2.4 Answers to 2.3 Exercise

Answer 1

$$\begin{aligned}
\left(x^2 + y^2\right)^2 &= 100\left(x^2 - y^2\right) \\
\left(r^2\right)^2 &= 100\left(r^2 \cos^2\theta - r^2 \sin^2\theta\right) \\
r^4 &= 100\, r^2\left(\cos^2\theta - \sin^2\theta\right) \\
r^2 &= 100 \cos(2\theta) \qquad \square
\end{aligned}$$

[3 marks]

Answer 2

(i)

$$\begin{aligned}
y^2\,(8 + x) &= x^2\,(8 - x) \\
r^2 \sin^2\theta\,(8 + r\cos\theta) &= r^2\cos^2\theta\,(8 - r\cos\theta) \\
\sin^2\theta\,(8 + r\cos\theta) &= \cos^2\theta\,(8 - r\cos\theta) \\
8\sin^2\theta + r\sin^2\theta\cos\theta &= 8\cos^2\theta - r\cos^3\theta \\
r\cos^3\theta + r\sin^2\theta\cos\theta &= 8\cos^2\theta - 8\sin^2\theta \\
r\cos\theta\left(\cos^2\theta + \sin^2\theta\right) &= 8\left(\cos^2\theta - \sin^2\theta\right) \\
r\cos\theta &= 8\cos(2\theta) \\
r &= \frac{8\cos(2\theta)}{\cos\theta} \qquad \square
\end{aligned}$$

[6 marks]

(ii)

$$\begin{aligned}
\text{LHS} &= y^2(8 + x) \\
&= \left(-4\sqrt{3}\right)^2\left(8 + (-4)\right) \qquad \text{when } \left(-4, -4\sqrt{3}\right) \text{ is inserted} \\
&= 16 \times 3 \times 4 \\
&= 192
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= x^2\,(8 - x) \\
&= (-4)^2 \times \left(8 - (-4)\right) \qquad \text{when } \left(-4, -4\sqrt{3}\right) \text{ is inserted} \\
&= 16 \times 12 \\
&= 192 \\
&= \text{LHS}
\end{aligned}$$

$\therefore \left(-4, -4\sqrt{3}\right)$ is on the curve $\qquad \square$

[2 marks]

(iii)

$$y^2(8+x) = x^2(8-x)$$
$$y^2(8+x) = 8x^2 - x^3$$

Differentiating Implicitly

$$y^2(1) + 2y\frac{dy}{dx}(8+x) = 16x - 3x^2$$
$$\frac{dy}{dx}(16y + 2xy) = 16x - 3x^2 - y^2$$
$$\frac{dy}{dx} = \frac{16x - 3x^2 - y^2}{16y + 2xy} \qquad \square$$

[4 marks]

(iv)

$$\frac{dy}{dx} = \frac{16(-4) - 3(-4)^2 - \left(-4\sqrt{3}\right)^2}{16\left(-4\sqrt{3}\right) + 2(-4)\left(-4\sqrt{3}\right)}$$
$$= \frac{-64 - 48 - 48}{-64\sqrt{3} + 32\sqrt{3}}$$
$$= \frac{-160}{-32\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{5\sqrt{3}}{3}$$

$$y = mx + c$$
$$-4\sqrt{3} = \frac{5\sqrt{3}}{3} \times (-4) + c$$
$$c = -\frac{12\sqrt{3}}{3} + \frac{20\sqrt{3}}{3}$$
$$= \frac{8\sqrt{3}}{3}$$

$\therefore$ Tangent is $y = \frac{5\sqrt{3}}{3}x + \frac{8\sqrt{3}}{3}$

[3 marks]

Answer 3

(i)

$$\left(x^2 + y^2\right)\left(y^2 + x\left(x + 10\right)\right) = 40\,x\,y^2$$

$$\left(x^2 + y^2\right)\left(y^2 + x^2 + 10x\right) = 40\,x\,y^2$$

$$r^2\left(r^2 + 10\,r\cos\theta\right) = 40\,r\cos\theta\,r^2\sin^2\theta$$

$$r^3\left(r + 10\cos\theta\right) = 40\,r^3\cos\theta\sin^2\theta$$

$$r + 10\cos\theta = 40\cos\theta\sin^2\theta$$

$$r = 40\cos\theta\sin^2\theta - 10\cos\theta$$

$$r = 10\cos\theta\left(4\sin^2\theta - 1\right) \qquad \square$$

[5 marks]

