6.6 Answers to 6.5 Exercise

Answers 1

(a) Clearly the pole is at the centre of the diagram in which case, also from the diagram, it would seem that r is 0.6 m when $\theta = 0$

$$r = 0.4 + a\cos 2\theta$$

$$0.6 = 0.4 + a\cos 0$$

$$a = 0.2 \qquad \square$$

[2 marks]

(b) The area of the table top will be,

$$Area\ Rectangle = 1.2 \times 0.6$$

$$= 0.72 \text{ m}^2$$

For the glass, this is a forgiving question in that there is no issue with a negative value of r as the curve would have to go through the pole to change sign which it clearly does not.
So any suitable pair of limits will do, although it'll be easier to use the polar axis with $\theta = 0$ as the lower limit,

Sensible upper limits would be $\theta = \dfrac{\pi}{2}$ or $\theta = \pi$ or $\theta = 2\pi$

$$Area = \frac{1}{2}\int (0.4 + 0.2\cos 2\theta)^2\, d\theta$$

$$= \frac{1}{2}\int (0.2(2 + \cos 2\theta))^2\, d\theta$$

$$= \frac{1}{2}0.2^2\int (2 + \cos 2\theta)^2\, d\theta$$

$$= \frac{1}{50}\int 4 + 4\cos 2\theta + \cos^2 2\theta\, d\theta$$

$$= \frac{1}{50}\left[\int 4\, d\theta + 2\int 2\cos 2\theta\, d\theta + \frac{1}{2}\int 1 + \cos 4\theta\, d\theta\right]$$

$$= \frac{1}{50}\left[4\theta + 2\sin 2\theta + \frac{\theta}{2} + \frac{1}{8}\int 4\cos 4\theta\, d\theta\right]$$

$$= \frac{1}{50}\left[4\theta + 2\sin 2\theta + \frac{\theta}{2} + \frac{1}{8}\sin 4\theta\right]$$

$$= \frac{1}{400}[32\theta + 16\sin 2\theta + 4\theta + \sin 4\theta]$$

$$= \frac{1}{400}[36\theta + 16\sin 2\theta + \sin 4\theta]$$

I'll use limits of 0 and 2π,

$$Area = \frac{1}{400}\left[72\pi + 16\sin 4\pi + \sin 8\pi - 0 - 0 - 0 \right]$$

$$= \frac{72\pi}{400}$$

$$= \frac{9\pi}{50}$$

$\therefore$ The area of the wood is $0.72 - \dfrac{9\pi}{50}$

$$= \frac{36 - 9\pi}{50}$$

$$= \frac{9(4 - \pi)}{50}$$

$$= 0.155\ \text{m}^2$$

[8 marks]

Answer 2

$$I = \frac{1}{2}\int r^2\, d\theta$$

$$= \frac{1}{2}\int (a + 3\cos\theta)^2\, d\theta$$

$$= \frac{1}{2}\int a^2 + 6a\cos\theta + 9\cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int a^2 + 6a\cos\theta + \frac{9}{2} + \frac{9}{4}\, 2\cos 2\theta\, d\theta$$

$$= \frac{1}{2}\left[a^2\theta + 6a\sin\theta + \frac{9\theta}{2} + \frac{9}{4}\sin 2\theta \right] + c$$

With lower limit 0 and upper limit 2π this will become,

$$= \frac{1}{2}\left[a^2\, 2\pi + 0 + 9\pi + 0 - 0 - 0 - 0 - 0 \right]$$

$$= \frac{\pi}{2}\left(2a^2 + 9 \right)$$

Using the given area,

$$\frac{\pi}{2}\left(2a^2 + 9 \right) = \frac{107\pi}{2}$$

$$2a^2 + 9 - 107 = 0$$

$$2a^2 = 98$$

$$a = 7 \qquad \text{as} \quad a > 0$$

[8 marks]

Answer 3

(i)

$$r = 2a \cos\theta + 2b \sin\theta$$

Multiply through by r

$$r^2 = 2a\, r \cos\theta + 2b\, r \sin\theta$$

$$x^2 + y^2 = 2ax + 2by$$

$$x^2 - 2ax + y^2 - 2by = 0$$

$$(x - a)^2 - a^2 + (y - b)^2 - b^2 = 0$$

$$(x - a)^2 + (y - b)^2 = a^2 + b^2$$

Which is recognized as a circle, centre $(a,\ b)$ and radius $\sqrt{a^2 + b^2}$

[5 marks]

