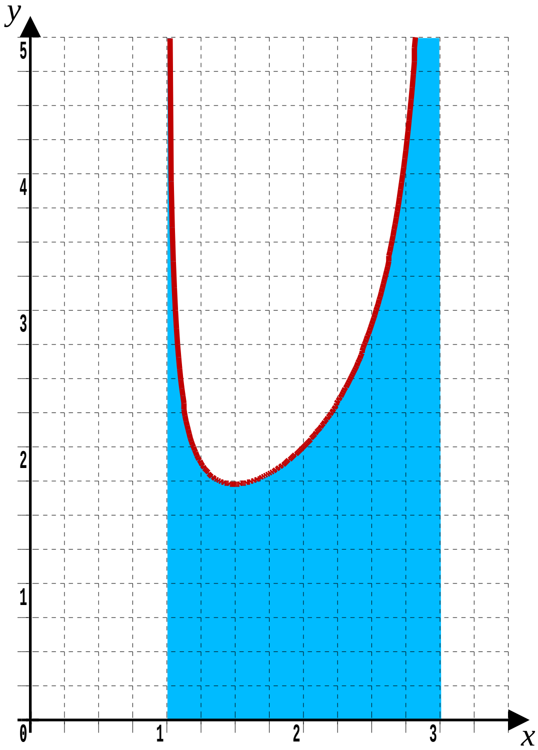
[5 marks]

(ii) From the graph, explain why $\displaystyle\int_7^\infty \frac{(x-4)(5x+2)}{\sqrt{x-6}}\,dx$ will be divergent

[2 marks]

Question 3

The graph is of the function $f(x) = \dfrac{x}{\sqrt{(x-1)(3-x)}}$

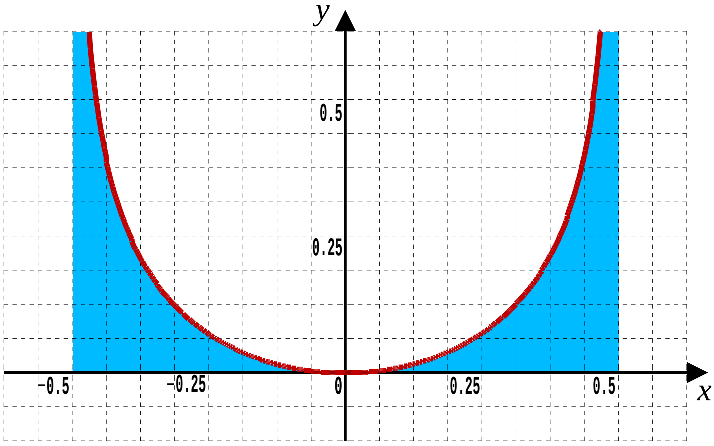


Use the substitution $x = \sin\theta + 2$ to determine the area shaded which is between the curve, the x-axis and the vertical asymptotes at $x = 1$ and $x = 3$

[7 marks]

Question 4

The graph is of the function $f(x) = \dfrac{x^2}{\sqrt{1 - 4x^2}}$



Use the substitution $x = \dfrac{1}{2}\sin\theta$ to determine the area shaded which is between the curve, the x-axis and the vertical asymptotes at $x = \pm\dfrac{1}{2}$

[7 marks]

Question 5

Faced with an improper integral of the form,

$$\int_a^b \frac{1}{\sqrt{(b-x)(x-a)}}\,dx \qquad \text{where } a \text{ and } b \text{ are constants with } a < b$$

the substitution to use is $x = a\cos^2\theta + b\sin^2\theta$

(i) Show that $(b-x)(x-a) = (b-a)^2 \cos^2\theta \sin^2\theta$

[3 marks]

(ii) Show that $\int_a^b \frac{1}{\sqrt{(b-x)(x-a)}}\,dx = \pi$

[8 marks]

Question 6

The graph is of the function $f(x) = \dfrac{1}{x\sqrt{x^2 - 1}}$

Use the substitution $x = sec\,\theta$ to determine the area between the curve, the x-axis, the vertical asymptote at $x = 2$ and extending indefinitely rightward as $x \rightarrow \infty$

[9 marks]

3.3 Answers

3.3.1 Solution to 3.1 Example (Teaching Video)

Let $x = 2\sin\theta$ in which case $\dfrac{dx}{d\theta} = 2\cos\theta$ and limits are $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

$$\begin{aligned}
\int_0^2 \frac{1}{\sqrt{4 - x^2}}\,dx &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{4 - (2\sin^2\theta)}} \times 2\cos\theta\ d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{2\cos\theta}{\sqrt{4(1 - \sin^2\theta)}}\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{2\cos\theta}{\sqrt{4}\sqrt{\cos^2\theta}}\,d\theta \\
&= \int_0^{\frac{\pi}{2}} 1\,d\theta \\
&= [\,\theta\,]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}
\end{aligned}$$

Notice that, in this case, the improper integral under the substitution became proper.

[6 marks]

3.3.2 Solution to 3.2 Exercise

Answer 1

(i) If the area from $x = 0$ to $x = \infty$ is bounded then, because the function is symmetrical, the area from $x = -\infty$ to $x = \infty$ will be double it.

[1 mark]

(ii)

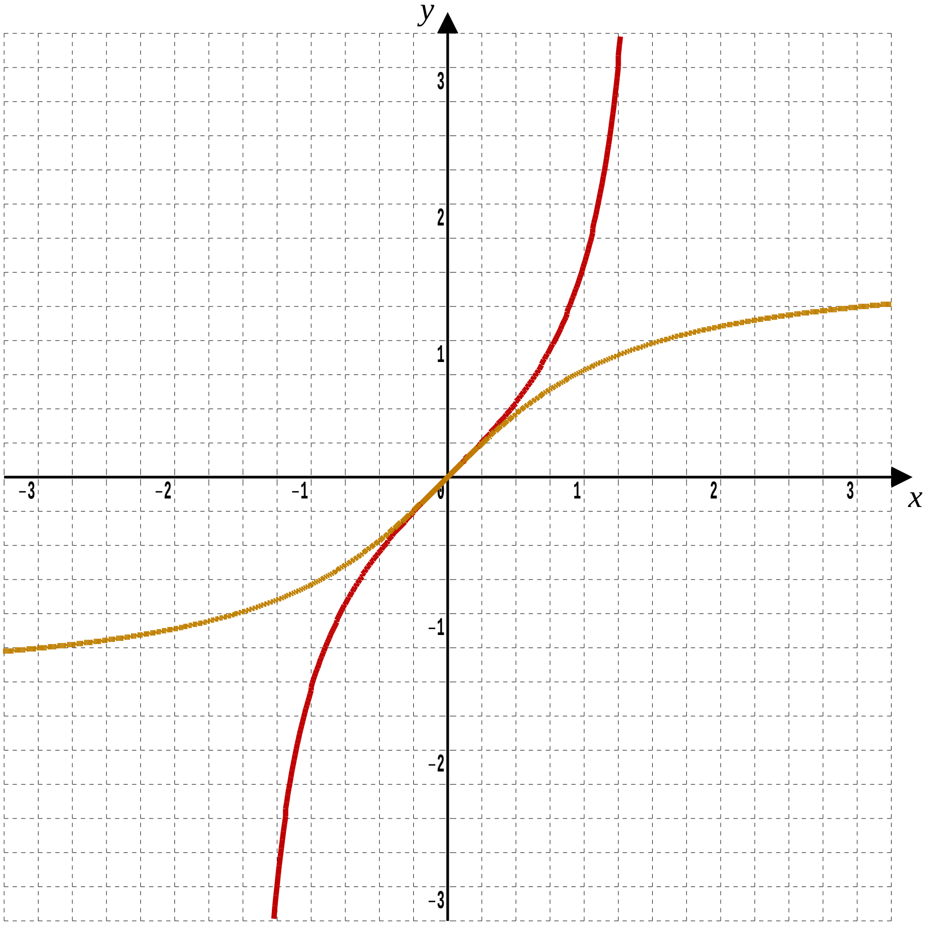
$$\begin{aligned}
x &= \tan\theta \\
&= \frac{\sin\theta}{\cos\theta} \\
\frac{dx}{d\theta} &= \frac{\cos\theta\cos\theta - (-\sin\theta)\sin\theta}{\cos^2\theta} \quad \text{by the quotient rule} \\
&= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} \\
&= \frac{1}{\cos^2\theta} \\
&= \sec^2\theta \qquad \square
\end{aligned}$$

[2 marks]

(iii)

Let $x = \tan\theta$ in which case $\dfrac{dx}{d\theta} = \sec^2\theta$

For the limits it may help to have the graph of $\theta = \arctan x$ handy;



Red : $y = \tan x$ Gold : $y = \arctan x$

The limits $x = 0$, $x = \infty$ become $\theta = 0$, $\theta = \dfrac{\pi}{2}$

$$\int_0^{\infty} \frac{1}{1 + x^2}\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^2\theta} \times \sec^2\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sec^2\theta} \times \sec^2\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} 1\ d\theta$$

$$= \left[\ \theta\ \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} \qquad\qquad \therefore \int_{-\infty}^{\infty} f(x)\ dx = \pi \qquad \square$$

[6 marks]

Answer 2

(i) Let $x = 6 + u^2$ in which case $\dfrac{dx}{du} = 2u$

The limits $x = 6$, $x = 7$ become $\theta = 0$, $\theta = 1$

$$\int_6^7 \frac{(x - 4)(5x + 2)}{\sqrt{x - 6}}\, dx = \int_0^1 \frac{\left(u^2 + 2\right)\left(5u^2 + 32\right)}{\sqrt{u^2}}\, 2u\, du$$

$$= 2\int_0^1 5u^4 + 42u^2 + 64\, du$$

$$= 2\left[\, u^5 + 14u^3 + 64u \,\right]_0^1$$

$$= 158$$

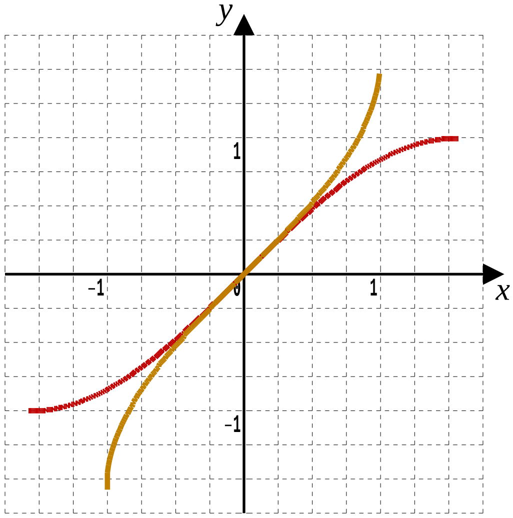
[5 marks]

(ii) The function looks to be both increasing and convex and so, as $x \to \infty$ the area $\to \infty$ indicating the function from 7 to ∞ is divergent.

