5.4 Answers

5.4.1 Solution to 5.2 Example (Teaching Video)

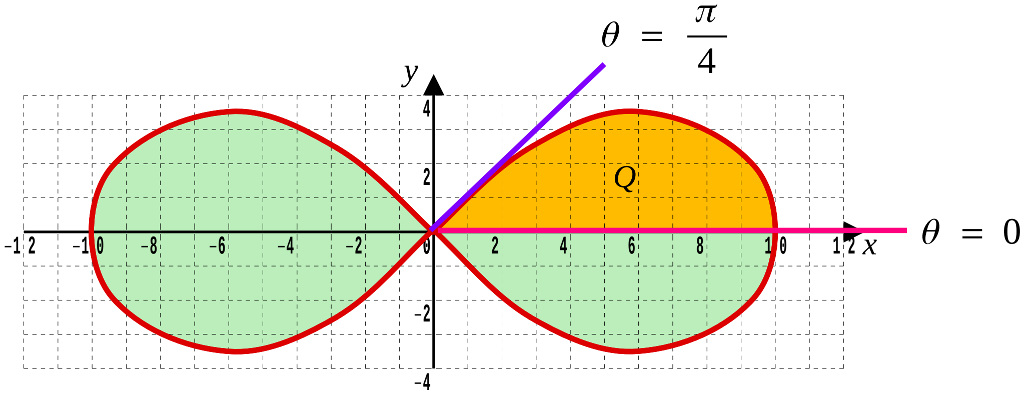
Integrations tend to be less work when the lower limit is zero, so a good tactic to get started on this problem is to work out where on the curve the point corresponding to $\theta = 0$ is.

As we have the graph, this is easy; simply look along the polar axis.
As there is only one point on the x-axis (not at the origin) $\theta = 0$ is $(\ 10, 0°\ )$
which is easily confirmed by putting $\theta = 0$ into the polar curve's equation.

With, along with the graph, the equation of the curve in mind,

$$r^2 = 100 \cos(2\theta)$$

it's reasonably obvious that the curve proceeds anticlockwise until at an angle of $\dfrac{\pi}{4}$ radians it's passing through the pole.



So one way of getting the required area is to do the following integration,

$$\begin{aligned}
Area &= 4 \times Area\ Q \\
&= 4 \times \frac{1}{2} \int_0^{\frac{\pi}{4}} 100 \cos(2\theta)\ d\theta \\
&= 200 \int_0^{\frac{\pi}{4}} \cos(2\theta)\ d\theta \\
&= 100 \int_0^{\frac{\pi}{4}} 2 \cos(2\theta)\ d\theta \\
&= 100\ \Big[\ \sin(2\theta)\ \Big]_0^{\frac{\pi}{4}} \\
&= 100 \left[\sin\left(\frac{\pi}{2}\right) - \sin(0) \right] \\
&= 100
\end{aligned}$$

[6 marks]

5.4.2 Solutions to 5.3 Exercise

Answer 1

(i)

$$Area\ P = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(10\sqrt{\sin\theta}\right)^2 d\theta$$

$$= 50\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin\theta\, d\theta$$

$$= 50\left[-\cos\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= 50\left[-\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right)\right]$$

$$= 50\left[-\frac{1}{2} + \frac{\sqrt{2}}{2}\right]$$

$$= 25\left(\sqrt{2} - 1\right)$$

[4 marks]

(ii)

$$Area\ Q = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(10\sqrt{\sin\theta}\right)^2 d\theta$$

$$= 50\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin\theta\, d\theta$$

$$= 50\left[-\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= 50\left[-\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{6}\right)\right]$$

$$= 50\left[-\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\right]$$

$$= 25\left(\sqrt{3} - \sqrt{2}\right)$$

(iii) Over the interval $\pi < \theta < 2\pi$, $\sin\theta$ is negative, so if that interval were part of the domain, the taking of a square root of a negative number would be an issue.