(ii) When $\theta = 60°$,

$$r = 10\cos 60\left(4\sin^2 60 - 1\right)$$

$$= 10 \times \frac{1}{2}\left(4 \times \left(\frac{\sqrt{3}}{2}\right)^2 - 1\right)$$

$$= 10 \times \frac{1}{2}\left(4 \times \frac{3}{4} - 1\right)$$

$$= 10 \times \frac{1}{2} \times 2$$

$= 10$ This is the tip of the loop in the first quadrant

[1 mark]

(iii) When $\theta = 30°$,

$$r = 10\cos 30\left(4\sin^2 30 - 1\right)$$

$$= 10 \times \frac{\sqrt{3}}{2}\left(4 \times \left(\frac{1}{2}\right)^2 - 1\right)$$

$$= 0$$

The implication is that this is the angle of the slope as the curve makes one of its passes through the origin.

$$m = \tan\theta$$

$$= \tan 30$$

$$= \frac{\sqrt{3}}{3}$$

$\therefore$ The tangent would be $y = \dfrac{\sqrt{3}}{3}x$

[3 marks]

Answer 4
(i)

$$y^2 = \frac{x^3}{72 - x}$$

$$y^2 (72 - x) = x^3$$

$$r^2 \sin^2 \theta \left(72 - r \cos \theta \right) = r^3 \cos^3 \theta$$

$$\sin^2 \theta \left(72 - r \cos \theta \right) = r \cos^3 \theta$$

$$72 \sin^2 \theta - r \sin^2 \theta \cos \theta = r \cos^3 \theta$$

$$r \cos^3 \theta + r \sin^2 \theta \cos \theta = 72 \sin^2 \theta$$

$$r \cos \theta \left(\cos^2 \theta + \sin^2 \theta \right) = 72 \sin^2 \theta$$

$$r \cos \theta = 72 \sin^2 \theta$$

$$r = 72 \times \frac{\sin \theta}{\cos \theta} \times \sin \theta$$

$$r = 72 \tan \theta \sin \theta \qquad \square$$

[4 marks]

(ii)

$$y^2 = \frac{x^3}{72 - x}$$

$$y^2 (72 - x) = x^3$$

$$\begin{aligned} \text{LHS} &= y^2 \left(72 - x \right) \\ &= \left(2\sqrt{2} \right)^2 \left(72 - 8 \right) \qquad \text{when } \left(8,\ 2\sqrt{2} \right) \text{ is inserted} \\ &= 4 \times 2 \times 64 \\ &= 512 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= x^3 \\ &= 8^3 \qquad \text{when } \left(8,\ 2\sqrt{2} \right) \text{ is inserted} \\ &= 512 \\ &= \text{LHS} \end{aligned}$$

$\therefore \left(8,\ 2\sqrt{2} \right)$ is on the curve $\square$

[2 marks]

(iii)

$$y^2 = \frac{x^3}{72 - x}$$

$$y^2 (72 - x) = x^3$$

Differentiate Implicitly

$$y^2 (-1) + 2y \frac{dy}{dx} (72 - x) = 3x^2$$

$$\frac{dy}{dx} (144y - 2xy) = 3x^2 + y^2$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{144y - 2xy}$$

[4 marks]

(iv)

$$\frac{dy}{dx} = \frac{3(8)^2 + (2\sqrt{2})^2}{144(2\sqrt{2}) - 2(8)(2\sqrt{2})}$$

$$= \frac{192 + 8}{288\sqrt{2} - 32\sqrt{2}}$$

$$= \frac{200}{256\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{25\sqrt{2}}{64}$$

$$y = mx + c$$

$$2\sqrt{2} = \frac{25\sqrt{2}}{64} \times (8) + c$$

$$2\sqrt{2} = \frac{25\sqrt{2}}{8} + c$$

$$c = \frac{16\sqrt{2}}{8} - \frac{25\sqrt{2}}{8}$$

$$= -\frac{9\sqrt{2}}{8}$$

$\therefore$ Tangent is $y = \dfrac{25\sqrt{2}}{64}x - \dfrac{9\sqrt{2}}{8}$

[3 marks]