[2 marks]

Answer 3

Let $x = \sin\theta + 2$ in which case $\dfrac{dx}{d\theta} = \cos\theta$



Red : $y = \sin x$ Gold : $y = \arcsin x$

The limits $x = 1$, $x = 3$ become $\theta = -\dfrac{\pi}{2}$, $\theta = \dfrac{\pi}{2}$

$$\int_1^3 \frac{x}{\sqrt{(x-1)(3-x)}}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\theta + 2}{\sqrt{(\sin\theta + 1)(1 - \sin\theta)}} \cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\theta + 2}{\sqrt{1 - \sin^2\theta}} \cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\theta + 2}{\sqrt{\cos^2\theta}} \cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta + 2\, d\theta$$

$$= \left[-\cos\theta + 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left[-\cos\left(\frac{\pi}{2}\right) + \pi + \cos\left(-\frac{\pi}{2}\right) + \pi\right]$$

$= 2\pi$ (As $\cos x$ is an even function)

[7 marks]

Answer 4

Let $x = \frac{1}{2} sin\,\theta$ in which case $\frac{dx}{d\theta} = \frac{1}{2} cos\,\theta$

The limits $x = \pm\frac{1}{2}$ become $-\frac{\pi}{2}, \frac{\pi}{2}$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^2}{\sqrt{1-4x^2}}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\frac{1}{2}\, sin\,\theta\right)^2}{\sqrt{1-4\left(\frac{1}{2}\, sin\,\theta\right)^2}}\,\frac{1}{2}\, cos\,\theta\, d\theta$$

$$= \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{sin^2\,\theta\, cos\,\theta}{\sqrt{1-sin^2\,\theta}}\, d\theta$$

$$= \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^2\,\theta\, d\theta$$

$$= \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\, cos\, 2\theta\right) d\theta$$

$$= \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 - cos\,2\theta\;\; d\theta$$

$$= 2 \times \frac{1}{16}\left[\theta - \frac{1}{2}\, sin\, 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left[\frac{\pi}{2} - 0 - 0 + 0\right]$$

$$= \frac{\pi}{16}$$

[7 marks]

Answer 5

(i)

$$\begin{aligned}
\text{LHS} &= (b-x)(x-a) \\
&= \left(b - a\,cos^2\,\theta - b\,sin^2\,\theta\right)\left(a\,cos^2\,\theta + b\,sin^2\,\theta - a\right) \\
&= \left(b\left(1 - sin^2\,\theta\right) - a\,cos^2\,\theta\right)\left((-a)\left(1 - cos^2\,\theta\right) + b\,sin^2\,\theta\right) \\
&= \left(b\,cos^2\,\theta - a\,cos^2\,\theta\right)\left(b\,sin^2\,\theta - a\,sin^2\,\theta\right) \\
&= \left((b-a)\,cos^2\,\theta\right)\left((b-a)\,sin^2\,\theta\right) \\
&= (b-a)^2\, cos^2\,\theta\, sin^2\,\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

(ii) For the lower limit, when $x = a$,

$$a = a\cos^2\theta + b\sin^2\theta$$

$$a\left(1 - \cos^2\theta\right) = b\sin^2\theta$$

$$b\sin^2\theta - a\sin^2\theta = 0$$

$$\left(b - a\right)\sin^2\theta = 0$$

Either $b - a = 0$ which is not so as $a < b$

Or $\sin^2\theta = 0$

$$\theta = 0$$

For the upper limit, when $x = b$

$$b = a\cos^2\theta + b\sin^2\theta$$

$$b\left(1 - \sin^2\theta\right) = a\cos^2\theta$$

$$b\cos^2\theta - a\cos^2\theta = 0$$

$$\left(b - a\right)\cos^2\theta = 0$$

Either $b - a = 0$ which is not so as $a < b$

Or $\cos^2\theta = 0$

$$\theta = \frac{\pi}{2}$$

Let $x = a\cos^2\theta + b\sin^2\theta$

$$\Rightarrow \frac{dx}{d\theta} = 2a\cos\theta\,(-\sin\theta) + 2b\sin\theta\cos\theta$$

$$= 2b\cos\theta\sin\theta - 2a\cos\theta\sin\theta$$

$$= 2(b - a)\cos\theta\sin\theta$$

The limits $x = a,\ b$ become $0,\ \frac{\pi}{2}$

$$\int_a^b \frac{1}{\sqrt{(b-x)(x-a)}}\,dx = \int_0^{\frac{\pi}{2}} \frac{2(b-a)\cos\theta\sin\theta}{\sqrt{(b-a)^2\cos^2\theta\sin^2\theta}}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1\,d\theta$$

$$= 2\left[\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[\frac{\pi}{2} - 0\right]$$

$$= \pi \qquad \square$$

[8 marks]

Answer 6

Let $x = sec\ \theta$ in which case $\dfrac{dx}{d\theta} = tan\ \theta\ sec\ \theta$

(Easily proven using the chain rule)

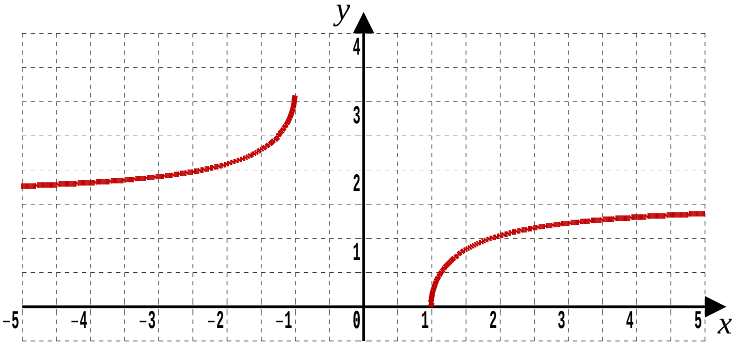
Graph Plotting

To draw the graphs of *arcsec x*, *arccsc x* and *arccot x*, or calculate their value at a particular point, use the relationships,

$$arcsec\ x = arccos\left(\frac{1}{x}\right)$$

$$arccsc\ x = arcsin\left(\frac{1}{x}\right)$$

$$arccot\ x = arctan\left(\frac{1}{x}\right)$$



Graph of $y = arcsec\ x$

The limits $x = 2,\ \infty$ become $\dfrac{\pi}{3},\ \dfrac{\pi}{2}$

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2 - 1}} dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{sec\ \theta\sqrt{sec^2\ \theta - 1}}\ tan\ \theta\ sec\ \theta\ d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{tan\ \theta}{\sqrt{tan^2\ \theta}}\ d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1\ d\theta$$

$$= \left[\ \theta\ \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

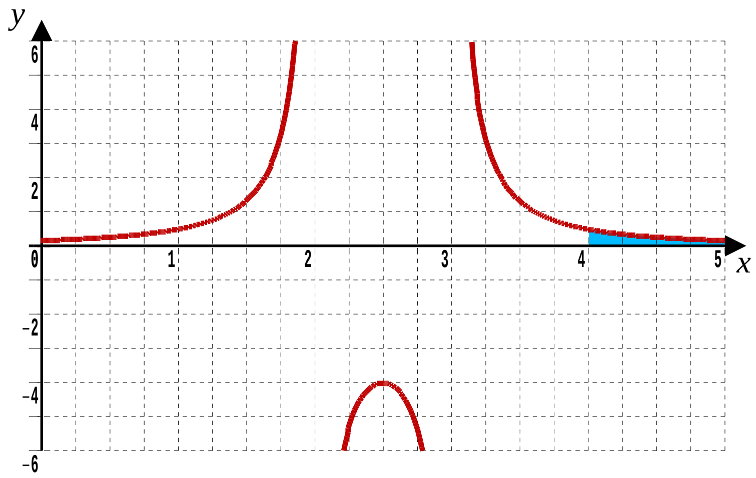
$$= \left[\ \frac{\pi}{2} - \frac{\pi}{3}\ \right]$$

$$= \frac{\pi}{6}$$

[9 marks]

4.1 Partial Fractions

The graph is of the function $f(x) = \dfrac{1}{x^2 - 5x + 6}$ and the task is to find the area from $x = 4$ and extending rightward as $x \to \infty$



Teaching video : http://www.NumberWonder.co.uk/v9100/4.mp4

[8 marks]

4.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 50

Question 1

Use the method of partial fractions to show that $\int_2^{\infty} \frac{1}{x^2 + x - 2}\, dx = \frac{2}{3} ln\, 2$

[8 marks]

Question 2

Use the method of partial fractions to show that $\int_0^{\infty} \frac{19}{6x^2 + 35x + 36}\, dx = 3\, ln\left(\frac{3}{2}\right)$

[8 marks]

Question 3

(i) Use the substitution $x = 2\tan\theta$ to show that,

$$\int \frac{1}{x^2 + 4}\,dx = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c$$

[3 marks]

(ii) Show that $\displaystyle\int_2^{\infty} \frac{32}{(x^2 + 4)(x + 2)}\,dx = \pi - 2\ln 2$

[9 marks]

Question 4

Show that $\displaystyle\int_{\frac{1}{2}}^{\infty} \frac{1 - x^2}{4x^4 + x^2}\, dx = 2 - \frac{5\pi}{8}$

[10 marks]

Question 5

Show that $96\int_{2\sqrt{3}}^{\infty} \frac{3 - x}{(x - 2)^2(x^2 + 4)}\,dx = 2\pi + 3 + 3\sqrt{3} + 9\,ln\left(1 - \frac{\sqrt{3}}{2}\right)$

[12 marks]

4.3 Answers

4.3.1 Solution to 4.1 Example (Teaching Video)

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{(x-3)(x-2)}$$

$$\frac{1}{[\,(x-3)(x-2)\,]} = \frac{A}{(x-3)} + \frac{B}{(x-2)}$$

Multiply through by the red brackets

$$\frac{1\,[\,\dots\dots\,]}{[\,(x-3)(x-2)\,]} = \frac{A\,[\,\dots\dots\,]}{(x-3)} + \frac{B\,[\,\dots\dots\,]}{(x-2)}$$

$$1 = A(x-2) + B(x-3)$$

$$\text{Let } x = 3 \Rightarrow A = 1$$

$$\text{Let } x = 2 \Rightarrow B = -1$$

$$\int_4^{\infty} \frac{1}{x^2 - 5x + 6}\,dx = \int_4^{\infty} \frac{1}{x-3} - \frac{1}{x-2}\,dx$$

$$= \lim_{t \to \infty} \int_4^{t} \frac{1}{x-3} - \frac{1}{x-2}\,dx$$

$$= \lim_{t \to \infty} \Big[\ln(x-3) - \ln(x-2)\Big]_4^t$$

$$= \lim_{t \to \infty} \left[\ln\left(\frac{x-3}{x-2}\right)\right]_4^t$$

$$= \lim_{t \to \infty} \left[\ln\left(\frac{t-3}{t-2}\right) - \ln\left(\frac{1}{2}\right)\right]$$

$$= \ln 1 - \ln 2^{-1}$$

$$= \ln 2$$

[8 marks]

4.3.2 Solutions to 4.2 Exercise

Answer 1

$$\frac{1}{x^2 + x - 2} = \frac{1}{(x+2)(x-1)}$$

$$\frac{1}{[\,(x-1)(x+2)\,]} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

Multiply through by the red brackets

$$\frac{1\,[\,\ldots\ldots\,]}{[\,(x-1)(x+2)\,]} = \frac{A\,[\,\ldots\ldots\,]}{(x-1)} + \frac{B\,[\,\ldots\ldots\,]}{(x+2)}$$

$$1 = A(x+2) + B(x-1)$$

$$\text{Let } x = 1 \Rightarrow A = \frac{1}{3}$$

$$\text{Let } x = -2 \Rightarrow B = -\frac{1}{3}$$

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x^2 + x - 2}\,dx &= \frac{1}{3}\int_2^{\infty} \frac{1}{x-1} - \frac{1}{x+2}\,dx \\
&= \frac{1}{3}\lim_{t\to\infty} \int_2^{t} \frac{1}{x-1} - \frac{1}{x+2}\,dx \\
&= \frac{1}{3}\lim_{t\to\infty} \Big[\, \ln(x-1) - \ln(x+2) \,\Big]_2^t \\
&= \frac{1}{3}\lim_{t\to\infty} \left[\ln\left(\frac{x-1}{x+2}\right) \right]_2^t \\
&= \frac{1}{3}\lim_{t\to\infty} \left[\ln\left(\frac{t-1}{t+2}\right) - \ln\left(\frac{1}{4}\right) \right] \\
&= \frac{1}{3}\ln 1 - \frac{1}{3}\ln 2^{-2} \\
&= \frac{2}{3}\ln 2 \qquad \square
\end{aligned}$$