(ii)

$$R = \frac{1}{2} \int (2 \cos\theta + 2 \sin\theta)^2 \, d\theta \quad \text{as} \quad a = b = 1$$

$$= 2 \int (\cos\theta + \sin\theta)^2 \, d\theta \qquad \text{as} \quad a = b = 1$$

$$= 2 \int \sin^2\theta + 2 \sin\theta \cos\theta + \cos^2\theta \, d\theta$$

$$= 2 \int 1 + \sin 2\theta \, d\theta$$

$$= 2\theta - \cos 2\theta + c$$

With the lower limit of $\frac{\pi}{6}$ and an upper limit of $\frac{\pi}{3}$ this becomes,

$$= 2\left(\frac{\pi}{3}\right) - \cos\left(\frac{2\pi}{3}\right) - 2\left(\frac{\pi}{6}\right) + \cos\left(\frac{2\pi}{6}\right)$$

$$= \frac{2\pi}{3} + \frac{1}{2} - \frac{\pi}{3} + \frac{1}{2}$$

$$= 1 + \frac{\pi}{3} \qquad \square$$

[5 marks]

Answer 4

A circle with centre $(a,\ b)$, radius r that passes through the pole has equation

$$: \text{ Cartesian } \quad (x - a)^2 + (y - b)^2 = r^2 \quad \text{with} \quad r^2 = a^2 + b^2$$

$$: \text{ Polar } \quad r = 2a \cos\theta + 2b \sin\theta$$

$$\therefore\ r = \sqrt{3}\cos\theta + \sin\theta \text{ has centre } \left(\frac{\sqrt{3}}{2},\ \frac{1}{2}\right) \text{ and radius 2}$$

When $r = 0$, $\quad \sqrt{3}\cos\theta + \sin\theta = 0$

$$\sin\theta = -\sqrt{3}\cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = -\sqrt{3}$$

$$\tan\theta = -\sqrt{3}$$

$$\theta = -\frac{\pi}{3},\ \frac{2\pi}{3}$$

Which the question pretty much told us anyway !

$$I = \frac{1}{2}\int \left(\sqrt{3}\cos\theta + \sin\theta\right)^2 d\theta$$

$$= \frac{1}{2}\int\left(3\cos^2\theta + 2\sqrt{3}\cos\theta\sin\theta + \sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int\left(2\cos^2\theta + \sqrt{3}\sin 2\theta + 1\right) d\theta$$

$$= \frac{1}{2}\int\left(1 + \cos 2\theta + \sqrt{3}\sin 2\theta + 1\right) d\theta$$

$$= \frac{1}{2}\int\left(2 + \cos 2\theta + \sqrt{3}\sin 2\theta\right) d\theta$$

$$= \int 1\, d\theta + \frac{1}{4}\int 2\cos 2\theta\, d\theta + \frac{\sqrt{3}}{4}\int 2\sin 2\theta\, d\theta$$

$$= \theta + \frac{1}{4}\sin 2\theta - \frac{\sqrt{3}}{4}\cos 2\theta + c$$

With lower limit $-\dfrac{\pi}{3}$ and upper limit 0 this becomes,

$$= 0 + \frac{1}{4}\sin 0 - \frac{\sqrt{3}}{4}\cos 0 - \left(-\frac{\pi}{3}\right) - \frac{1}{4}\sin\left(-\frac{2\pi}{3}\right) + \frac{\sqrt{3}}{4}\cos\left(-\frac{2\pi}{3}\right)$$

$$= -\frac{\sqrt{3}}{4} + \frac{\pi}{3} + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8}$$

$$= \frac{1}{12}\left(4\pi - 3\sqrt{3}\right) \qquad \square$$

[7 marks]

Answer 5

$$Area = \frac{1}{2}\int r^2\, d\theta$$

$$= 2 \times \frac{1}{2}\int_0^{\frac{\pi}{2}} 4\, cos^3\,\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\, cos\,\theta\, cos^2\,\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} cos\,\theta\,\left(1 - sin^2\,\theta\right)\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} cos\,\theta\, d\theta - 4\int_0^{\frac{\pi}{2}} \left(cos\,\theta\right)\left(sin\,\theta\right)^2\, d\theta$$

$$= 4\left[\, sin\,\theta\,\right]_0^{\frac{\pi}{2}} - 4\left[\frac{sin^3\,\theta}{3}\right]_0^{\frac{\pi}{2}}$$

$$= 4\left[\, sin\left(\frac{\pi}{2}\right) - sin\, 0\,\right] - \frac{4}{3}\left[\, sin^3\left(\frac{\pi}{2}\right) - sin^3\, 0\,\right]$$

$$= 4 - \frac{4}{3}$$

$$= \frac{8}{3}$$

[7 marks]