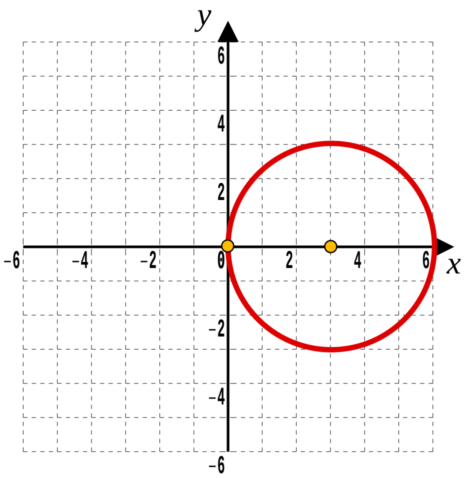
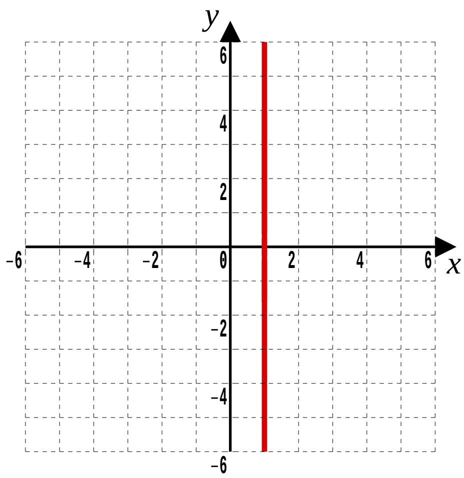
10.2 Answers to 10.1 Test

Answer 1

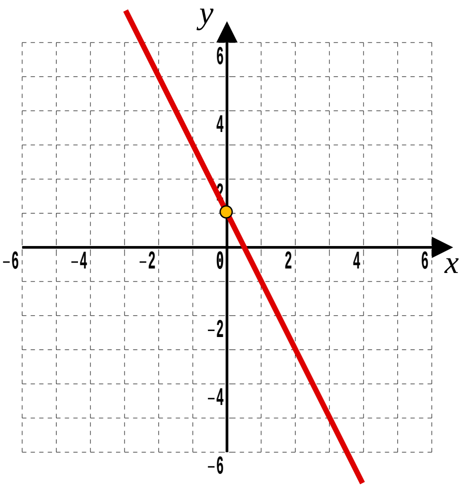


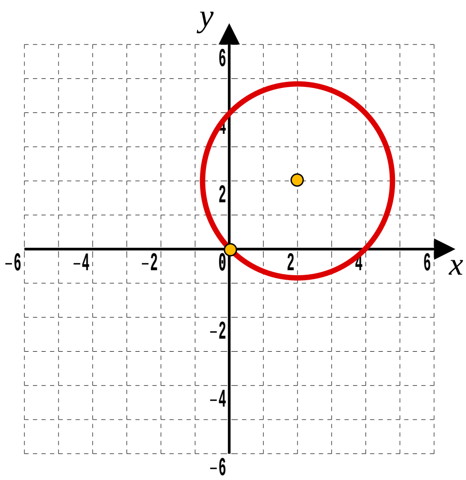


(i) $r = 6 \cos \theta$

(ii) $r = \sec \theta$

[2, 2 marks]





(iii) $r = \dfrac{1}{2 \cos \theta + \sin \theta}$

(iv) $r = 4\sqrt{2} \cos (\theta - 45°)$

[2, 2 marks]

Answer 2

(i)

$$y^2\ (4 + x) = x^2 (12 - x)$$

$$r^2 \sin^2\theta\ (4 + r\cos\theta) = r^2 \cos^2\theta\ (12 - r\cos\theta)$$

$$\sin^2\theta\ (4 + r\cos\theta) = \cos^2\theta\ (12 - r\cos\theta)$$

$$4\sin^2\theta + r\sin^2\theta\cos\theta = 12\cos^2\theta - r\cos^3\theta$$

$$r\cos^3\theta + r\sin^2\theta\cos\theta = 12\cos^2\theta - 4\sin^2\theta$$

$$r\cos\theta\left(\cos^2\theta + \sin^2\theta\right) = 4\left(3\cos^2\theta - \sin^2\theta\right)$$

$$r\cos\theta = 4\left(\cos^2\theta + \sin^2\theta + 2\cos^2\theta - 2\sin^2\theta\right)$$

$$r\cos\theta = 4(1 + 2\cos(2\theta))$$

$$r = \frac{4\ (2\cos(2\theta) + 1)}{\cos\theta} \qquad \square$$

[4 marks]

(ii)

$$\text{LHS} = y^2\left(4 + x\right)$$

$$= 4^2\left(4 + 4\right) \qquad \text{when } \left(4,\ 4\right) \text{ is inserted}$$

$$= 16 \times 8$$

$$= 128$$

$$\text{RHS} = x^2\ (12 - x)$$

$$= 4^2 \times \left(12 - 4\right) \qquad \text{when } \left(4,\ 4\right) \text{ is inserted}$$

$$= 16 \times 8$$

$$= 128$$

$$= \text{LHS}$$

$$\therefore\ \left(4,\ 4\right) \text{ is on the curve} \qquad \square$$

[2 marks]