[8 marks]

Answer 2

$$\frac{19}{6x^2 + 35x + 36} = \frac{19}{(3x+4)(2x+9)}$$

$$\frac{19}{[\,(3x+4)(2x+9)\,]} = \frac{A}{(3x+4)} + \frac{B}{(2x+9)}$$

Multiply through by the red brackets

$$\frac{19\,[\,\dots\dots\,]}{[\,(3x+4)(2x+9)\,]} = \frac{A\,[\,\dots\dots\,]}{(3x+4)} + \frac{B\,[\,\dots\dots\,]}{(2x+9)}$$

$$19 = A(2x+9) + B(3x+4)$$

$$\text{Let } x = -\frac{4}{3} \Rightarrow A = 3$$

$$\text{Let } x = -\frac{9}{2} \Rightarrow B = -2$$

$$\int_0^{\infty} \frac{19}{6x^2 + 35x + 36}\,dx = \int_0^{\infty} \frac{3}{3x+4} - \frac{2}{2x+9}\,dx$$

$$= \lim_{t \to \infty} \int_0^{t} \frac{3}{3x+4} - \frac{2}{2x+9}\,dx$$

$$= \lim_{t \to \infty} \Big[\, ln(3x+4) - ln(2x+9) \,\Big]_0^t$$

$$= \lim_{t \to \infty} \left[ln\left(\frac{3x+4}{2x+9} \right) \right]_0^t$$

$$= \lim_{t \to \infty} \left[ln\left(\frac{3t+4}{2t+9} \right) - ln\left(\frac{4}{9} \right) \right]$$

$$= ln\left(\frac{3}{2} \right) - ln\left(\frac{4}{9} \right)$$

$$= ln\left(\frac{3}{2} \div \frac{4}{9} \right)$$

$$= ln\left(\frac{3}{2} \times \frac{9}{4} \right)$$

$$= ln\left(\frac{3}{2} \right)^3$$

$$= 3\,ln\left(\frac{3}{2} \right) \qquad \square$$

[8 marks]

Answer 3

(i) Let $x = 2\,tan\,\theta$ in which case $\dfrac{dx}{d\theta} = 2\,sec^2\,\theta$

Making the substitution,

$$\int \frac{1}{x^2 + 4}\,dx = \int \frac{1}{4\,tan^2\,x + 4}\,2\,sec^2\,\theta\,d\theta$$

$$= \frac{2}{4}\int \frac{1}{tan^2\,\theta + 1}\,sec^2\,\theta\,d\theta$$

$$= \frac{1}{2}\int \frac{1}{sec^2\,\theta}\,sec^2\,\theta\,d\theta$$

$$= \frac{1}{2}\int 1\,d\theta$$

$$= \frac{1}{2}\,\theta + c$$

$$= \frac{1}{2}\,arctan\left(\frac{x}{2}\right) + c \qquad \square$$

[3 marks]

(ii)

$$\frac{32}{[\,(x^2 + 4)(x + 2)\,]} = \frac{Ax + B}{(x^2 + 4)} + \frac{C}{(x + 2)}$$

Multiply through by the red brackets

$$\frac{32\,[\,\dots\dots\,]}{[\,(x^2 + 4)(x + 2)\,]} = \frac{(Ax + B)\;[\,\dots\dots\,]}{(x^2 + 4)} + \frac{C\,[\,\dots\dots\,]}{(x + 2)}$$

$$32 = (Ax + B)(x + 2) + C\left(x^2 + 4\right)$$

Let $x = -2 \Rightarrow C = 4$

Let $x = 0 \Rightarrow 32 = 2B + 4C \Rightarrow B = 8$

Let $x = 1 \Rightarrow 32 = (A + 8)(3) + 4(5) \Rightarrow A = -4$

$$\int_2^{\infty} \frac{32}{(x^2 + 4)(x + 2)}\,dx = \int_2^{\infty} \frac{8 - 4x}{x^2 + 4} + \frac{4}{x + 2}\,dx$$

$$= \lim_{t \to \infty} \int_2^{t} 8\,\frac{1}{x^2 + 4} - 2\,\frac{2x}{x^2 + 4} + 4\,\frac{1}{x + 2}\,dx$$

$$= \lim_{t \to \infty} \left[4\,arctan\left(\frac{x}{2}\right) - 2ln\left(x^2 + 4\right) + 4ln\,(x + 2)\right]_2^t$$

$$= \lim_{t \to \infty} \left[4\,arctan\left(\frac{t}{2}\right) + 2ln\left(\frac{(\,t + 2\,)^2}{t^2 + 4}\right) - \pi - 2ln\,(2)\right]$$

$$= 4 \times \frac{\pi}{2} + 2\,ln\,(1) - \pi - 2ln\,(2)$$

$$= \pi - 2ln\,(2)$$

[9 marks]

Answer 4

$$\int_{\frac{1}{2}}^{\infty} \frac{1-x^2}{4x^4+x^2}\,dx = \int_{\frac{1}{2}}^{\infty} \frac{1-x^2}{x^2(4x^2+1)}\,dx$$

Partial fractions,

$$\frac{1-x^2}{[\,x^2(4x^2+1)\,]} = \frac{Ax+B}{x^2} + \frac{Cx+D}{4x^2+1}$$

$$\frac{(1-x^2)[\,\ldots\ldots\,]}{[\,x^2(4x^2+1)\,]} = \frac{(Ax+B)[\,\ldots\ldots]}{x^2} + \frac{(Cx+D)[\,\ldots\ldots\,]}{4x^2+1}$$

$$1-x^2 = (Ax+B)(4x^2+1) + (Cx+D)x^2$$

Coefficients of x^0 : $1 = B$

Coefficients of x : $0 = A$

Coefficients of x^2 : $(-1) = 4B + D \Rightarrow D = -5$

Coefficients of x^3 : $0 = A + C \Rightarrow C = 0$

$$\int_{\frac{1}{2}}^{\infty} \frac{1-x^2}{4x^4+x^2}\,dx = \int_{\frac{1}{2}}^{\infty} \frac{1}{x^2}\,dx - \int_{\frac{1}{2}}^{\infty} \frac{5}{4x^2+1}\,dx$$

$$\int_{\frac{1}{2}}^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_{\frac{1}{2}}^{t} x^{-2}\,dt$$

$$= \lim_{t\to\infty} \left[-\frac{1}{x}\right]_{\frac{1}{2}}^{t}$$

$$= \lim_{t\to\infty} \left[-\frac{1}{t} + 2\right]$$

$$= 2$$

Let $x = \frac{1}{2}\tan\theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{2}\sec^2\theta$

$$\int_{\frac{1}{2}}^{\infty} \frac{5}{4x^2+1}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{5}{\tan^2\theta+1} \times \frac{1}{2}\sec^2\theta\,d\theta$$

$$= \frac{5}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1\,d\theta$$

$$= \frac{5}{2}[\,\theta\,]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{5\pi}{8}$$

$$\therefore \int_{\frac{1}{2}}^{\infty} \frac{1-x^2}{4x^4+x^2}\,dx = 2 - \frac{5\pi}{8} \qquad \square$$

[10 marks]

Answer 5

$$\frac{3-x}{[(x-2)^2(x^2+4)]} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+4}$$

$$\frac{(3-x)[\ldots\ldots]}{[(x-2)^2(x^2+4)]} = \frac{A[\ldots\ldots]}{(x-2)} + \frac{B[\ldots\ldots]}{(x-2)^2} + \frac{(Cx+D)[\ldots\ldots]}{x^2+4}$$

$$3-x = A(x-2)\left(x^2+4\right) + B\left(x^2+4\right) + (Cx+D)(x-2)^2$$

$$\text{Let } x = 2 \Rightarrow 1 = 8B \Rightarrow B = \frac{1}{8}$$

$$\begin{aligned}
\text{Let } x = 2\mathrm{i} \Rightarrow 3-2\mathrm{i} &= (2\mathrm{i}C+D)(2\mathrm{i}-2)^2 \\
&= (2\mathrm{i}C+D)(-4-8\mathrm{i}+4) \\
&= (2\mathrm{i}C+D)(-8\mathrm{i}) \\
&= 16C - 8\mathrm{i}D
\end{aligned}$$

$$\therefore C = \frac{3}{16} \text{ and } D = \frac{1}{4}$$

$$\text{Coefficients of } x^3 : 0 = A + C \Rightarrow A = -\frac{3}{16}$$

$$\begin{aligned}
96\int_{2\sqrt{3}}^{\infty} \frac{3-x}{(x-2)^2(x^2+4)}\,dx &= \int_{2\sqrt{3}}^{\infty} \frac{-18}{x-2} + \frac{12}{(x-2)^2} + \frac{18x+24}{x^2+4}\,dx \\
&= \int_{2\sqrt{3}}^{\infty} \frac{-18}{x-2} + \frac{12}{(x-2)^2} + \frac{18x}{x^2+4} + \frac{24}{x^2+4}\,dx
\end{aligned}$$

$$\begin{aligned}
\int_{2\sqrt{3}}^{\infty} \frac{12}{(x-2)^2}\,dx &= 12 \lim_{t\to\infty} \int_{2\sqrt{3}}^{t} (x-2)^{-2}\,dx \\
&= 12 \lim_{t\to\infty} \left[-(x-2)^{-1}\right]_{2\sqrt{3}}^{t} \\
&= 12 \lim_{t\to\infty} \left[-\frac{1}{t-2} + \frac{1}{2\sqrt{3}-2}\right] \\
&= \frac{6}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} \\
&= \frac{6(\sqrt{3}+1)}{3-1} \\
&= 3\sqrt{3}+3
\end{aligned}$$

$$\int_{2\sqrt{3}}^{\infty} \frac{18x}{x^2 + 4} - \frac{18}{x - 2}\, dx = \underset{t \to \infty}{limit} \int_{2\sqrt{3}}^{t} 9 \times \frac{2x}{x^2 + 4} - 18 \times \frac{1}{x - 2}\, dx$$

$$= \underset{t \to \infty}{limit} \left[9\, ln\left(x^2 + 4 \right) - 18\, ln\left(x - 2 \right) \right]_{2\sqrt{3}}^{t}$$

$$= 9 \underset{t \to \infty}{limit} \left[ln\left(\frac{x^2 + 4}{(x - 2)^2} \right) \right]_{2\sqrt{3}}^{t}$$

$$= 9 \underset{t \to \infty}{limit} \left[ln\left(\frac{t^2 + 4}{(t - 2)^2} \right) - ln\left(\frac{16}{4(\sqrt{3} - 1)^2} \right) \right]$$

$$= 9 \left[ln\, 1 - ln\left(\frac{4}{4 - 2\sqrt{3}} \right) \right]$$

$$= 9 \left[0 - ln\left(\frac{2}{2 - \sqrt{3}} \right) \right]$$

$$= 9\, ln \left(\frac{2 - \sqrt{3}}{2} \right)$$

$$= 9\, ln\left(1 - \frac{\sqrt{3}}{2} \right)$$

Let $x = 2\, tan\, \theta$ in which case $\frac{dx}{d\theta} = 2\, sec^2\, \theta$

