7.4 Answers to 7.3 Exercise

Further A-Level Pure Mathematics, Core 2
Polar Coordinates

Answer 1

(a) When the curves intersect,

$$2\sin\theta = 1.5 - \sin\theta$$
$$3\sin\theta = 1.5$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

With the diagram as a guide, $P\left(1, \frac{\pi}{6}\right)$, $Q\left(1, \frac{5\pi}{6}\right)$

[3 marks]

(b)

$$Area = 2 \times \frac{1}{2}\int_0^{\frac{\pi}{6}} (2\sin\theta)^2\, d\theta + 2 \times \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{3}{2} - \sin\theta\right)^2 d\theta$$

$$= 4\int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{9}{4} - 3\sin\theta + \sin^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} 2 - 2\cos 2\theta\, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{9}{4} - 3\sin\theta + \frac{2}{4} - \frac{1}{4} \times 2\cos 2\theta\, d\theta$$

$$= \left[2\theta - \sin 2\theta\right]_0^{\frac{\pi}{6}} + \left[\frac{11\theta}{4} + 3\cos\theta - \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{11\pi}{8} - \frac{11\pi}{24} - \frac{3\sqrt{3}}{2} + \frac{1}{4} \times \frac{\sqrt{3}}{2}$$

$$= \frac{5}{4}\pi - \frac{15}{8}\sqrt{3}$$

[8 marks]

Answer 2

(a) When the poles intersect,

$$7\cos\theta = 3(1+\cos\theta)$$

$$4\cos\theta = 3$$

$$\cos\theta = \frac{3}{4}$$

$$\theta = \pm 0.7227...$$

With the diagram as a guide, $P\left(\frac{21}{4},\ 0.7227...\right)$, $Q\left(\frac{21}{4},\ -0.7227...\right)$

[3 marks]

(b) Let $\phi = \arccos\left(\frac{3}{4}\right) \Rightarrow \cos\phi = \frac{3}{4},\ \sin\phi = \frac{\sqrt{7}}{4}$

$$Area = 2\times\frac{1}{2}\int_0^{\phi} 3^2(1+\cos\theta)^2\,d\theta + 2\times\frac{1}{2}\int_{\phi}^{\frac{\pi}{2}} 7^2\cos^2\theta\,d\theta$$

$$= 9\int_0^{\phi} 1+2\cos\theta+\cos^2\theta\,d\theta + 49\int_{\phi}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$= 9\int_0^{\phi} 1+2\cos\theta+\frac{1}{2}+\frac{1}{4}\times 2\cos 2\theta\,d\theta + 49\int_{\phi}^{\frac{\pi}{2}}\frac{1}{2}+\frac{1}{4}\times 2\cos 2\theta\,d\theta$$

$$= 9\left[\frac{3\theta}{2}+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\phi} + 49\left[\frac{\theta}{2}+\frac{1}{4}\sin 2\theta\right]_{\phi}^{\frac{\pi}{2}}$$

$$= \frac{27\phi}{2}+18\sin\phi+\frac{9}{4}\sin 2\phi+\frac{49\pi}{4}-\frac{49\phi}{2}-\frac{49}{4}\sin 2\phi$$

$$= \frac{49\pi}{4}-11\phi+18\sin\phi-10\sin 2\phi$$

$$= \frac{49\pi}{4}-11\phi+18\sin\phi-20\sin\phi\cos\phi$$

$$= \frac{49\pi}{4}-11\arccos\left(\frac{3}{4}\right)+18\times\frac{\sqrt{7}}{4}-20\times\frac{\sqrt{7}}{4}\times\frac{3}{4}$$

$$= \frac{49\pi}{4}-11\arccos\left(\frac{3}{4}\right)+\frac{3\sqrt{7}}{4} \qquad * \text{ The exact answer } *$$

$$= 32.5187....$$

[7 marks]