Had the equation been $r^2 = 100\sin\theta$ the graph would have had a second petal that was a mirror reflection of the first in the polar axis.

[2 marks]

(iv)

$$Area\ R = \frac{1}{2} \int_0^{\pi} \left(10 \sqrt{sin\ \theta} \right)^2 d\theta$$

$$= 50 \int_0^{\pi} sin\ \theta\ d\theta$$

$$= 50 \left[- cos\ \theta \right]_0^{\pi}$$

$$= 50 \left[- cos\ \pi + cos\ 0 \right]$$

$$= 50 \left[- (-1) + 1 \right]$$

$$= 100$$

[3 marks]

(v)

$$r = 10 \sqrt{sin\ \theta}$$

Square both sides

(This will make the second 'ghost' loop visible)

$$r^2 = 100\ sin\ \theta$$

Multiply both sides by r

$$r^3 = 100\ r\ sin\ \theta$$

$$r^3 = 100\ y$$

Square both sides

$$\left(r^3 \right)^2 = 10000\ y^2$$

$$\left(r^2 \right)^3 = 10000\ y^2$$

$$\left(x^2 + y^2 \right)^3 = 10000\ y^2$$

Add in a restriction to stop the 'ghost' loop being visible

$$\left(x^2 + y^2 \right)^3 = 10000\ y^2, \qquad y \geqslant 0$$

[3 marks]

Answer 2

(i)

$$I = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 36\ (\sin\theta - \cos\theta)\ d\theta$$

$$= 18\left[-\cos\theta - \sin\theta\ \right]_0^{2\pi}$$

$$= 18\left[-\cos(2\pi) - \sin(2\pi) + \cos 0 + \sin 0\right]$$

$$= 18\left[\ -1 - 0 + 1 + 0\right]$$

$$= 0 \qquad \square$$

[4 marks]

(ii)

$$r^2 = 0$$

$$36\ (\sin\theta - \cos\theta) = 0$$

$$\sin\theta - \cos\theta = 0$$

$$\sin\theta = \cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = 1$$

$$\tan\theta = 1$$

$$\theta = \frac{\pi}{4},\ \frac{5\pi}{4}$$

$$\therefore\ \alpha = \frac{\pi}{4},\ \beta = \frac{5\pi}{4}$$

[3 marks]

(iii)

$$Area\ R = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 36\ (\sin\theta - \cos\theta)\ d\theta$$

$$= 18\left[-\cos\theta - \sin\theta\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= 18\left[-\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\ \right]$$

$$= 18\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right]$$

$$= 36\sqrt{2}$$

[4 marks]

(iv) Total Area $= 72\sqrt{2}$

[1 mark]

Answer 3
(i)

$$Area\ R = \frac{1}{2}\int_0^{\frac{\pi}{3}} 25\, sec\,\theta\, d\theta$$

$$= \frac{25}{2}\int_0^{\frac{\pi}{3}} sec\,\theta\, d\theta$$

$$= \frac{25}{2}\left[\, ln\left| sec\,\theta + tan\,\theta \right| \,\right]_0^{\frac{\pi}{3}}$$

$$= \frac{25}{2}\left[ln\left| \frac{1}{cos\left(\frac{\pi}{3}\right)} + tan\left(\frac{\pi}{3}\right) \right| - ln\left| \frac{1}{cos\,0} + tan\,0 \right| \right]$$

$$= \frac{25}{2}\left[\, ln\left| 2 + \sqrt{3} \right| - ln\left| 1 + 0 \right| \,\right]$$

$$= \frac{25}{2} ln\left(2 + \sqrt{3} \right)$$

[7 marks]

(ii)

$$r^2 = 25\, sec\,\theta$$

$$r^2 = \frac{25}{cos\,\theta}$$

$$r\, r\, cos\,\theta = 25$$

$$r\, x = 25$$

Square both sides

$$r^2\, x^2 = 625$$

$$(x^2 + y^2)\, x^2 = 625$$

[4 marks]