The limits $2\sqrt{3}, \infty$ become $\frac{\pi}{3}, \frac{\pi}{2}$

$$\int_{2\sqrt{3}}^{\infty} \frac{24}{x^2 + 4}\, dx = 6 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{4}{4\, tan^2\, \theta + 4}\, 2\, sec^2\, \theta\, d\theta$$

$$= 12 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{sec^2\, \theta}\, sec^2\, \theta\, d\theta$$

$$= 12 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1\, d\theta$$

$$= 12 \left[\theta \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

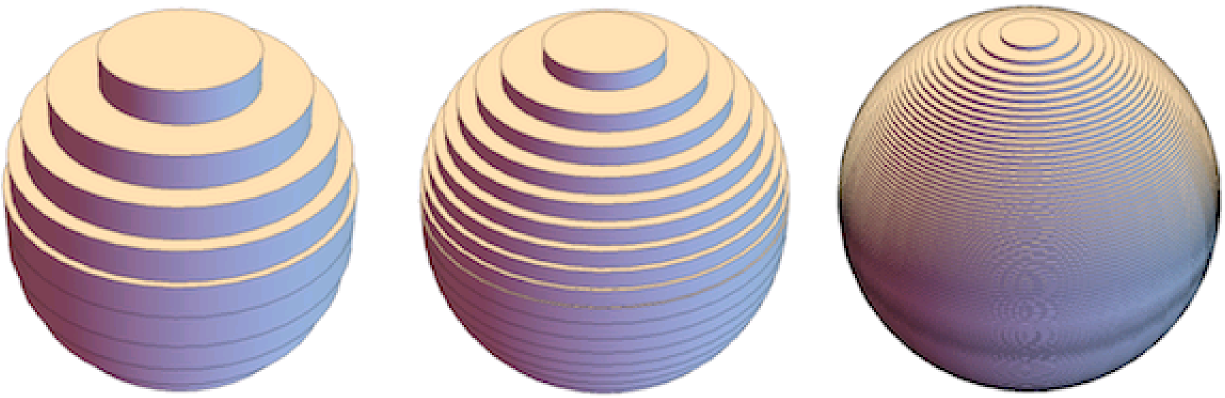
$$= 12 \left[\frac{\pi}{2} - \frac{\pi}{3} \right] = 2\pi$$

Thus $96 \int_{2\sqrt{3}}^{\infty} \frac{3 - x}{(x - 2)^2 (x^2 + 4)}\, dx = 2\pi + 3 + 3\sqrt{3} + 9\, ln\left(1 - \frac{\sqrt{3}}{2} \right)$ □

[12 marks]

5.1 Volume of Revolution

If a solid, when sliced, has cross sections that are circular then its volume may be found using the idea of a profile curve being spun about either the x or the y-axis. This is called the "discs" method of finding a volume as the integration is finding the sum of a number cylinders, as that number tends to infinity, and as the thickness of the discs involved correspondingly tends to zero. In the limit, the volume found is exact.



As the number of discs in increased the volume of this sphere is better approximated by the sum of the volumes of those discs
From left to right: 10 discs, 20 discs and 100 discs
Image made in Wolfram Alpha by Martin Hansen

In the A-Level Further Mathematics course this is the only method taught but there other techniques such as the "shell" method or the "slab" method that can tackle a variety of other shapes; They are easily learnt should the need arise.

For the "discs" method the key result is the following;

The Volume of Revolution Formulae

- When $y = f(x)$ is rotated $2\pi^c$ about the x-axis on interval $a \leqslant x \leqslant b$

$$Volume = \pi \int_a^b y^2 \, dx$$

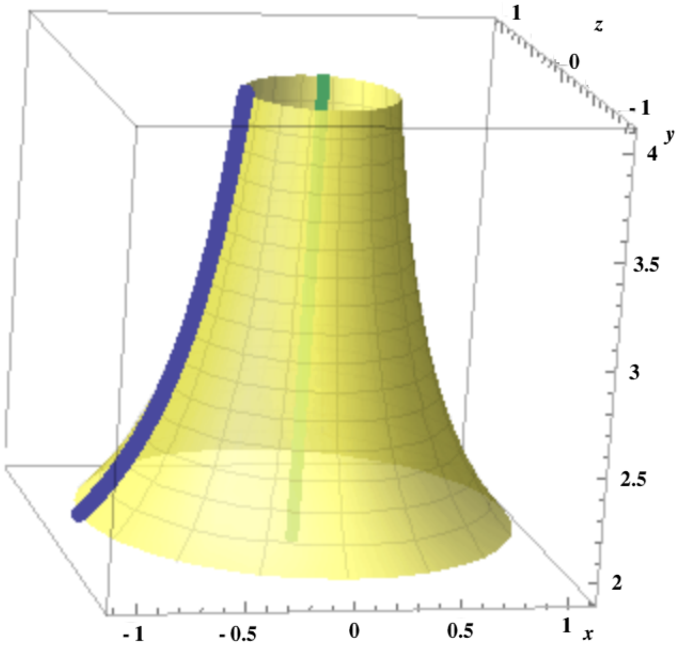
- When $x = f(y)$ is rotated $2\pi^c$ about the y-axis on interval $a \leqslant y \leqslant b$

$$Volume = \pi \int_a^b x^2 \, dy$$

In this lesson the focus is upon questions that involve improper integrals.
Any of the integration techniques studied previously may be called upon.

5.2 Example

Find the volume swept out when the profile curve $x = \dfrac{1}{\sqrt{y}\ ln\, y}$ is rotated $2\pi^c$ about the y-axis, $2 \leqslant y < \infty$. The lower end of the solid is depicted below.



Teaching Video : http://www.NumberWonder.co.uk/v9100/5.mp4

[5 marks]

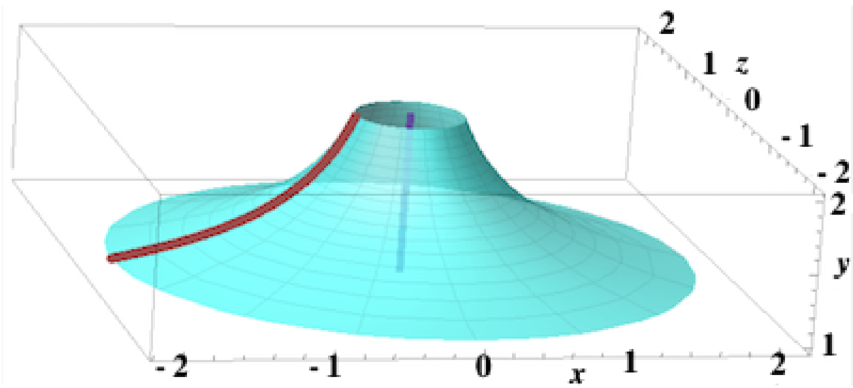
5.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

Find the volume swept out when the profile curve $x = \dfrac{y+1}{y^3}$ is rotated $2\pi^c$ about the y-axis, $1 \leqslant y < \infty$. The lower end of the solid is depicted below.



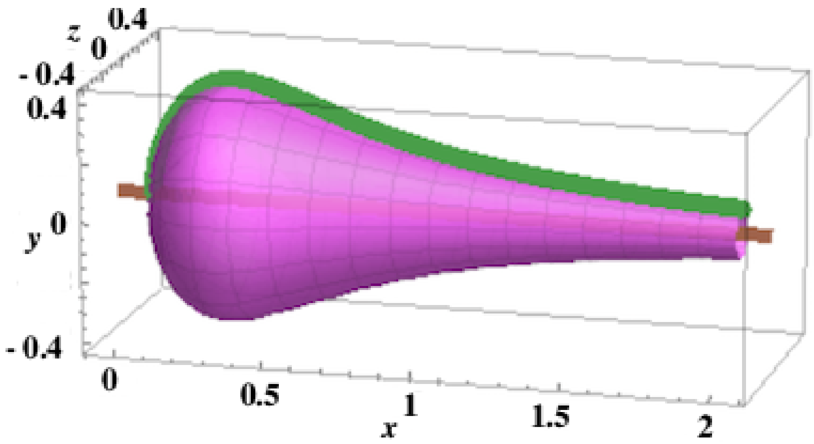
[5 marks]

Question 2

Find the volume swept out when the profile curve $y = \dfrac{\sqrt{x}}{1 + 4x^2}$ is rotated $2\pi^c$ about the x-axis, $0 \leqslant x < \infty$.

The integration involved is a "chain rule backwards".

The lower end of the solid is depicted below.

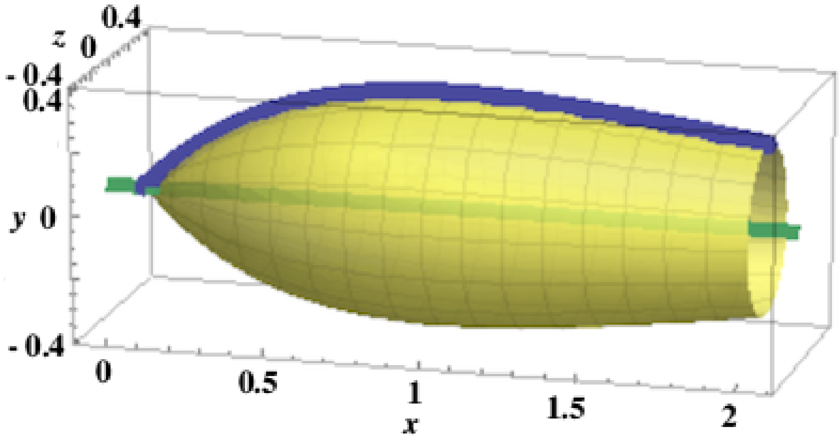


[5 marks]

Question 3

Find the volume swept out when the profile curve $y = x\,e^{-x}$ is rotated $2\pi^c$ about the x-axis, $0 \leqslant x < \infty$. The lower end of the solid is depicted below.