(iii)

$$y^2\left(4 + x\right) = x^2\ (12 - x)$$

$$y^2\left(4 + x\right) = 12x^2 - x^3$$

Differentiating Implicitly with respect to x

$$y^2(1) + 2y\frac{dy}{dx}\left(4 + x\right) = 24x - 3x^2$$

$$\frac{dy}{dx}\left(8y + 2xy\right) = 24x - 3x^2 - y^2$$

$$\frac{dy}{dx} = \frac{24x - 3x^2 - y^2}{8y + 2xy} \qquad \square$$

[4 marks]

(iv)

$$\frac{dy}{dx} = \frac{24(4) - 3(4)^2 - (4)^2}{8(4) + 2(4)(4)}$$

$$= \frac{96 - 48 - 16}{32 + 32}$$

$$= \frac{32}{64}$$

$$= \frac{1}{2}$$

$$y = mx + c$$

$$4 = \frac{1}{2} \times (4) + c$$

$$c = 2$$

$$\therefore \text{ Tangent is } y = \frac{1}{2}x + 2, \quad \text{that is, } x - 2y + 4 = 0$$

[3 marks]

Answer 3

$$I = \frac{1}{2}\int r^2 \, d\theta$$

$$= \frac{1}{2}\int (6 + a \sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int 36 + 12a\sin\theta + a^2 \sin^2\theta \, d\theta$$

$$= \frac{1}{2}\int 36 + 12a\sin\theta + \frac{a^2}{2} - \frac{a^2}{4} 2\cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left[36\theta - 12a\cos\theta + \frac{a^2\theta}{2} - \frac{a^2}{4}\sin 2\theta \right] + c$$

With lower limit 0 and upper limit 2π this will become,

$$= \frac{1}{2}\left[72\pi - 12a + a^2\pi - 0 - 0 + 12a - 0 + 0 \right]$$

$$= \frac{\pi}{2}\left(72 - a^2 \right)$$

Using the given area,

$$\frac{\pi}{2}\left(72 - a^2 \right) = \frac{97\pi}{2}$$

$$a^2 + 97 - 72 = 0$$

$$a^2 = 25$$

$$a = 5 \qquad \text{as} \quad 0 < a < 6$$

[8 marks]

Answer 4

(a)

$$y = r \sin\theta$$
$$= 3a\ (1 + \cos\theta)\ \sin\theta$$
$$= 3a \sin\theta + 3a \sin\theta \cos\theta$$
$$= 3a \sin\theta + 3a\,\frac{1}{2}\sin 2\theta$$

$$\frac{dy}{d\theta} = 3a\cos\theta + 3a\cos 2\theta$$
$$= 3a\left(\cos\theta + \cos^2\theta - \sin^2\theta\right)$$
$$= 3a\left(\cos\theta + \cos^2\theta - 1 + \cos^2\theta\right)$$
$$= 3a\left(2\cos^2\theta + \cos\theta - 1\right)$$
$$= 3a\left(2\cos\theta - 1\right)\left(\cos\theta + 1\right)$$

But $\dfrac{dy}{d\theta} = 0$ for a tangent to be parallel to the polar axis

Either $\cos\theta = \dfrac{1}{2}$ or $\cos\theta = -1$

The $\cos\theta = -1$ result shows that the angle of the curve at the origin is π^{c}

For $\cos\theta = \dfrac{1}{2}$

$$\theta = \frac{\pi}{3} \Rightarrow A\left(\frac{9a}{2}, \frac{\pi}{3}\right)$$

[6 marks]

(b) Ignoring limits for the time being....

$$I = \frac{1}{2}\int r^2 \, d\theta$$

$$= \frac{1}{2}\int (3a)^2 \; (1 + \cos\theta)^2 \; d\theta$$

$$= \frac{9a^2}{2}\int 1 + 2\cos\theta + \cos^2\theta \, d\theta$$

$$= \frac{9a^2}{2}\int 1 + 2\cos\theta + \frac{1}{2} + \frac{1}{4} 2\cos 2\theta \, d\theta$$

$$= \frac{9a^2}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta \right]$$

$$= \frac{9a^2}{8}[\, 6\theta + 8\sin\theta + \sin 2\theta \,] + c$$

With lower limit 0 and upper limit $\frac{\pi}{3}$ this will become,

$$= \frac{9a^2}{8}\left[2\pi + 4\sqrt{3} + \frac{\sqrt{3}}{2} - 0 - 0 - 0 \right]$$

$$= \frac{9a^2}{16}[\, 4\pi + 8\sqrt{3} + \sqrt{3} \,]$$

$$= \frac{9a^2}{16}[\, 4\pi + 9\sqrt{3} \,]$$

$$= a^2\left(\frac{9}{4}\pi + \frac{81}{16}\sqrt{3} \right)$$

That is, $p = \frac{9}{4}$ and $q = \frac{81}{16}$

[5 marks]