Answer 4

(i) For the petal highlighted in a peach colour,

$$Area = \frac{1}{2} \int_0^{\frac{\pi}{3}} 4\, sin(3\theta)\ d\theta$$

$$= \frac{2}{3} \int_0^{\frac{\pi}{3}} 3\, sin(3\theta)\ d\theta \qquad \text{Setting up a "chain rule backwards"}$$

$$= \frac{2}{3} \left[-cos(3\theta) \right]_0^{\frac{\pi}{3}}$$

$$= \frac{2}{3} \left[-cos\,\pi + cos\,0 \right]$$

$$= \frac{2}{3} \left[-(-1) + 1 \right]$$

$$= \frac{4}{3}$$

$\therefore$ Total area of all six petals is 8

[6 marks]

(ii)

$$\begin{aligned}
\text{LHS} &= sin\,3\theta \\
&= sin\ (2\theta + \theta) \\
&= sin\,2\theta\, cos\,\theta + cos\,2\theta\, sin\,\theta \\
&= 2\, sin\,\theta\, cos\,\theta\, cos\,\theta + \left(cos^2\theta - sin^2\theta \right) sin\,\theta \\
&= 2\, sin\,\theta\, cos^2\theta + cos^2\theta\, sin\,\theta - sin^3\theta \\
&= 3\, sin\,\theta\, cos^2\theta - sin^3\theta \\
&= 3\, sin\,\theta \left(1 - sin^2\theta \right) - sin^3\theta \\
&= 3\, sin\,\theta - 3\, sin^3\theta - sin^3\theta \\
&= 3\, sin\,\theta - 4\, sin^3\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

(iii)

$$r^2 = 4\, sin(3\theta)$$

$$r^2 = 4\left(3\, sin\,\theta - 4\, sin^3\theta \right)$$

Multiply both sides by r^3

$$r^5 = 12\, r^2\, r\, sin\,\theta - 16\, r^3\, sin^3\theta$$

$$\left(x^2 + y^2 \right)^{\frac{5}{2}} = 12\left(x^2 + y^2 \right) y - 16\, y^3$$

$$\left(x^2 + y^2 \right)^{\frac{5}{2}} = 12\, x^2 y - 4\, y^3$$

[4 marks]

Answer 5

(i) To get the limit lines,

$$r^2 = 0$$

$$64\,(\,sin\,(4\theta) - cos\,(2\theta)\,) = 0$$

$$sin\,(4\theta) - cos\,(2\theta) = 0$$

$$2\,sin\,(2\theta)\,cos\,(2\theta) - cos\,(2\theta) = 0$$

$$cos\,(2\theta)\,(2\,sin\,(2\theta) - 1\,) = 0$$

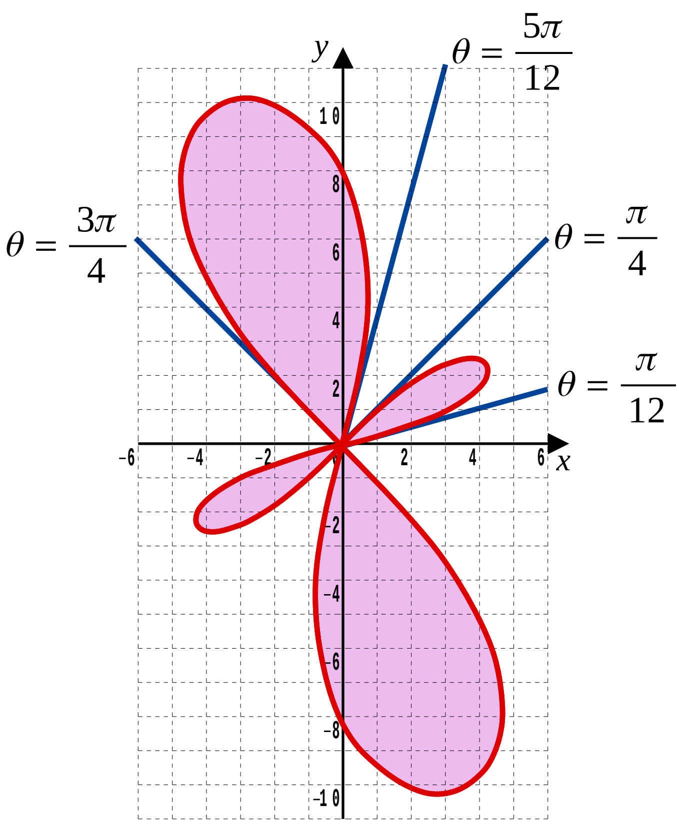
$$\text{Either } cos\,(2\theta) = 0 \text{ or } sin\,(2\theta) = \frac{1}{2}$$