You may use without proof the fact that $\underset{t \to \infty}{limit}\; t^n e^{-2t} = 0$ for $n \in \mathbb{Z}^+$



[7 marks]

Question 4

(i) By means of a suitable trigonometric substitution, which you should clearly state, determine the value of the improper integral,

$$\int_0^\infty \frac{1}{x^2 + 9}\,dx$$

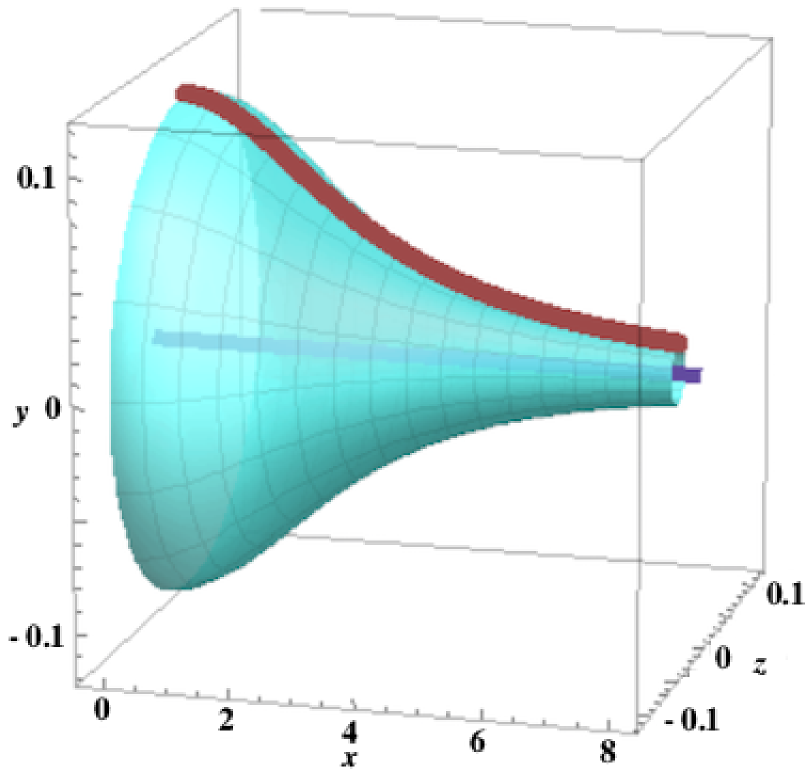
[4 marks]

(ii) By thinking of $\int \frac{1}{x^2 + 9}\,dx$ as $\int \left(\frac{1}{x^2 + 9}\right)(1)\,dx$ and using integration by parts show that,

$$\int \frac{1}{x^2 + 9}\,dx = \frac{x}{x^2 + 9} + 2\int \frac{1}{x^2 + 9}\,dx - 18\int \frac{1}{(x^2 + 9)^2}\,dx$$

[4 marks]

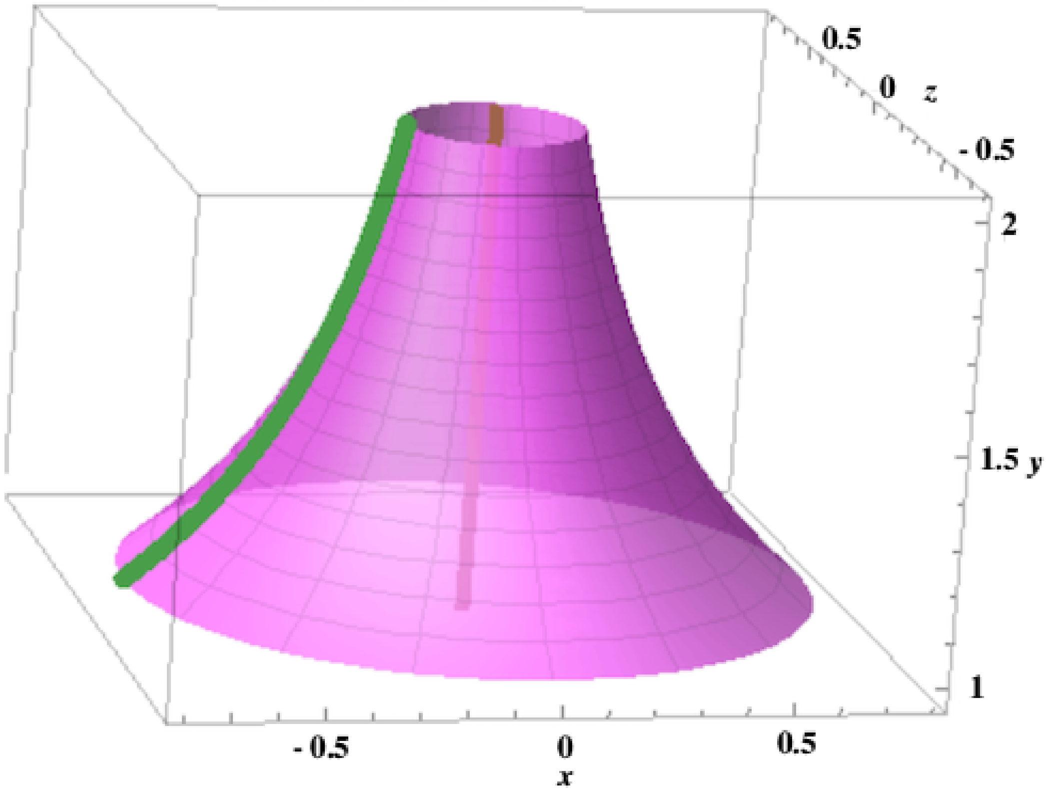
(iii) Find the volume swept out when the profile curve $y = \dfrac{1}{x^2 + 9}$ is rotated by $2\pi^c$ about the x-axis, $0 \leqslant x < \infty$.



[4 marks]

Question 5

Find the volume swept out when the profile curve $x = \dfrac{\sqrt{4y+1}}{y(y+1)}$ is rotated $2\pi^c$ about the y-axis, $1 \leqslant x < \infty$. The lower end of the solid is depicted below.



[11 marks]

5.4 Answers

5.4.1 Solution to Example 5.2 (Teaching Video)

$$\begin{aligned} Volume &= \pi \int_a^b x^2 \, dy \\ &= \pi \int_2^{\infty} \frac{1}{y\,(\ln y)^2} \, dy \\ &= \pi \lim_{t \to \infty} \int_2^t \left(\frac{1}{y}\right)(\ln y)^{-2} \, dy \end{aligned}$$

Notice that this is of the form $f'(y)\,[\,f(y)\,]^n$ which is a "chain rule backwards"

$$\begin{aligned} &= \pi \lim_{t \to \infty} \Big[-(\ln y)^{-1} \Big]_2^t \\ &= \pi \lim_{t \to \infty} \left[-\frac{1}{\ln t} + \frac{1}{\ln 2} \right] \\ &= \frac{\pi}{\ln 2} \end{aligned}$$

[5 marks]

5.4.2 Solutions to Exercise 5.3

Answer 1

$$\begin{aligned} Volume &= \pi \int_a^b x^2 \, dy \\ &= \pi \int_1^{\infty} \frac{(y+1)^2}{y^6} \, dy \\ &= \pi \lim_{t \to \infty} \int_1^t \frac{y^2 + 2y + 1}{y^6} \, dy \\ &= \pi \lim_{t \to \infty} \int_1^t y^{-4} + 2y^{-5} + y^{-6} \, dy \\ &= \pi \lim_{t \to \infty} \left[-\frac{1}{3y^3} - \frac{2}{4y^4} - \frac{1}{5y^5} \right]_1^t \\ &= \pi \lim_{t \to \infty} \left[-\frac{1}{3t^3} - \frac{1}{2t^4} - \frac{1}{5t^5} + \frac{1}{3} + \frac{1}{2} + \frac{1}{5} \right] \\ &= \frac{31\pi}{30} \end{aligned}$$

[5 marks]

Answer 2

$$
\begin{aligned}
Volume &= \pi \int_a^b y^2 \, dx \\
&= \pi \int_0^\infty \frac{x}{(1 + 4x^2)^2} \, dx \\
&= \frac{\pi}{8} \lim_{t \to \infty} \int_0^t 8x \left(1 + 4x^2 \right)^{-2} \, dx
\end{aligned}
$$

Notice that this is of the form $f'(x) [f(x)]^n$ which is a "chain rule backwards"

$$
\begin{aligned}
&= \frac{\pi}{8} \lim_{t \to \infty} \left[- \frac{1}{1 + 4x^2} \right]_0^t \\
&= \frac{\pi}{8} \left[- \frac{1}{1 + 4t^2} + 1 \right] \\
&= \frac{\pi}{8}
\end{aligned}
$$

[5 marks]

Answer 3

$$
\begin{aligned}
Volume &= \pi \int_a^b y^2 \, dx \\
&= \pi \int x^2 e^{-2x} \, dx \\
& \qquad \textbf{\textit{L}} \qquad \textbf{\textit{I}} \qquad\qquad \textbf{\textit{D}} \qquad \textbf{\textit{I}} \\
&= \pi \left[x^2 \left(-\frac{1}{2} \right) e^{-2x} - \int 2x \left(-\frac{1}{2} \right) e^{-2x} \, dx \right] \\
&= \pi \left[-\frac{1}{2} x^2 e^{-2x} + \int x \, e^{-2x} \, dx \right] \\
& \qquad\qquad\qquad \textbf{\textit{L}} \qquad \textbf{\textit{I}} \qquad\qquad \textbf{\textit{D}} \qquad \textbf{\textit{I}} \\
&= \pi \left[-\frac{1}{2} x^2 e^{-2x} + x \left(-\frac{1}{2} \right) e^{-2x} - \int 1 \left(-\frac{1}{2} \right) e^{-2x} \, dx \right] \\
&= \frac{\pi}{2} \left[-x^2 e^{-2x} - x e^{-2x} + \int e^{-2x} \, dx \right] \\
&= \frac{\pi}{2} \left[-x^2 e^{-2x} - x e^{-2x} - \frac{1}{2} e^{-2x} + c \right]
\end{aligned}
$$

$$
\begin{aligned}
\therefore \ \pi \int_0^\infty x^2 e^{-2x} \, dx &= \frac{\pi}{2} \lim_{t \to \infty} \left[-t^2 e^{-2t} - t e^{-2t} - \frac{1}{2} e^{-2t} + \frac{1}{2} \right]_0^t \\
&= \frac{\pi}{4}
\end{aligned}
$$

[7 marks]

Use L'Hôspital's Rule to prove that $\lim_{t \to \infty} t^n e^{-2t} = 0$ for $n \in \mathbb{Z}^+$

Answer 4

(i) Let $x = 3\tan\theta$ in which case $\dfrac{dx}{d\theta} = 3\sec^2\theta$

Making the substitution,

$$\int \frac{1}{x^2 + 9}\,dx = \int \frac{1}{9\tan^2 x + 9}\,3\sec^2\theta\,d\theta$$
$$= \frac{3}{9}\int \frac{1}{\tan^2\theta + 1}\sec^2\theta\,d\theta$$
$$= \frac{1}{3}\int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta$$
$$= \frac{1}{3}\int 1\,d\theta$$
$$= \frac{1}{3}\theta + c$$
$$= \frac{1}{3}\arctan\left(\frac{x}{3}\right) + c$$

Thus,

$$\int_0^{\infty} \frac{1}{x^2 + 9}\,dx = \frac{1}{3}\underset{t\to\infty}{limit}\left[\arctan\left(\frac{x}{3}\right)\right]_0^t$$
$$= \frac{1}{3}\underset{t\to\infty}{limit}\left[\arctan\left(\frac{t}{3}\right) - \arctan\left(\frac{0}{3}\right)\right]$$
$$= \frac{1}{3}\left[\frac{\pi}{2} - 0\right]$$
$$= \frac{\pi}{6}$$

[4 marks]

(ii)

$$\int \frac{1}{x^2 + 9}\,dx = \int\left(\frac{1}{x^2 + 9}\right)(1)\,dx$$

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$$= \left(\frac{1}{x^2 + 9}\right)x - \int(-1)\left(x^2 + 9\right)^{-2} 2x \times x\,dx$$
$$= \left(\frac{x}{x^2 + 9}\right) + 2\int \frac{x^2 + 9 - 9}{(x^2 + 9)^2}\,dx$$
$$= \left(\frac{x}{x^2 + 9}\right) + 2\int \frac{x^2 + 9}{(x^2 + 9)^2}\,dx - 2\int \frac{9}{(x^2 + 9)^2}\,dx$$

$$\int \frac{1}{x^2 + 9}\,dx = \frac{x}{x^2 + 9} + 2\int \frac{1}{x^2 + 9}\,dx - 18\int \frac{1}{(x^2 + 9)^2}\,dx \qquad \square$$

