8.5 Answers to Exercise 8.4

Answer 1

(i)

$$\begin{aligned} y &= r \sin\theta \\ &= (7 + 2\cos\theta)\sin\theta \\ &= 7\sin\theta + 2\sin\theta\cos\theta \\ &= 7\sin\theta + \sin 2\theta \end{aligned}$$

$$\frac{dy}{d\theta} = 7\cos\theta + 2\cos 2\theta$$

But $\frac{dy}{d\theta} = 0$ for a tangent to be parallel to the polar axis

$$\begin{aligned} \therefore\ 7\cos\theta + 2\cos 2\theta &= 0 \\ 7\cos\theta + 2(\cos^2\theta - \sin^2\theta) &= 0 \\ 7\cos\theta + 2(2\cos^2\theta - 1) &= 0 \\ 4\cos^2\theta + 7\cos\theta - 2 &= 0 \\ (4\cos\theta - 1)(\cos\theta + 2) &= 0 \end{aligned}$$

Either $\cos\theta = \frac{1}{4}$ or $\cos\theta = -2$

As $\cos\theta = -2$ has no solutions, the only solution is $\cos\theta = \frac{1}{4}$ □

[5 marks]

(ii)

$$\begin{aligned} r &= 7 + 2\cos\theta \\ &= 7 + 2 \times \frac{1}{4} \\ &= 7.5 \end{aligned}$$

[1 mark]

Answer 2

(a)

$$x = r\cos\theta$$
$$= (1 + \tan\theta)\cos\theta$$
$$= \cos\theta + \sin\theta$$

$$\frac{dx}{d\theta} = -\sin\theta + \cos\theta$$

But $\frac{dx}{d\theta} = 0$ for a tangent to be perpendicular to the polar axis

$$\therefore\ -\sin\theta + \cos\theta = 0$$
$$\sin\theta = \cos\theta$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4} \Rightarrow r = 2 \Rightarrow P\left(2,\ \frac{\pi}{4}\right)$$

[5 marks]

(b) Without worrying about the limits to begin with...

$$Area\ R = \frac{1}{2}\int (1 + \tan\theta)^2\ d\theta$$
$$= \frac{1}{2}\int 1 + 2\tan\theta + \tan^2\theta\ d\theta$$
$$= \frac{1}{2}\int 1 + 2\tan\theta + \sec^2\theta - 1$$
$$= \int \tan\theta\ d\theta + \frac{1}{2}\int \sec^2\theta\ d\theta$$
$$= \ln|\sec\theta| + \frac{1}{2}\tan\theta + c$$

With a lower limit of $\frac{\pi}{4}$ and an upper limit of $\frac{\pi}{3}$ this becomes,

$$Area\ R = \ln\left|\frac{1}{\cos\left(\frac{\pi}{3}\right)}\right| + \frac{1}{2}\tan\left(\frac{\pi}{3}\right) - \ln\left|\frac{1}{\cos\left(\frac{\pi}{4}\right)}\right| - \frac{1}{2}\tan\left(\frac{\pi}{4}\right)$$
$$= \ln 2 + \frac{\sqrt{3}}{2} - \ln\sqrt{2} - \frac{1}{2}$$
$$= \ln 2 - \frac{1}{2}\ln 2 + \frac{1}{2}\sqrt{3} - \frac{1}{2}$$
$$= \frac{1}{2}\left(\ln 2 + \sqrt{3} - 1\right)$$

That is, $p = 2,\ q = 3$ and $r = -1$

[7 marks]

Answer 3

(a)

$$y = r \sin\theta$$
$$= a\,(\sin 2\theta \sin\theta)$$

$$\frac{dy}{d\theta} = a\,(\sin 2\theta \cos\theta + 2\cos 2\theta \sin\theta)$$
$$= a\left(2\sin\theta\cos^2\theta + 2\left(\cos^2\theta - \sin^2\theta\right)\sin\theta\right)$$
$$= a\left(2\sin\theta\cos^2\theta + 2\sin\theta\cos^2\theta - 2\sin^3\theta\right)$$
$$= a\left(4\sin\theta\cos^2\theta - 2\sin^3\theta\right)$$
$$= 2a\sin\theta\left(2\cos^2\theta - \sin^2\theta\right)$$
$$= 2a\sin\theta\left(2\cos^2\theta - (1 - \cos^2\theta)\right)$$
$$= 2a\sin\theta\left(3\cos^2\theta - 1\right)$$

But $\frac{dy}{d\theta} = 0$ for a tangent to be parallel to the polar axis

$$\therefore\ 2a\sin\theta\left(3\cos^2\theta - 1\right) = 0$$

Either $\sin\theta = 0$ or $3\cos^2\theta - 1 = 0$

The $\sin\theta = 0$ result shows that the initial angle of the loop at the origin is 0

For $3\cos^2\theta - 1 = 0$

$$\cos^2\theta = \frac{1}{3}$$
$$\cos\theta = \pm\frac{1}{\sqrt{3}}$$

The $\cos\theta = -\frac{1}{3}$ result corresponds to a loop not shown in the diagram that is for values of θ outside of the domain specified.

For the angle of interest, where $\theta = \phi$, $\cos\phi = \frac{1}{\sqrt{3}}$ □

[6 marks]

(b) $r = a\sin 2\theta$ with $\theta = \phi$, $\cos\phi = \frac{1}{\sqrt{3}}$, $\sin\phi = \frac{\sqrt{2}\sqrt{3}}{3}$

$$r = 2a\sin\phi\cos\phi$$
$$= 2a\sqrt{1 - \left(\frac{1}{\sqrt{3}}\right)^2} \times \frac{1}{\sqrt{3}}$$
$$= \frac{2\sqrt{2}}{3}a$$

[2 marks]

(c) Preparation...

$$\cos^2 2\theta + \sin^2 2\theta = 1$$

$$\text{Subtract} \quad \cos^2 2\theta - \sin^2 2\theta = \cos 4\theta$$

$$2\sin^2\theta = 1 - \cos 4\theta$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 4\theta$$

$$\begin{aligned}
Area\ S &= \frac{1}{2}\int_0^{\phi} a^2 \sin^2 2\theta \, d\theta \\
&= \frac{a^2}{4}\int_0^{\phi} 1 - \cos 4\theta \, d\theta \\
&= \frac{a^2}{16}\int_0^{\phi} 4 - 4\cos 4\theta \, d\theta \\
&= \frac{a^2}{16}\left[\, 4\theta - \sin 4\theta \,\right]_0^{\phi} \\
&= \frac{a^2\phi}{4} - \frac{a^2 \sin 4\phi}{16} \\
&= \frac{a^2\phi}{4} - \frac{a^2\, 2\sin 2\phi \cos 2\phi}{16} \\
&= \frac{a^2\phi}{4} - \frac{a^2\, 4\sin\phi\cos\phi\left(\cos^2\phi - \sin^2\phi\right)}{16} \\
&= \frac{a^2\phi}{4} - \frac{a^2}{4} \times \frac{\sqrt{2}\sqrt{3}}{3} \times \frac{1}{\sqrt{3}}\left(\frac{1}{3} - \frac{2}{3}\right) \\
&= \frac{a^2\phi}{4} - \frac{a^2}{4} \times \frac{\sqrt{2}}{3}\left(-\frac{1}{3}\right) \\
&= \frac{1}{36}a^2\left(9\phi + \sqrt{2}\right) \\
&= \frac{1}{36}a^2\left(9\arccos\left(\frac{1}{\sqrt{3}} + \sqrt{2}\right)\right)
\end{aligned}$$

[7 marks]