$$\text{For } cos\,(2\theta) = 0 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\text{For } sin\,(2\theta) = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$



So the areas sought are,

$$\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} r^2\,d\theta \quad \text{and} \quad \frac{1}{2}\int_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} r^2\,d\theta$$

Area of loop wholly in 1st Quadrant:

$$Area = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} r^2\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} 64\left(sin(4\theta) - cos(2\theta) \right) d\theta$$

$$= 32\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} sin(4\theta) - cos(2\theta)\ d\theta$$

$$= 8\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} 4\,sin(4\theta)\ d\theta - 16\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} 2\,cos(2\theta)\ d\theta$$

$$= 8\left[-cos(4\theta)\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}} - 16\left[sin(2\theta)\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$$

$$= 8\left[-cos\,\pi + cos\left(\frac{\pi}{3}\right)\right] - 16\left[sin\left(\frac{\pi}{2}\right) - sin\left(\frac{\pi}{6}\right)\right]$$

$$= 8\left[-(-1) + \frac{1}{2}\right] - 16\left[1 - \frac{1}{2}\right]$$

$$= 8 \times \frac{3}{2} - 16 \times \frac{1}{2}$$

$$= 12 - 8$$

$$= 4$$

For the larger loop, mostly in the 2nd Quadrant

$$Area = \frac{1}{2}\int_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} r^2\, d\theta$$

$$= 32\int_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} sin(4\theta) - cos(2\theta)\ d\theta$$

$$= 8\int_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} 4\,sin(4\theta)\ d\theta - 16\int_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} 2\,cos(2\theta)\ d\theta$$

$$= 8\left[-cos(4\theta)\right]_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} - 16\left[sin(2\theta)\right]_{\frac{5\pi}{12}}^{\frac{3\pi}{4}}$$

$$= 8\left[-cos\,3\pi + cos\left(\frac{5\pi}{3}\right)\right] - 16\left[sin\left(\frac{3\pi}{2}\right) - sin\left(\frac{5\pi}{6}\right)\right]$$

$$= 8\left[-(-1) + \frac{1}{2}\right] - 16\left[-1 - \frac{1}{2}\right]$$

$$= 36$$

$$\therefore\ \text{Total Area} = 2(4 + 36)$$

$$= 80$$

[12 marks]

(ii)

$$r^2 = 64\,(\,sin\,(4\theta) - cos\,(2\theta)\,)$$
$$= 64\,(\,2\,sin\,(2\theta)\,cos\,(2\theta) - cos\,(2\theta)\,)$$
$$= 64\,cos\,(2\theta)\,(\,2\,sin\,(2\theta) - 1\,)$$
$$= 64\,\left(\,cos^2\,\theta - sin^2\,\theta\,\right)(\,4\,sin\,\theta\,cos\,\theta - 1\,)$$
$$r^6 = 64\left(\,r^2\,cos^2\,\theta - r^2\,sin^2\,\theta\,\right)\left(\,4\,r\,sin\,\theta\,r\,cos\,\theta - r^2\,\right)$$
$$\left(\,x^2 + y^2\,\right)^3 = 64\,\left(\,x^2 - y^2\,\right)\left(\,4yx - x^2 - y^2\,\right)$$

[6 marks]

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