[4 marks]

(iii)

$$Volume = \pi \int_a^b y^2\, dx$$

$$= \pi \int_0^\infty \frac{1}{(x^2 + 9)^2}\, dx$$

$$= \frac{\pi}{18} \lim_{t \to \infty} \left[\frac{x}{x^2 + 9} \right]_0^t + \frac{\pi}{18} \lim_{t \to \infty} \int_0^t \frac{1}{x^2 + 9}\, dx \qquad \text{from part (ii)}$$

$$= \frac{\pi}{18} \lim_{t \to \infty} \left[\frac{t}{t^2 + 9} - 0 \right] + \frac{\pi}{18} \times \frac{\pi}{6} \qquad \text{from part (i)}$$

$$= \frac{\pi^2}{108}$$

[4 marks]

Answer 5

Using Partial Fractions,

$$\frac{4y + 1}{\color{red}{[}\, y^2 (y + 1)^2 \,\color{red}{]}} = \frac{Ay + B}{y^2} + \frac{Cy + D}{(y + 1)^2}$$

Multiplying through by the red bracket

$$\frac{(4y + 1)\color{red}{[\,\dots\dots\,]}}{\color{red}{[}\, y^2 (y + 1)^2 \,\color{red}{]}} = \frac{(Ay + B)\color{red}{[\,\dots\dots\,]}}{y^2} + \frac{(Cy + D)\,\color{red}{[\,\dots\dots\,]}}{(y + 1)^2}$$

$$4y + 1 = (Ay + B)(y + 1)^2 + (Cy + D)y^2$$

$$\text{Let } y = 0 \Rightarrow 1 = B$$

$$\text{Let } y = -1 \Rightarrow -3 = (-C + D)(1) \Rightarrow D = C - 3$$

$$\text{Coefficients of } y^3 : 0 = A + C$$

$$\text{Let } y = 1 \Rightarrow 5 = (A + B)\,4 + (C + D)$$

$$5 = -4C + 4 + (C + C - 3)$$

$$4 = -2C$$

$$C = -2$$

$$\therefore A = 2,\ B = 1,\ C = -2 \text{ and } D = -5$$

$$\therefore \frac{4y + 1}{y^2 (y + 1)^2} = \frac{2y + 1}{y^2} - \frac{2y + 5}{(y + 1)^2}$$

$$= \frac{2}{y} + \frac{1}{y^2} - \frac{2y + 2}{y^2 + 2y + 1} - 3(1)(y + 1)^{-2}$$

$$\pi \int_1^{\infty} \frac{4y + 1}{y^2 (y + 1)^2}\, dy = \pi \underset{t \to \infty}{limit} \int_1^{t} \frac{2}{y} + \frac{1}{y^2} - \frac{2y + 2}{y^2 + 2y + 1} - 3(1)(y + 1)^{-2}\, dy$$

$$= \pi \underset{t \to \infty}{limit} \left[2\, ln\, y - \frac{1}{y} - ln\left(y^2 + 2y + 1 \right) + \frac{3}{y + 1} \right]_1^t$$

$$= \pi \underset{t \to \infty}{limit} \left[ln\left(\frac{y^2}{y^2 + 2y + 1} \right) - \frac{1}{y} + \frac{3}{y + 1} \right]_1^t$$

$$= \pi \underset{t \to \infty}{limit} \left[ln\left(\frac{t^2}{t^2 + 2t + 1} \right) - \frac{1}{t} + \frac{3}{t + 1} - ln\left(\frac{1}{4} \right) + 1 - \frac{3}{2} \right]$$

$$= \pi \left[ln\, 1 - 0 + 0 - ln\left(\frac{1}{4} \right) + 1 - \frac{3}{2} \right]$$

$$= \pi \left[-ln\, 4^{-1} - \frac{1}{2} \right]$$

$$= \pi \left(ln\, 4 - \frac{1}{2} \right)$$

$$= \pi \left(2\, ln\, 2 - \frac{1}{2} \right)$$

[11 marks]

Lesson 6

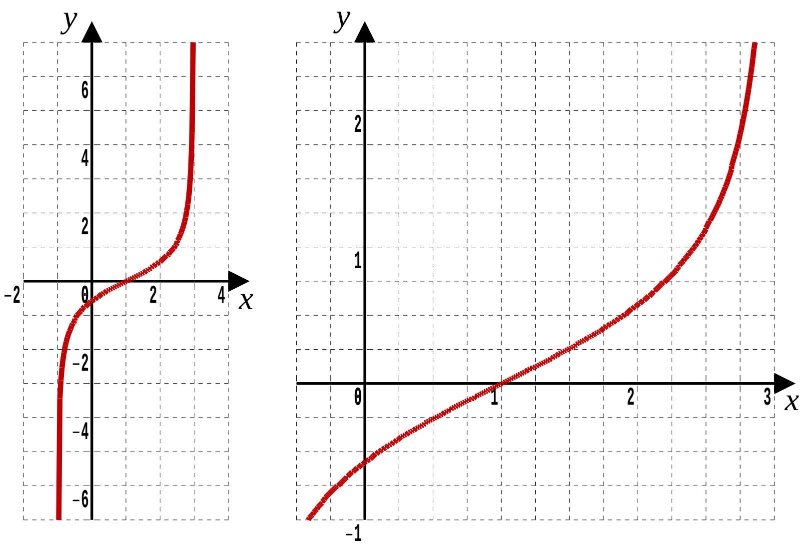
6.1 Revision

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

Two views of the graph of $f(x) = \dfrac{x - 1}{\sqrt{3 + 2x - x^2}} dx$ are presented below



Determine the value of the improper integral $\displaystyle\int_1^3 \frac{x - 1}{\sqrt{3 + 2x - x^2}} dx$

[6 marks]

Question 2

Further A-Level Examination Question from June 2016, FP2, Q1 (b), (OCR)

Find, in exact form, the value of the following integral,

$$\int_0^{\frac{3}{4}} \frac{1}{\sqrt{3 - 4x^2}}\, dx$$

Hint : Use the substitution $x = \frac{\sqrt{3}}{2} \sin\theta$

[6 marks]

Question 3

(i) Find $\int \frac{1}{x(x+5)}\, dx$

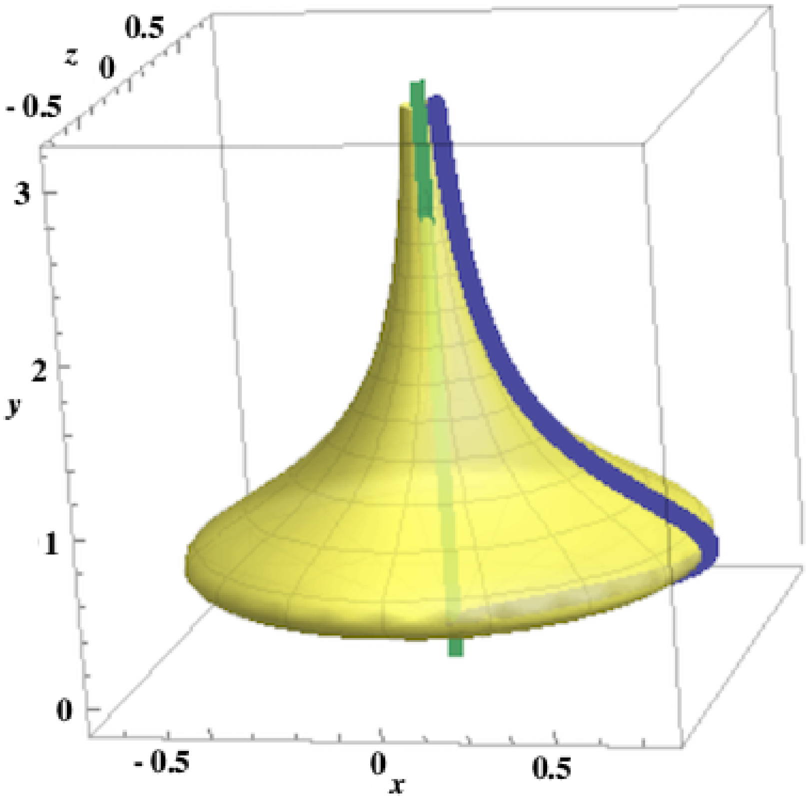
[4 marks]

(ii) Hence show that $\int_5^{\infty} \frac{1}{x^2 + 5x}\, dx$ converges and find its value

[3 marks]

Question 4

Find the volume swept out when the profile curve $x = \dfrac{y}{1 + 5y^3}$ is rotated $2\pi^c$ about the y-axis, $0 \leqslant y < \infty$. The lower end of the solid is depicted below.



[6 marks]

Question 5

Further A-Level Examination Question from January 2013, FP2, Q1(a) (OCR)

(i) Differentiate with respect to x the equation $a \tan y = x$ (where a is a constant) and hence show that the derivative of $\arctan\left(\frac{x}{a}\right)$ is $\frac{a}{a^2 + x^2}$

[3 marks]

(ii) By first expressing $x^2 - 4x + 8$ in completed square form, evaluate the integral $\int_0^4 \frac{1}{x^2 - 4x + 8}\, dx$ giving your answer exactly.

[4 marks]

(iii) Use integration by parts to find $\int arctan\, x\, dx$

[4 marks]

(iv) What does your part (iii) answer allow you to deduce regarding the improper integral $\int_0^{\infty} arctan\, x\, dx$? Give a reason for your answer.

[2 marks]

(v) Sketch the graph of $y = arctan\, x$ and explain how this backs up your part (iv) answer.

[2 marks]

6.2 Answers

Answer 1

$$\int_1^3 \frac{x - 1}{\sqrt{3 + 2x - x^2}}dx = \left(-\frac{1}{2}\right) \underset{t \to 3^-}{limit} \int_1^t \frac{2 - 2x}{\sqrt{3 + 2x - x^2}}dx$$

$$= \left(-\frac{1}{2}\right) \underset{t \to 3^-}{limit} \int_1^t (2 - 2x)(3 + 2x - x^2)^{-\frac{1}{2}} dx$$

Notice that this is of the form $f'(x)[f(x)]^n$ which is a "chain rule backwards".

$$= \left(-\frac{1}{2}\right) \underset{t \to 3^-}{limit} \left[2(3 + 2x - x^2)^{\frac{1}{2}} \right]_1^t$$

$$= (-1) \underset{t \to 3^-}{limit} \left[\sqrt{3 + 2t - t^2} - \sqrt{3 + 2 - 1} \right]$$

$$= (-1)\left[0 - \sqrt{4} \right]$$

$$= 2$$

[6 marks]

Answer 2

Let $x = \frac{\sqrt{3}}{2} sin\,\theta$ in which case $\frac{dx}{d\theta} = \frac{\sqrt{3}}{2} cos\,\theta$

$$\int \frac{1}{\sqrt{3 - 4x^2}} dx = \int \frac{1}{\sqrt{3 - 3\,sin^2\theta}} \times \frac{\sqrt{3}}{2} cos\,\theta\; d\theta$$

$$= \int \frac{1}{\sqrt{3(1 - sin^2\theta)}} \times \frac{\sqrt{3}}{2} cos\,\theta\; d\theta$$

$$= \int \frac{1}{\sqrt{3(cos^2\theta)}} \times \frac{\sqrt{3}}{2} cos\,\theta\; d\theta$$

$$= \frac{1}{2} \int 1\, d\theta$$

$$= \frac{1}{2}\theta + c$$

$$= \frac{1}{2} arcsin\left(\frac{2x}{\sqrt{3}}\right) + c$$

$$\int_0^{\frac{3}{4}} \frac{1}{\sqrt{3 - 4x^2}} dx = \frac{1}{2}\left[arcsin\left(\frac{2x}{\sqrt{3}}\right) \right]_0^{\frac{3}{4}}$$

$$= \frac{1}{2}\left[arcsin\left(\frac{\sqrt{3}}{2}\right) - arcsin(0) \right]$$

$$= \frac{\pi}{6}$$

[6 marks]

Answer 3

(i) Making use of Partial Fractions,

$$\frac{1}{[\,x(x+5)\,]} = \frac{A}{x} + \frac{B}{x+5}$$

Multiply through by the red bracket

$$\frac{1\,[\,\dots\dots\,]}{[\,x(x+5)\,]} = \frac{A\,[\,\dots\dots\,]}{x} + \frac{B\,[\,\dots\dots\,]}{x+5}$$

$$1 = A(x+5) + Bx$$

$$\text{Let } x = 0 \Rightarrow A = \frac{1}{5}$$

$$\text{Let } x = -5 \Rightarrow B = -\frac{1}{5}$$

$$\int \frac{1}{x(x+5)}\,dx = \frac{1}{5}\int \frac{1}{x} - \frac{1}{x+5}\,dx$$

$$= \frac{1}{5}\left[\, ln\,x - ln(x+5)\,\right]$$

$$= \frac{1}{5}\,ln\left(\frac{x}{x+5}\right) + c$$

[4 marks]

(ii)

$$\int_5^{\infty} \frac{1}{x^2+5x}\,dx = \frac{1}{5}\lim_{t\to\infty}\left[\, ln\left(\frac{x}{x+5}\right)\right]_5^t$$

$$= \frac{1}{5}\lim_{t\to\infty}\left[\, ln\left(\frac{t}{t+5}\right) - ln\left(\frac{5}{5+5}\right)\right]$$

$$= \frac{1}{5}\left[\, ln\,1 - ln\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{5}\,ln\,2$$

That is, the integral converges and has a value of $\frac{1}{5}\,ln\,2$

[3 marks]

Answer 4

$$Volume \;=\; \pi \int_a^b x^2 \, dy$$

$$= \; \pi \int_0^{\infty} \frac{y^2}{(\,1 \,+\, 5\,y^3\,)^2} \; dy$$

$$= \; \frac{\pi}{15} \; \underset{t \to \infty}{limit} \; \int_0^t 15\,y^2 \left(\,1 \,+\, 5\,y^3\right)^{-2} \; dy$$

Notice that this is of the form $f'(x)\,[\,f(x)\,]^n$ which is a "chain rule backwards"

$$= \; \frac{\pi}{15} \; \underset{t \to \infty}{limit} \; \Big[-\left(\,1 \,+\, 5\,y^3\right)^{-1} \Big]_0^t$$

$$= \; \frac{\pi}{15} \; \underset{t \to \infty}{limit} \; \left[-\,\frac{1}{1 \,+\, 5\,t^3} \,+\, \frac{1}{1} \right]$$

$$= \; \frac{\pi}{15}$$

[6 marks]

Answer 5

(i)

$$a\,tan\,y \;=\; x$$

Differentiate implicitly with respect to *x*, as instructed

$$a\,sec^2\,y\,\frac{dy}{dx} \;=\; 1$$

$$\frac{dy}{dx} \;=\; \frac{1}{a\,sec^2\,y} \qquad \text{use } 1 \,+\, tan^2\,y \;=\; sec^2\,y$$

$$= \; \frac{1}{a \,+\, a\,tan^2\,y} \times \frac{a}{a}$$

$$= \; \frac{a}{a^2 \,+\, (\,a\,tan\,y\,)^2} \qquad \text{use } a\,tan\,y \;=\; x$$

$$= \; \frac{a}{a^2 \,+\, x^2}$$

$$\therefore \text{ If } \; y \;=\; arctan\left(\frac{x}{a}\right) \; \text{ then } \; \frac{dy}{dx} \;=\; \frac{a}{a^2 \,+\, x^2} \qquad \square$$

[3 marks]

(ii) From part (i),

$$\int \frac{a}{a^2 + x^2}\, dx = arctan\left(\frac{x}{a} \right) + c$$

$$\int \frac{1}{x^2 - 4x + 8}\, dx = \int \frac{1}{(x-2)^2 + 4}\, dx$$

$$= \frac{1}{2} \int \frac{2}{2^2 + (x-2)^2}\, dx$$

$$= \frac{1}{2}\, arctan\left(\frac{x-2}{2} \right) + c$$

$$\int_0^4 \frac{1}{x^2 - 4x + 8}\, dx = \frac{1}{2} \left[arctan\left(\frac{x-2}{2} \right) \right]_0^4$$

$$= \frac{1}{2} \left[arctan\, 1 - arctan\,(-1) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right]$$

$$= \frac{\pi}{4}$$

[4 marks]

(iii)

$$\int arctan\, x\, dx = \int arctan\, x\ (1)\ dx$$

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$$= arctan\, x\ (x) - \int \frac{1}{1 + x^2}\ (x)\ dx$$

$$= x\, arctan\, x - \frac{1}{2} \int 2x \left(1 + x^2 \right)^{-1} dx$$

Notice that this is of the form $f'(x)[\,f(x)\,]^n$ which is a “chain rule backwards”

$$= x\, arctan\, x - \frac{1}{2}\, ln\left(1 + x^2 \right) + c$$

[4 marks]

(iv)

$$\int_0^{\infty} arctan\, x\, dx = \underset{t \to \infty}{limit} \left[x\, arctan\, x - \frac{1}{2}\, ln\left(1 + x^2 \right) \right]_0^t$$

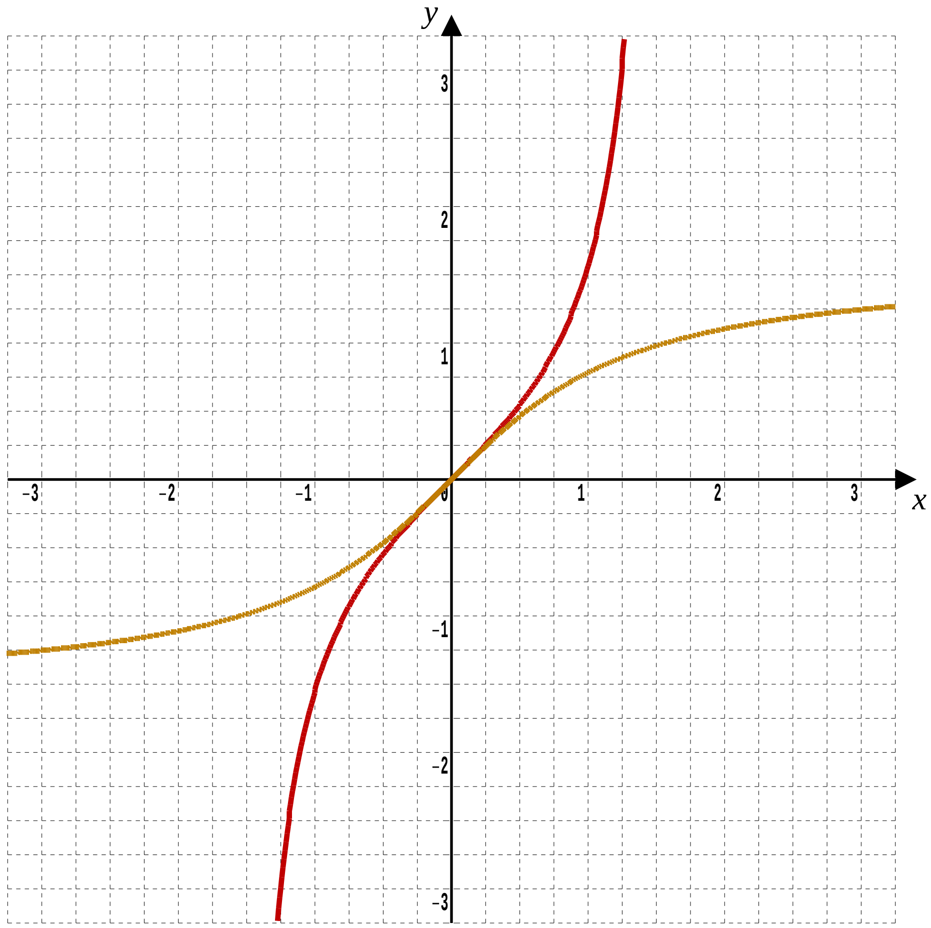
$$= \underset{t \to \infty}{limit} \left[t\, arctan\, t - \frac{1}{2}\, ln\left(1 + t^2 \right) - 0 - 0 \right]$$

As $t \to \infty$, $t\, arctan\, t \to \infty$ and $ln\left(1 + t^2 \right) \to \infty$

Thus this improper integral is not convergent and so has no value

[2 marks]

(v)



Red : $y = \tan x$ Gold : $y = \arctan x$

The graph of $\arctan x \;\rightarrow\; \dfrac{\pi}{2}$ as $x \;\rightarrow\; \infty$ with the result that the area given by $\int_0^t \arctan x \, dx$ will be roughly the same as that of a rectangle measuring $\left(\dfrac{\pi}{2}\right)$ by t for large values of t

Thus $Area \;\rightarrow\; \infty$ as $t \;\rightarrow\; \infty$ further endorsing the fact that the integral is divergent.

[2 marks]

Lesson 7

Further A-Level Pure Mathematics, Core 2

Improper Integrals

7.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 33

Question 1

Further A-Level Examination Question from June 2011, FP2, Q1 (b) (ii) (OCR)

Find, in exact form, the value of the integral,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(1 + 4x^2)^{\frac{3}{2}}}\, dx$$

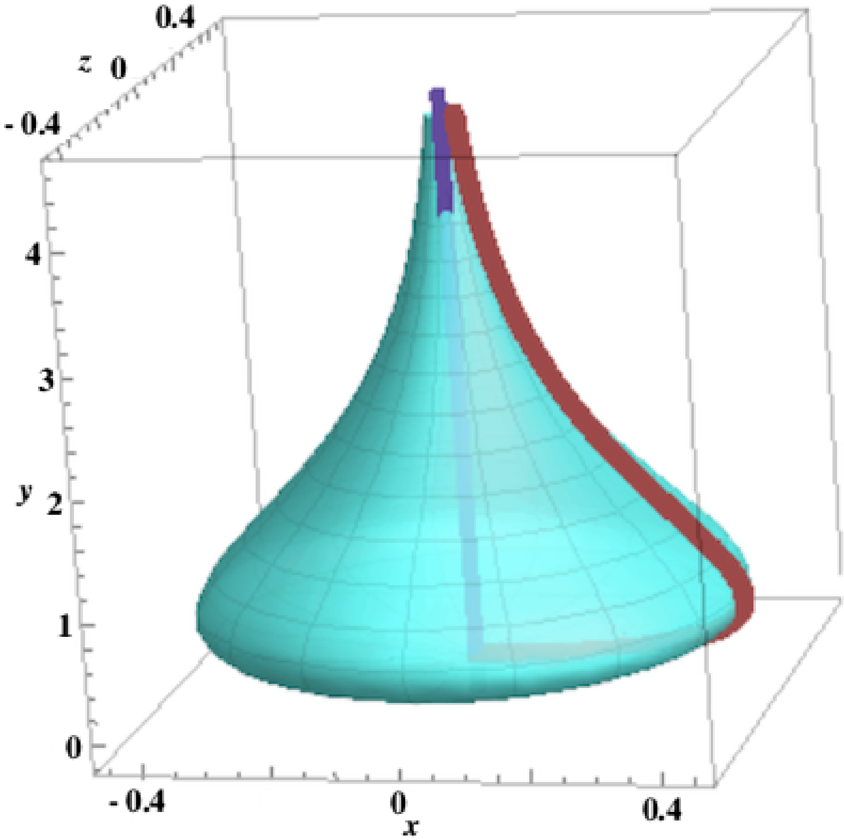
Hint : Use the substitution $x = \frac{1}{2} \tan \theta$

[6 marks]

Question 2

Find the volume swept out when the profile curve $x = \sqrt{y}\, e^{-y}$ is rotated $2\pi^c$ about the y-axis, $0 \leqslant y < \infty$. The lower end of the solid is depicted below.

You may use without proof the fact that $\underset{t \to \infty}{limit}\ t\, e^{-2t} = 0$



[6 marks]

Question 3

Further A-Level Examination Question from June 2019, Paper 1, Q2 (Edexcel)

Show that,

$$\int_0^{\infty} \frac{8x - 12}{(2x^2 + 3)(x + 1)}\, dx = \ln k$$

where k is a rational number to be found

[7 marks]

Question 4

Further A-Level Examination Question from January 2010, FP2, Q1(a) (OCR)

(i) Given that $y = arctan\sqrt{x}$ find $\dfrac{dy}{dx}$ giving your answer in terms of x

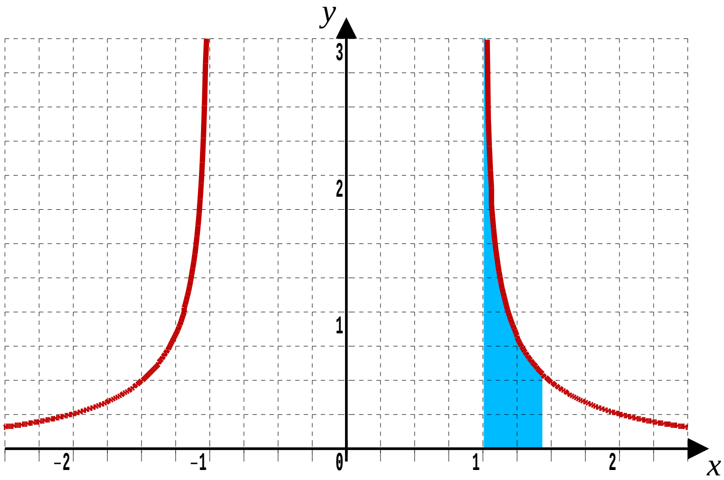
[3 marks]

(ii) Hence show that $\displaystyle\int_0^1 \frac{1}{\sqrt{x}\,(x+1)}\,dx = \frac{\pi}{2}$

[3 marks]

Question 5

The area shaded blue, and extending upwards indefinitely, is represented by the improper integral $\int_1^{\sqrt{2}} \frac{1}{\sqrt{x^4 - 1}}\, dx$



Use the substitution $x^2 = \sec\theta$ to show that the improper integral can be converted to a proper one with limits that show the original integral is convergent.

Note : You are not asked to find the value of the improper integral, which is difficult.

[8 marks]

Answer 1

Let $x = \frac{1}{2} tan\,\theta$ in which case $\frac{dx}{d\theta} = \frac{1}{2} sec^2\,\theta$

$$\int \frac{1}{(1 + 4x^2)^{\frac{3}{2}}}\,dx = \int \frac{1}{(1 + tan^2\,\theta)^{\frac{3}{2}}} \times \frac{1}{2} sec^2\,\theta\ d\theta$$

$$= \int \frac{1}{(sec^2\,\theta)^{\frac{3}{2}}} \times \frac{1}{2} sec^2\,\theta\ d\theta$$

$$= \frac{1}{2} \int \frac{1}{sec^3\,\theta} \times sec^2\,\theta\,d\theta$$

$$= \frac{1}{2} \int cos\,\theta\,d\theta$$

$$= \frac{1}{2} sin\,\theta + c$$

$$= \frac{1}{2} sin(arctan(2x)) + c$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(1 + 4x^2)^{\frac{3}{2}}}\,dx = \frac{1}{2}\left[sin(arctan(2x)) \right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[sin\left(\frac{\pi}{4}\right) - sin\left(-\frac{\pi}{4}\right) \right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right]$$

$$= \frac{\sqrt{2}}{2}$$

[6 marks]

Answer 2

$$Volume = \pi \int_a^b x^2 \, dy$$

$$= \pi \int_0^\infty y\, e^{-2y} \, dy$$

$$= \pi \lim_{t \to \infty} \int_0^t y\, e^{-2y} \, dy$$

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$$= \pi \lim_{t \to \infty} \left[y \left(-\frac{1}{2}\right) e^{-2y} - \int 1 \left(-\frac{1}{2}\right) e^{-2y} \, dy \right]_0^t$$

$$= \frac{\pi}{2} \lim_{t \to \infty} \left[-y\, e^{-2y} + \int e^{-2y} \, dy \right]_0^t$$

$$= \frac{\pi}{2} \lim_{t \to \infty} \left[-y\, e^{-2y} + \left(-\frac{1}{2}\right) e^{-2y} \right]_0^t$$

$$= \frac{\pi}{2} \lim_{t \to \infty} \left[-t\, e^{-2t} - \frac{1}{2} e^{-2t} + 0 + \frac{1}{2} \right]$$

$$= \frac{\pi}{4}$$

[6 marks]

Answer 3

Partial fractions;

$$\frac{8x - 12}{[\ (2x^2 + 3)(x + 1)\ ]} = \frac{Ax + B}{2x^2 + 3} + \frac{C}{x + 1}$$

Multiply through by the red bracket

$$\frac{(8x - 12)[\ \ldots\ldots\ ]}{[\ (2x^2 + 3)(x + 1)\ ]} = \frac{(Ax + B)\ [\ \ldots\ldots\ ]}{2x^2 + 3} + \frac{C\ [\ \ldots\ldots\ ]}{x + 1}$$

$$8x - 12 = \left(Ax + B\right)\left(x + 1\right) + C\left(2x^2 + 3\right)$$

$$\text{Let } x = -1 \Rightarrow C = -4$$

$$\text{Let } x = 0 \Rightarrow B = 0$$

$$\text{Let } x = 1 \Rightarrow A = 8$$

$$\int \frac{8x - 12}{(2x^2 + 3)(x + 1)}\, dx = \int \frac{8x}{2x^2 + 3} - \frac{4}{x + 1}\, dx$$

$$= 2\int 4x\left(2x^2 + 3\right)^{-1} - \frac{4}{x + 1}\, dx$$

$$= 2\, ln\left(2x^2 + 3\right) - 4\, ln(x + 1) + c$$

$$= 2\left[\ ln\left(2x^2 + 3\right) - ln\ (x + 1)^2\ \right] + c$$

$$= 2\, ln\left(\frac{2x^2 + 3}{(x + 1)^2}\right) + c$$

$$\int_0^{\infty} \frac{8x - 12}{(2x^2 + 3)(x + 1)}\, dx = 2 \lim_{t \to \infty} \left[\ ln\left(\frac{2x^2 + 3}{(x + 1)^2}\right)\right]_0^t$$

$$= 2 \lim_{t \to \infty} \left[\ ln\left(\frac{2t^2 + 3}{(t + 1)^2}\right) - ln\left(\frac{3}{1}\right)\right]$$

$$= 2\left[\ ln\, 2 - ln\, 3\right]$$

$$= 2\, ln\left(\frac{2}{3}\right)$$

$$= ln\left(\frac{4}{9}\right)$$

$$\text{That is, } k = \frac{4}{9}$$

[7 marks]

Answer 4

(i)

$$y = arctan\sqrt{x}$$
$$tan\, y = \sqrt{x}$$
$$(tan\, y)^2 = x$$

Differentiate implicitly with respect to x

$$2\, tan\, y\, sec^2 y\,\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{2\, tan\, y\, sec^2 y}$$
$$= \frac{1}{2\, tan\, y\,(1 + tan^2 y)}$$
$$= \frac{1}{2\sqrt{x}\,(1 + x)}$$

[3 marks]

(ii) From part (i),

$$\int_0^1 \frac{1}{\sqrt{x}\,(x + 1)}\,dx = 2\int_0^1 \frac{1}{2\sqrt{x}\,(x + 1)}\,dx$$
$$= 2\Big[arctan\sqrt{x}\Big]_0^1$$
$$= 2\Big[arctan\sqrt{1} - arctan\, 0\Big]$$
$$= 2 \times \frac{\pi}{4}$$
$$= \frac{\pi}{2}$$

[3 marks]

Answer 5

Let $x^2 = sec\,\theta$, and differentiate implicitly with respect to θ to get,

$$2\,x\,\frac{dx}{d\theta} = sec\,\theta\, tan\,\theta$$
$$\frac{dx}{d\theta} = \frac{sec\,\theta\, tan\,\theta}{2\sqrt{sec\,\theta}}$$
$$= \frac{1}{2}\sqrt{sec\,\theta}\; tan\,\theta$$

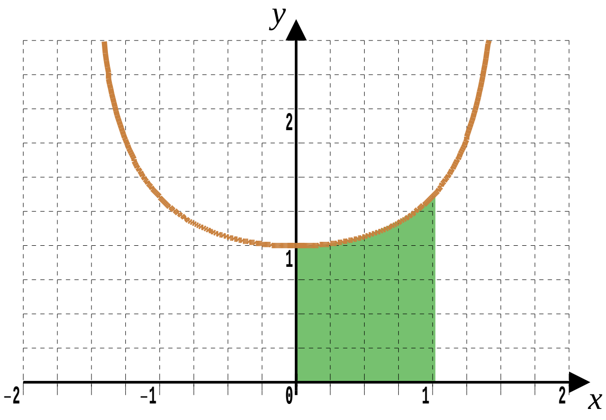
For the lower limit
$$\begin{aligned} x^2 &= \sec\theta \\ 1 &= \frac{1}{\cos\theta} \\ \cos\theta &= 1 \\ \theta &= 0 \end{aligned}$$

For the upper limit
$$\begin{aligned} x^2 &= \sec\theta \\ (\sqrt{2})^2 &= \frac{1}{\cos\theta} \\ \cos\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{3} \end{aligned}$$

Making the substitution,

$$\begin{aligned} \int_1^{\sqrt{2}} \frac{1}{\sqrt{x^4 - 1}}\,dx &= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{\sec^2\theta - 1}} \times \frac{1}{2}\sqrt{\sec\theta}\,\tan\theta\,d\theta \\ &= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{\tan^2\theta}} \times \frac{1}{2}\sqrt{\sec\theta}\,\tan\theta\,d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{\cos\theta}}\,d\theta \end{aligned}$$

This integral is a proper one and could be interpreted as finding the finite area shaded green under the curve $y = \dfrac{1}{\sqrt{\cos x}}$



Thus the original improper integral $\displaystyle\int_1^{\sqrt{2}} \frac{1}{\sqrt{x^4 - 1}}\,dx$ must be convergent □

[8 marks]